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CRITICAL NOTICE

MICHAEL POTTER

Reason's Nearest Kin: Philosophies of Arithmetic from Kant to Carnap
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William Demopoulos
Department of Philosophy
University of Western Ontario

This is an excellent book: informative, insightful, suggestive, and a genuine pleasure to read. Potter offers analyses of Kant, Frege, Dedekind, Russell, Wittgenstein, Ramsey, Hilbert, Gödel, and Carnap that are always elegantly presented and often brilliantly clear. The discussion of Hilbert and Gödel is particularly elegant; that of Ramsey breaks new ground; the account of the *Tractatus*, with its illuminating comparison of the work to Hilbert's program, is one of the most accessible I have read. Perhaps the centerpiece of the exposition of philosophical views—the discussion of Hilbert and Gödel focuses, as it must, mainly on technical aspects of Hilbert's brand of formalism—is the discussion of Russell, to whom Potter devotes two chapters (as he does to Frege). I would have preferred a more detailed presentation of type theory, but perhaps Potter will undertake this in future work.[1] Only Potter's discussions of the views of Frege and Carnap struck me as contentious.

In this review, I will focus first on Potter's discussion of Frege, partly for its contentiousness, but also because I believe Frege comes closest to resolving Potter's formulation of the central problem of the philosophy of arithmetic. I will then turn to an aspect of Potter's account of Russell, namely, the 'propositional paradox' and its significance for later developments; the issues raised by this relatively little-known paradox are historically important and I hope to bring them to wider attention by addressing them in this review.

[1] For example, Gregory Landini's claim (Landini [1996]) to have sustained Russell's assertion that full induction is derivable in the 1925 *Principia* without Reducibility is noted (p. 200). But the reader is left in the dark regarding its relevance to John Myhill's well-known theorem that the natural numbers are *not* definable in *Ramified Principia* without Reducibility. Unfortunately, Landini's paper does not address the issue at all as directly as it should; one suspects that an equivalent of Reducibility is implicit in Landini's extensionality assumptions, but it is difficult to be sure without further study. I wish Potter had addressed these issues more fully.

According to Potter, a philosophy of arithmetic must fulfill two desiderata. It must first of all account for the applicability of arithmetic in everyday and scientific practice, a task that 'evidently participates in the wider puzzle of explaining the link between experience, language, thought and the world' (p. 18). In addressing this desideratum, philosophers have sought to base arithmetic on one or another of these four elements, a practice that leads to the second desideratum: 'to explain how arithmetical propositions can depend on such a source for their truth without their truth thereby being rendered contingent on inessential features of that source' (p. 19).

Given the argument structure of *Grundlagen*, it is clear that the central fact for which Frege sought to give an account is our knowledge of the Dedekind infinity of the natural numbers, our knowledge that the concept *natural number* is equinumerous with one of its proper subconcepts. Like Kant before him, Frege took it for granted that we have arithmetical knowledge and that there are therefore arithmetical truths. The task that *Grundlagen* set itself was to show on what this knowledge rests, and to show along the way that this issue can be successfully addressed without an appeal to Kantian intuition. Regarding Frege's relation to Kant, Potter makes a highly perceptive observation:

> Kant had conceived of the objects we make judgements about as necessarily subject to the structure sensibility imposes, and had therefore conceived of us as constrained in reasoning about these objects to construct them in intuition in order to warrant to ourselves their possibility relative to this structure. According to Frege we are subject to no such constraint: there is no further criterion that needs to be satisfied by a term in order for it to denote an object than that it should function linguistically in the correct manner, since objects are not required to be subject to any circumambient structure (p. 68).

In the Introduction to *Grundgesetze* Frege tells us that the 'fundamental thought' of *Grundlagen* is contained in the claim of Section 46 that the content of a statement of number—a statement like 'The Apostles were twelve', for example—involves the predication of something of a concept. If we set aside the move to classes, *Grundlagen*'s development of this idea is carried out in two stages. Echoing his earlier treatment of generality, Frege is led first to the notion of a numerically definite concept, a concept of second level that is (first order) definable in terms of the ordinary first order quantifiers and identity. To say that the apostles were twelve is to predicate such a numerically definite concept of the concept of being an apostle. Frege quickly dismisses the adequacy of this suggestion for its failure to provide a *complete* account of our statements of number: since it restricts itself to concepts, it misses the fact that many of our judgements regarding number employ numerical singular terms in 'recognition statements'. Such statements involve the relation of identity and thus treat numbers as objects rather than concepts. This is especially the case for our 'pure' arithmetical knowledge, since it rests on the use of equations and inequations. The second stage of Frege's development of the fundamental thought is intended to accommodate our use of recognition statements. If we did not take this step, if our understanding of the fundamental thought concluded with the introduction of numerically definite concepts, we could not claim to have given a complete account of our arithmetical knowledge. But if our practices sanction the use of numerical singular terms, there must be implicit in this practice a criterion by which we are able to judge whether a recognition statement is true—to judge whether the same number is alternatively represented as the number of one or another concept. Frege advances the 'numerical equivalence',

$$\mathrm{NxFx} = \mathrm{NxGx}\ .\equiv.\ \mathrm{F} \approx \mathrm{G},$$

—for any concepts F and G, the number of Fs is the same as the number of Gs if and only if the Fs and the Gs are in one-one correspondence—as the solution to this problem. It constitutes the criterion by which we are able to say whether the same number has been given to us in two different ways, as the number of possibly distinct concepts. The numerical equivalence does not *justify* the contention that numbers are objects; it does, however, elucidate a practice that, like ours, incorporates such an assumption. As such, the numerical equivalence expresses the central assumption in Frege's analysis of the content of statements of number.

If Frege's analysis is to be judged correct, it must account for the Dedekind infinity of the numbers and it must expose the unity, which underlies our disparate uses of the concept of number. Recent work[2] has shown that *Grundlagen* contains an outline of how, within second order logic, the Dedekind infinity of the numbers can be derived from the numerical equivalence. But of course the Dedekind infinity of the numbers can be obtained directly from Dedekind's and Peano's axiomatizations. What is the advantage of Frege's development of arithmetic from the numerical equivalence? The answer, I think, is that by contrast with Dedekind and Peano, Frege derives the number-theoretic or pure properties of the numbers from an analysis of their application in the practice of counting. It thus shows how a mathematical analysis such as Dedekind's or Peano's arises out of the most common everyday applications we make of the numbers. By basing his approach on the numerical equivalence, the applicability of arithmetic is therefore built into Frege's account from the very start. The question of the adequacy of Frege's philosophy of arithmetic therefore turns on the adequacy of the account of our knowledge of the numerical equivalence. The

[2] See especially the papers reprinted in the section entitled 'Frege Studies' of Boolos ([1998]).

explanation that Frege himself gave is problematic, since it consisted in deriving the equivalence from an inconsistent theory of classes. But if an alternative justification of the numerical equivalence can be extracted from *Grundlagen*, Frege would seem to have come very close to providing what Potter demands of a satisfactory philosophy of arithmetic.

Potter's discussion of Frege leaves little doubt that he does not believe such a justification is forthcoming. In Potter's view, the numerical equivalence is either a simple stipulation of the truth conditions for identity statements involving numbers or a special kind of analytic truth, one that is peculiar to the introduction of singular terms by means of an equivalence relation (what Crispin Wright has called concept introduction via an abstraction principle). Potter argues that the first alternative is problematic because of the failure of the numerical equivalence to specify the sense of '$NxFx = \ldots$' for all contexts '. . .'. There seems little chance that this lacuna can be filled in a satisfactory manner; for example, Frege's own proposal, the explicit definition of the numbers in terms of classes, leads to well-known difficulties. But the prospects for making the case for the 'analytic truth' of the numerical equivalence are equally dim. Its formal similarity to the 'extensional equivalence':

$$ExFx = ExGx \;.\equiv.\; (x).Fx + Gx$$

—the class of Fs is the same as the class of Gs if and only if all and only Fs are Gs—should make us wary of our grasp of the notion of analyticity being appealed to, the notion, suggested in Section 64 of *Grundlagen*, that the sense of the right-hand side of the numerical equivalence merely 'recarves' the sense of the left. If we can 'see' that the numerical equivalence represents such a recarving, and as such is an analytic truth, why can't we also 'see' this in the case of the inconsistent extensional equivalence? In the absence of a criterion for distinguishing good from bad recarvings, the claim that the numerical equivalence is an analytic truth seems hollow and unconvincing.

I will restrict myself to three brief comments: First, the significance one attaches to the failure to specify truth conditions for all identity contexts is highly sensitive to what understanding one has of the putative *analyticity* of the numerical equivalence. Although the numerical equivalence fails to specify truth conditions for all identity contexts—it does not address the case where one of the terms in a recognition statement is not given to us (either explicitly or implicitly) as the number of a concept—this affects only the robustness of the sense of 'object' that attaches to the thesis that numbers are objects. In short, the numerical equivalence is incapable of specifying a *unique* sequence of objects as 'the natural sequence of numbers'.

My second comment is related to the previous one. In my view, Potter's discussion of analyticity misses the alternative of understanding Frege to have given a correct conceptual analysis of the content of our statements of number. If the analyticity of the numerical equivalence consists in the fact that it is the expression of a conceptual analysis of recognition statements of the form 'NxFx = NxGx,' then to justify its analyticity in this sense is to justify its correctness as an analysis—to justify the claim that the numerical equivalence is *analytic of* the concept of numerical identity which recognition statements of this form express. I have argued that it is correct given the clarity and systematicity it lends to the total store of our arithmetical knowledge. But this is an argument it would not have been possible to make without Frege's proof of the Dedekind infinity of the numbers and his articulation of the substantial framework that proof presupposes. That Frege's analysis is correct is therefore not something to be judged by the simple recognition of the sameness of a sense under a 'recarving'; indeed, it is a virtue of this notion of analysis that it owes nothing to this metaphor.

Lastly, there emerges from Frege's analysis an account of what our knowledge of arithmetic *rests upon*. But by contrast with the notion of analyticity that Potter is assuming, understanding what our knowledge of arithmetic rests upon does not carry with it a means of securing that knowledge against someone who would question the concept of number on which it is based. Frege's philosophy of arithmetic does not therefore provide a *foundation* for arithmetic in one traditional sense of that term. (But neither, for that matter, did Kant's.) Certainly part of the attraction of the re-carving metaphor is that it holds out the promise of a foundation in this sense.

I turn now to the propositional paradox. At the conclusion of *The Principles of Mathematics* (indeed, on the last two pages of the final Appendix of the book), Russell articulated a paradox that has come to be known as the propositional paradox. Potter expounds the paradox, discusses it in the context of Frege's theory of thoughts and shows how Russell discovered that a slightly modified form of the paradox vitiates his substitutional theory of classes and relations. The issues raised are so central to the interpretation of Frege and Russell, and Potter's reconstruction so concise, that it will be worthwhile to go into these matters in some detail.

The only mention of the propositional paradox in Russell's published writings appears to have been the discussion in *Principles*. However, it occupies a significant position in his correspondence with Frege, where it led Frege to extend the theory of sense and reference which he had articulated in *Über Sinn und Bedeutung*. Like Russell's paradox of the class of all classes that are not members of themselves, the propositional paradox illustrates how '[f]rom Cantor's proposition that any class contains more subclasses than objects we can elicit constantly new contradictions' (Russell [1902], p. 147). While one can extract from the derivation of the paradox a proof that there are more properties of propositions than there are propositions, its main interest, like the interest of the Russell paradox, consists in the fact that it challenges a number of 'obvious' assumptions, in this case, assumptions about the nature of propositions and propositional identity. These assumptions would be shown to be problematic even if the class of properties of propositions were somehow restricted so that its cardinality did not exceed that of the class of propositions.

The paradox arises as follows. Let φ be a property of propositions; in Russell's terminology, φ is a propositional function that takes propositions as arguments and yields propositions as values. Consider a function f from properties of propositions to propositions defined by

$$\mathrm{f}\varphi :=_{\mathbf{Df}} : (\mathrm{p}).\varphi\mathrm{p} \supset \mathrm{p}.$$

The notation is justified by Russell's theory of propositions: since φ is a constituent of fφ, f is a many–one relation, and on the assumption that for every φ there is such a proposition fφ, f is a function.

In a propositional theory such as Russell's, '$(\mathrm{p}).\varphi\mathrm{p} \supset \mathrm{p}$' is naturally read as saying that every proposition with property φ is true. It should be noticed, however, that the notion of a truth predicate does not enter into the derivation of the paradox; nor does the paradox require for its formulation the availability of a semantic relation of designation, a point first made explicit in Church ([1984]). In terms of a well-known division of the paradoxes into those belonging to logic and mathematics 'in their role as symbolic systems' vs. those which arise from the bearing of logic on 'the analysis of thought' (Ramsey [1931], p. 21), the propositional paradox falls squarely into the second category. However I am unsure how best to resolve the largely terminological question of whether or not the paradox is *semantic*. Perhaps the most one can say is that it is neither more nor less a semantic paradox than an algebraic semantics is a semantic theory.

Again, since φ is a constituent of the proposition fφ, it also follows that

$$\mathrm{f}\varphi = \mathrm{f}\psi. \supset . \varphi = \psi \tag{1}$$

Now let ψ be a property of propositions given by

$$\psi(\mathrm{p}) :=_{\mathrm{Df}} : \exists\varphi.\ \mathrm{p} = \mathrm{f}\varphi\ .\ \sim\varphi(\mathrm{p})$$

and suppose $\sim\psi(\mathrm{f}\psi)$:

$$\sim\exists\varphi.\ \mathrm{f}\psi = \mathrm{f}\varphi\ .\ \sim\varphi(\mathrm{f}\psi);$$

i.e.

$$(\varphi).\ \mathrm{f}\psi = \mathrm{f}\varphi. \supset .\varphi(\mathrm{f}\psi),$$

and therefore, $\psi(\mathrm{f}\psi)$.

Next suppose $\psi(f\psi)$:

$\exists\varphi.\ f\psi = f\varphi\ .\ \sim\varphi(f\psi)$.

By (1), $\varphi = \psi$, and so $\sim\psi(f\psi)$, which, together with our previous result, implies a contradiction.

Potter's presentation (which has guided our exposition) brings out very clearly how little the paradox requires from the theory of Russellian propositions developed in *Principles*. Thus, (1) is implied by the Russellian assumption that the propositional function φ is a constituent of $f\varphi$; and the assumption that for every φ there is a proposition $f\varphi$—the proposition, $(p).\varphi p \supset p$—is so intuitively obvious that it would seem to be a necessary component of any theory of propositions. Notice, by the way, that the argument would go through without the assumption that f is a one-one function: assuming that for every φ there is a proposition of the form $(p).\varphi p \supset p$, all we require, in addition to implicit principles of propositional identity, is that the correspondence

$(p).\varphi p \supset p \mapsto \varphi$

be functional.

The solution mandated by the ramified theory of types depends on observing the restrictions it imposes on the 'orders' of the propositions and propositional functions to which the derivation of the paradox appeals.[3] The technical idea behind the ramified theory's stratification of propositions into orders is that the order of a proposition, p, must exceed the order of any proposition that falls within the domain of one of its quantified propositional variables, and therefore, by the restrictions the theory imposes, p cannot itself fall within the range of one of its quantified propositional variables. By an extension of terminology, the order of a propositional variable is the order of the propositions that belong to its range of values. According to the ramified theory, the proposition $q =_{Df} (p).\psi p \supset p$ and the propositional function ψ, defined in the course of the derivation of the paradox, are such that if the order of the bound propositional variable in q is n, the orders of q and ψ must be $n + 1$, so that q is not itself a possible argument for the function ψ. Since the contradiction was derived from the supposition that $\psi(q)$ or $\sim\psi(q)$, the argument collapses once the restrictions imposed by ramification are taken into account. Notice that this solution preserves the idea that φ is a constituent of $f\varphi$, as well as the idea that there always is a proposition $f\varphi$ corresponding to φ. However, the generality of these assumptions is the restricted generality of the ramified theory: for every ψ of order $n + 1$, there

[3] The remarks that follow are merely illustrative. The subject of the ramified theory is subtle and complex and deserves a careful reconsideration.

is a proposition of order $n+1$ which asserts the truth of every proposition of order n for which ψ holds.

As noted earlier, Potter applies his discussion of the propositional paradox to Frege's hierarchy of senses. As is well known, there are two hierarchies that emerge from Frege's philosophical reflections on logic and language: the hierarchy of functions (or concepts) and objects, and the hierarchy of senses. The first hierarchy has a simple type-theoretical structure and, like the simple theory of types, is naturally motivated by considerations of predicability: first level functions are predicated of objects, second level functions of first level functions, third level functions of second level functions, etc. There is therefore no analogue in Frege's hierarchy of functions of Russell's paradox of the propositional function that is predicable of all and only those propositional functions that are not predicable of themselves. But as Church ([1984]) makes completely explicit, and as Russell himself certainly perceived, the propositional paradox can be derived from the theory of Russellian propositions of *Principles* even when this theory is formulated within a simple type-theoretical formalism. It is therefore natural to ask whether Frege's theory of sense, and in particular, his theory of thoughts, fares just as badly, to ask in other words whether Frege's theory of thoughts admits the construction of an analogue of the propositional paradox. Following Frege ([1902]), Potter argues that it does not and that the reason why it does not is to be found in the hierarchy of senses. Potter further claims that Dummett's proposal to modify Frege's theory of sense by dropping the category of indirect sense renders the theory unable to deal with the propositional paradox, and thus leaves it irremediably inconsistent. As will become clear in a moment, if Potter is right, the propositional paradox would also pose a difficulty for the two-level theory of sense advanced in Parsons ([1981])—a theory that admits one degree of indirectness but eschews degrees of indirectness of two or more.

Recall that the hierarchy of senses arises from the doctrine of indirect reference: When a sentence is imbedded in an oblique context it does not have its customary reference, but refers instead to its customary sense, it refers, that is, to the thought it is normally taken to *express* when it stands outside such a context. Now so far as his discussion in *Über Sinn und Bedeutung* is concerned, Frege can be understood as proposing that in every oblique context, an expression refers to its customary sense. And since the sense of an expression determines its reference, in an oblique context, the sense of the sentence must also change. This however yields only *one* indirect sense, the same indirect sense for *all* oblique contexts in which the expression occurs, so that if a sentence is singly imbedded in an oblique context, it has an indirect reference, determined by its indirect sense, and if it is doubly imbedded, its reference is again indirect, but is identical with its reference when it is singly imbedded. Thus Frege's discussion in *Über Sinn und Bedeutung* can consistently be understood as assuming that the hierarchy closes off after the first degree of indirectness.

Now in his response to the propositional paradox (in the letter to Russell previously cited), Frege extends his earlier theory to one that sanctions an infinite hierarchy of indirect senses. In this hierarchy each imbedding into an oblique context is associated with a new indirect reference and a new indirect sense, which determines it as the reference of the expression thus imbedded. There are, therefore, arbitrarily high degrees of indirectness, corresponding to arbitrarily deep imbeddings into oblique contexts. As we shall see, Frege's extension allows for an elegant solution to the propositional paradox. However, this solution comes at a cost: since our grasp of the notion of a sense begins to waiver after the first degree of indirectness, this extension lacks the intrinsic plausibility and intuitiveness of the unextended and zero-level theories.

According to Potter, Frege saw that in order to avoid the propositional paradox

> we need only label the occurrences of 'ψ' and 'φ' in the argument according to the degree of indirectness of their reference [. . .] The supposition of the first part of the argument is that $\sim\psi_0(\mathrm{f}\psi_2)$. It follows that
>
> $(\varphi).\ \mathrm{f}\psi_2 = \mathrm{f}\varphi_1\ .\supset .\ \varphi_0(\mathrm{f}\psi_2),$
>
> from which we cannot conclude a contradiction because of the difference in degree between ψ_2 and φ_1 [. . .] [Moreover,] the only way to resist [the paradox] on a Fregean conception of sense is to distinguish between indirect meanings of the first and second degrees (p. 135).

Potter is making two quite separate claims here. First there is the claim of Frege's which Potter is expounding, namely that the stratification of sense and reference into two or more degrees of indirectness blocks the propositional paradox; and secondly, there is Potter's claim that this is the only way of blocking the paradox on a Fregean conception of sense and reference.

Potter's exposition (which I have quoted almost in its entirety) is highly compressed and might easily elude even an attentive reader. It may therefore be worthwhile to see just how the assignment of degrees resolves the paradox and to understand a bit more fully why this particular assignment of degrees is justified.

Potter's labeling of the supposition of the first half of the argument is to be understood as requiring that instantiation on the bound variable must preserve degree indices. The instantiation from $(\varphi).\ \mathrm{f}\psi_2 = \mathrm{f}\varphi_1\ .\supset .\ \varphi_0(\mathrm{f}\psi_2)$ to ψ therefore yields

$f\psi_2 = f\psi_1 . \supset . \psi_0(f\psi_2)$.

But since the thoughts $f\psi_i$ ($i = 1, 2$) may differ, we cannot conclude that $\psi_0(f\psi_2)$ and the contradiction is blocked. This resolution therefore hinges on the claim that in $\sim\psi_0(f\psi_2)$ the second occurrence of 'ψ' has an indirect sense (and reference) of the second degree. This is why the paradox is a difficulty even for a view like Parsons's where the level indices do not exceed one.

According to Frege ([1902]), the assignment of degrees of indirectness is explained as follows. If 'Every thought having the property ψ is true'—more simply and hereafter, 'Every ψ is true'—is understood to *express* a thought, 'ψ' has a direct occurrence and refers to its ordinary reference, which, in this case, is a property of thoughts. Using Potter's device of indices, we would write,

Every ψ_0 is true,

—where the index tells us that the occurrence of 'ψ' is direct and that our expression is a sentence. If, however we mean to *designate* the thought this sentence expresses, we typically do so by imbedding it in an oblique context yielding the designating expression,

The thought that every ψ is true.

On Potter's use of indices this is expressed by

Every ψ_1 is true,

—which is potentially misleading since it certainly looks like a sentence rather than a designating expression. Now the supposition we are considering in the first step of the argument is that the thought that every ψ_1 is true is not a ψ. When we delete the oblique-context-forming expression 'the thought that' and replace this formulation with one given wholly in terms of indices, we 'trace' the occurrence of 'the thought that' by increasing by one the index on the first occurrence of 'ψ,' and we have,

Every ψ_2 is true is not a ψ_0

—which is, of course, just Potter's $\sim\psi_0$ $(f\psi_2)$.

The infinite hierarchy of senses is reminiscent of Dedekind's notorious Theorem 66 of *Was Sind und Was Sollen die Zahlen?* which seeks to establish the existence of the 'simply infinite system' consisting of Dedekind's thoughts and his Ego. The similarity to Frege's hierarchy of thoughts may explain why in one of his posthumous writings (Frege [1897], p. 136) Frege refers favorably to Dedekind's argument and the conception of thoughts that it employs. The favorable reference is otherwise puzzling in view of the differences in their respective accounts of the infinity of the numbers. For

Dedekind our knowledge of the infinity of the numbers is secured once we are in possession of *some* simply infinite system or other, since by his categoricity result (Theorem 132), all such systems are isomorphic. By contrast, if our reconstruction of Frege's analysis in terms of the numerical equivalence is accepted, then Frege's account of our knowledge of an infinite concept derives specifically from our knowledge of the numbers and the applications we make of them when we apply them to concepts.

Does the theory of sense and reference fall into inconsistency if it is emended by excluding the notion of indirect sense (Dummett's proposal) or by excluding all indirect senses of degree higher than one (Parsons's proposal)? The propositional paradox would seem to show that it does. Whether these emendations can be supplemented with devices which would allow them to block the paradox, and whether such a supplementation leaves the theory 'Fregean' in conception, are issues I hope to have clarified but about which I have no settled opinion.

Let us turn now to Russell's substitutional theory of classes and relations and to Potter's account of the connection between the propositional and 'substitutional' paradoxes.[4]

The central virtue of the substitutional theory is that it promises to be a type-free theory that employs only propositions and the individuals and propositions, which occur in them. The notion of a Russellian proposition that is peculiar to the substitution theory must differ from that of the *Principles* notion of a proposition by virtue of the fact that the substitution theory proceeds from a more Spartan conception of what are the possible constituents of propositions. (See Urquhart and Pelham [1994]) for a characterization of the concept of Russellian proposition that is employed in the substitution theory.) For example, on the substitution theory, propositional functions are not possible constituents of propositions since they are treated as 'incomplete symbols' after the manner of 'On denoting'. Informally, the substitution theory may be thought of as the theory of a four-termed relation defined on a domain of Russellian propositions and individuals, namely the relation, *q results from p by substituting b for a wherever a occurs in p*. It is important to emphasize that the domain on which the substitution relation is defined is not necessarily (or even usually) a domain of linguistic symbols. In the cases of greatest interest, p and q are propositions, a and b, propositions or individuals. But substitutions can be meaningfully carried out on *any* combination of propositions and individuals. A suggestive comparison from Urquhart and Pelham ([1994]): consider a

[4] The paradox is formulated in a letter of Russell's to Ralph Hawtrey (22.i.07); this letter was recently uncovered by Landini, who reproduces it on the frontispage of his book, (Landini [1998]). (Potter incorrectly dates the letter as 1909.) The terminology 'substitutional paradox' comes from Urquhart and Pelham ([1994]).

domain of individuals and the sets defined on them. Every permutation of the individuals induces a permutation on the sets; substitutions are akin to the permutations, propositions to the sets, of this example; and the substitution theory is an investigation of the properties of substitutions of the same sort as an investigation of the properties of various permutations of such a domain. The analogy is imperfect since it is restricted to substitutions in propositions of individuals by individuals. But it succeeds in making the point that Russell's conception of a substitutional theory—according to which a substitution is not a syntactic operation—is a coherent one.

The substitution-theoretic analogue of a propositional function is the incomplete symbol, *the result of substituting a in p by*, for which we use the notation, '(p|a),' where p is a proposition and a one of its possible constituents. The incomplete symbol (p|a) is called a matrix (not to be confused with the matrices of *Principia*). The substitution theory countenances neither classes nor propositional functions—hence the terminology of incomplete symbols—but quantifies over only propositions and individuals. Every pair p and a, with p a proposition, is associated with such a propositional function analogue and, in accordance with the usual association of classes with propositional functions, with a class. On Russell's development of the substitution theory, identity is defined in terms of the primitive notion of substitution. Departing from Russell, and employing a notation that highlights its connection with the propositional functions of earlier and later work, Potter writes '(p|a)b = q' for 'q results from p by the substitution of b for a wherever a occurs in p.' In the cases of interest, p and q will be propositions, and we are to think of the 'class' determined by the matrix (p|a) as consisting of all those objects b for which the proposition q is true. The substitutional theory carries out most fully the idea of using the doctrine of incomplete symbols for the avoidance of the paradoxes. Consider, for example, the sentence '$x \in \alpha$.' This appears to treat α as an entity to which x is related by the membership relation. This appearance is dispelled in the substitution theory by choosing a matrix—(p|a), say—to 'represent' α and then transcribing '$x \in \alpha$' as

$$(\exists p, a).\ (p|a)x = q\ .\ q.$$

This substitution-theoretic sentence quantifies over only p and a, which therefore occur as genuine constituents of the proposition it expresses. But while the notation '(p|a)' is meaningful in context, it is not 'meaningful in isolation,' and therefore the problematic locutions, '$\alpha \in \alpha$' and '$\alpha \notin \alpha$,' which are essential to the derivation of the Russell paradox, cannot even be meaningfully formulated in the substitutional theory. This is achieved, moreover, without any appeal to the notion of a type.

But there is a variant of the propositional paradox that compromises the substitutional theory, and in so doing, makes Russell's move toward ramification that much more compelling. Following our earlier discussion, we may ask whether there is a function f, from *pairs of propositions* to propositions, satisfying the condition,

$$f(p, q) = f(r, s) \,.\supset .\, (p|q)f(p, q) \,.\equiv .\, (r|s)f(p, q). \qquad (2)$$

Clearly, (2) is an analogue of (1) which was employed in the derivation of the propositional paradox as a constraint on the mapping from propositional functions to propositions. (2) would follow if

$$f(p, q) = f(r, s) \,.\supset .\, (x).\, (p|q)x \,.\equiv .\, (r|s)x. \qquad (3)$$

Now (3) will follow if from $f(p, q) = f(r, s)$ we can infer

$$p = r \,.\, q = s.$$

So suppose that f(p,q) *is* the same proposition as f(r,s) and let f(p,q) be $(p|q)b = t$, and f(r,s) be $(r|s)b = t$.[5] Clearly the substitutional theory is committed to the existence of such a proposition, *since it simply expresses the fundamental relation of the theory*. But if $[(p|q)b = t] = [(r|s)b = t]$, $p = r \,.\, q = s$, and (2) follows.[6] Potter shows how, with (2) established, the argument proceeds along the usual Cantorian lines by defining a proposition

$$p = (\exists r, s).\, q = f(r, s) \,.\, \sim(r|s)q$$

and then considering the result of substituting f(p, q) for q in p.

As I said at the beginning of this review, Potter's is an excellent book, one that enhances our understanding of the tradition with which it deals while raising our appreciation of the clarity and depth of that tradition.[7]

References

Boolos, G. [1998]: *Logic, Logic and Logic*, Cambridge: Harvard University Press.

Church, A. [1984]: 'Russell's Theory of Identity of Propositions', *Philosophia naturalis*, **21**, pp. 513–22.

Frege, G. [1897]: 'Logic', in H. Hermes *et al.* (*eds*) [1980], *Posthumous Writings*, Chicago: University of Chicago Press.

[5] As Potter points out, aside from notational differences, this is the example Russell uses in his letter to Hawtrey.

[6] Notice that the argument proceeds from the assumption that the two Russellian propositions, $[(p|q)b = t]$ and $[(r|s)b = t]$, are *identical*. This is stronger than the claim that two substitutions yield the same result, which would allow, e.g. (Alys|Edith)Alys = (Alys|Dora)Alys, but Edith $\neq$ Dora. (Thanks to Alasdair Urquhart for clarification on this matter.)

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Frege, G. [1902]: 'Letter to Russell of 28.xii.1902', in B. McGuinness (*ed.*) [1980], *Philosophical and Mathematical Correspondence of Gottlob Frege*, Hans Kaal (tr.), Chicago: University of Chicago Press, pp. 152–54.

Landini, G. [1996]: 'The *Definability* of the Set of Natural Numbers in the 1925 *Principia*', *Journal of Philosophical Logic*, **25**, pp. 597–614.

Landini, G. [1998]: *Russell's Hidden Substitution Theory*, Oxford: Oxford University Press.

Parsons, T. [1981]: 'Frege's Hierarchies of Indirect Senses and the Paradox of Analysis', in Peter A. French *et al.* (*eds*), *Midwest Studies in* Philosophy, Vol. VI, Minneapolis: University of Minnesota Press, pp. 37–58.

Ramsey, F. P. [1931]: *The Foundations of Mathematics and Other Logical Essays*, R. B. Braithwaite (ed.), London: Kegan Paul.

Russell, B. [1902]: 'Letter to Frege of 29.ix.1902', in B. McGuinness (*ed.*) [1980], *Philosophical and Mathematical Correspondence of Gottlob Frege*, Chicago: University of Chicago Press, pp. 147–48.

Urquhart, A. and Pelham, J. [1994]: 'Russellian Propositions', in D. Prawitz *et al.* (*eds*), *Logic, Methodology and Philosophy of Science*, Elsevier: North Holland, pp. 307–26.