VOLKER HALBACH

TARSKIAN AND KRIPKEAN TRUTH

ABSTRACT. A theory of the transfinite Tarskian hierarchy of languages is outlined and compared to a notion of partial truth by Kripke. It is shown that the hierarchy can be embedded into Kripke's minimal fixed point model. From this results on the expressive power of both approaches are obtained.

1. INTRODUCTION

In his celebrated paper Kripke (1975) complained that there was not any detailed treatment of a transfinite Tarskian hierarchy of languages. Twenty years later the situation has improved[1], but Kripke's own approach of a type-free theory of truth is now far better worked out than the Tarskian hierarchy. As already noticed by Kripke both approaches have much in common, and it is the aim of this paper to clarify some similarities and differences and thereby to develop a theory of the Tarskian hierarchies.

Before being able to start I have to fix some notation. The expressions of all languages considered here are identified with natural numbers. For this purpose choose a standard sequence coding $\langle \ldots \rangle$ for natural numbers and assign a number to every symbol of the considered language in a natural way. Then formulas and terms of the language are certain sequences, i.e. numbers. The starting point for the discussion will be a recursive language $\mathcal{L}$ with only finitely many non-logical symbols, that is a language such that the properties of being a variable etc. is recursive. Hence the set of all terms, formulas, sentences, and so on will be recursive, too. Furthermore I assume that $\mathcal{L}$ has no predicate symbol of the form T_n where $n \in \omega$.[2]

Let $\mathcal{A}$ be an acceptable structure in the sense of (Moschovakis, 1974) for $\mathcal{L}$. Note that therefore $\mathcal{A}$ contains the natural numbers, and the recursive functions and predicates are expressible in $\mathcal{L}$ which therefore must have suitable predicate or function symbols. If h is a recursive function I shall write $\underset{\cdot}{h}$ for a function expression representing h in the language $\mathcal{L}$. In the case $\mathcal{L}$ has no suitable function expressions $\underset{\cdot}{h}$ is thought to be defined away by predicates in the usual way.

The fact that a formula ϕ is valid in a model $\mathcal{B}$ at an assignment a is expressed by $\mathcal{B} \models \phi[a]$. If ϕ does not contain free variables, i.e. ϕ is a sentence, $\mathcal{B} \models \phi$ abbreviates $\mathcal{B} \models \phi[a]$ for arbitrary a.

2. TARSKIAN HIERARCHIES[3]

To the language $\mathcal{L}$ a new one-place predicate symbol T_0 is added to obtain the language $\mathcal{L}_1$ which is interpreted by extending $\mathcal{A}$ in such a way that T_0 is the truth predicate for the initial language $\mathcal{L}$. Now $\mathcal{L}_1$ is expanded by T_1 suitably interpreted and so on. At limit stages λ one gets a language $\mathcal{L}_\lambda$ containing a predicate symbol T_α for any $\alpha < \lambda$ where each predicate symbol T_α is interpreted as truth predicate for the preceding language. The construction is then carried on by adding a new symbol T_λ to be interpreted in the obvious way. This is the rough idea how to construct transfinite (well-founded) Tarskian hierarchies, but there is still a problem. It has to be explained what is meant by the symbols T_α. If there is a T_α for each ordinal α these symbols cannot be natural numbers. It may be argued that the identification of expressions with natural numbers should be given up in consequence. But then it would no longer make sense to ask whether a set of formulas or a whole language is recursive because properties like being recursive can only be applied to sets of natural numbers and certainly not to uncountable sets. Furthermore languages which are not recursive or at least recursively enumerable cannot be considered as being really usable (or being analogues of usable languages). Therefore I shall use codes, i.e. natural numbers, as indices of the truth predicates instead of ordinals. Obviously, this limits the height of Tarskian hierarchies to countable ordinals. I shall also put another condition on the indices: It should be readily recognizable whether an index is higher than another and whether a given number is an index at all. This reflects the idea that the hierarchies should be somehow constructible and consist of recursive languages. Call such hierarchies posessing such a recursive ordering of their indices *recursive* Tarskian hierarchies. As every recursive ordering of natural numbers not embracing all natural numbers can be extended there is not any recursive Tarskian hierarchy which is not included by a larger recursive Tarskian hierarchy. It can even be proved, that if a recursive well-ordering is given and there are infinitely many numbers not in the field of this well-ordering it can be expanded to a well-ordering of length[4] ω_1^{CK} such that any initial segment of it is recursive. Instead of talking about increasing sequences of recursive Tarskian hierarchies I shall simplify discussion by considering hierarchies which are unions of maximal sequences of recursive Tarskian hierarchies. Obviously, every initial part of the ordering of the truth predicates in such a hierarchy has to be recursive, but the whole ordering will in general not be recursive. It follows from well known results of recursion theory that the maximal height of such an ordering is ω_1^{CK} because any recursive well-ordering has order type smaller than ω_1^{CK}. This leads to the following definition:

Let $\prec$ in the following be a linear well-ordering of natural numbers of length ω_1^{CK} such that $\prec$ restricted to any initial segment $\{n\colon\ n \prec i\}$ for $i \in \mathrm{Fld}(\prec)$ is a recursive relation.

I shall refer to $\prec$ as the *index system*. Let $\alpha_\prec$ be the α-th element of the well-ordering $\prec$. Hence $0_\prec$ is the smallest element in the index system. Using Kleene's ordinal notation system[5] $\mathcal{O}$ it can be shown that there are indeed well-orderings satisfying the above conditions. In general, initial segments of paths in $\mathcal{O}$ are recursively enumerable, but Jokusch has shown in (Jokusch, 1975) that there are paths *through* (i.e. of length ω_1^{CK}) such that any initial segment is recursive. Alternatively one could simply use Girard's version of $\mathcal{O}$ in (Girard, 1987) which is a modification of Kleene's such that any initial segment of a path is not only r.e. but even recursive. Note that an index system has at least complexity Π_1^1.

After having introduced the index system the Tarskian hierarchy with $\mathcal{L}$ as initial language is easily defined.

DEFINITION 1. Let L be the function from ω_1^{CK}, that is, from the set of all ordinals smaller than ω_1^{CK}, into the power set of ω. Let $L(\alpha)$ be the language[6] $\mathcal{L}$ expanded by all predicate symbols T_k such that $k \prec \alpha_\prec$.

Models for the languages $L(n)$ of the hierarchy are defined inductively in the obvious way.

DEFINITION 2. A sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ of length ω_1^{CK} is defined such that $\mathcal{A}_\alpha$ is an expansion of $\mathcal{A}$ to the language $L(\alpha)$. $\mathcal{A}_0$ is $\mathcal{A}$ itself. In order to define $\mathcal{A}_{\alpha+1}$ it is sufficient to fix the interpretation (extension) $\mathcal{A}_{\alpha+1}(\mathrm{T}_{\alpha_\prec})$ of the predicate symbol $\mathrm{T}_{\alpha_\prec}$, so let $\mathcal{A}_{\alpha+1}(\mathrm{T}_{\alpha_\prec})$ be the set of all $\mathcal{A}_\alpha$-valid sentences of $L(\alpha)$. If λ is a limit number $\mathcal{A}_\lambda$ is simply the union $\bigcup_{\alpha<\lambda} \mathcal{A}_\alpha$ of all preceding models (here the union is to be defined in a suitable way on models).

Sometimes I shall write $L(\omega_1^{\mathrm{CK}})$ for the union of all languages. Similarly, $\mathcal{A}_{\omega_1^{\mathrm{CK}}}$ is the union of all models.

The hierarchy may be considered as a variant of the ramified analytical hierarchy, built up over the acceptable model $\mathcal{A}$, because each level of the Tarskian hierarchy has exactly the same strength as the corresponding level in the ramified analytical hierarchy[7]. To define this hierarchy let $\mathcal{L}_\alpha^2$ be the extension of $\mathcal{L}$ having all second-order variables X_n^β where $\beta < \alpha$ and $n \in \omega$. The languages $\mathcal{L}_\alpha^2$ are closed under the usual operations of first-order logic and arbitrary quantification of the second-order quantifiers. Let the variable $\mathcal{L}^2_\alpha$ range over all subsets of the universe of $\mathcal{A}$ definable in a language $\mathcal{L}^2_\beta$ with $\beta < \alpha$. All first-order vocabulary is interpreted as before in $\mathcal{A}$. Then one has:

THEOREM 3. *Suppose $\alpha < \omega_1^{\mathrm{CK}}$. Then the sets definable in $L(\alpha)$ are exactly the sets definable in $\mathcal{L}^2_\alpha$.*

The proof is carried out by showing that membership may be defined using the truth predicate. Then the second-order variables range under this interpretation over sentences of the language $L(\beta)$ having exactly one fixed variable. The proof of the opposite direction is also carried out by induction on the index system. The induction basis is a well-known result, namely that truth in $\mathcal{L}$ may be defined by second-order variables of level 0. For the theorem above this proof is inductively iterated for $\alpha < \omega_1^{\mathrm{CK}}$.

Because of Theorem 3 the expressive power of the Tarski hierarchy does not depend on its index system.

In the case $\mathcal{L}$ is the language of arithmetic and $\mathcal{A}$ the standard model, a more usual hierarchy is the hyperarithmetic hierarchy. By results of Kleene, the sets definable in $L(\alpha)$ and in the $\omega(1+\alpha)$-th level are the same (see also (Feferman, 1991) and (Halbach, 1995)).

In the following sections I shall relate the Tarskian hierarchy to an unramified language, namely that containing only one additional truth predicate interpreted in way that Kripke proposed in (1975).

3. KRIPKE'S THEORY OF TRUTH

I shall recall (a variant of) the definition and some facts about the minimal fixed point model based on the strong Kleene evaluation scheme. The language for which models will be investigated is $\mathcal{L}_\mathrm{T}$, that is, $\mathcal{L}$ expanded by an additional one-place predicate symbol T. The models to be described in this section are partial, that is, there are sentences neither true nor false in the model, more specific, the extension $\mathrm{S}_1 := \mathcal{A}(\mathrm{T})$ of the truth predicate is not the complement of its antiextension $\mathrm{S}_2 := \mathcal{A}(\neg\mathrm{T})$. All other vocabulary is interpreted as in the acceptable model $\mathcal{A}$, and for complex formulas involving formulas containing T the strong Kleene 3-valued logic applies, or more formally speaking, depending on the extension S_1 and antiextension S_2 the partial model $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2)$ is defined as follows. S_1 will always be assumed to consist of sentences only and the pair of S_1 and S_2 is presupposed to be disjoint. Note that in the special case $\mathrm{S}_1 = \mathrm{S}_2 = \emptyset$ these requirements are satisfied.

(i) The partial model $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2)$ is an expansion of $\mathcal{A}$ to the language $\mathcal{L}_\mathrm{T}$.

(ii) $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \mathrm{T}t[a]$, if the value of t in $\mathcal{A}$ under the assignment a is a sentence of $\mathcal{L}$ and an element of S_1.
(iii) $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \neg\mathrm{T}t[a]$, if the value of t in $\mathcal{A}$ under the assignment a is no sentence of $\mathcal{L}$ or an element of S_2.
(iv) $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \neg\neg\phi[a]$, if $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \phi[a]$.
(v) $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \phi\wedge\psi[a]$, if $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \phi[a]$ and $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \psi[a]$.
(vi) $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \neg(\phi \wedge \psi)[a]$, if $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \neg\phi[a]$ or $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \neg\psi[a]$.
(vii) $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \exists x\phi$, if $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \phi[a']$ for an assignment a' differing from a at most in the value of the variable x.
(viii) $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \neg\exists x\phi[a]$, if $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \neg\phi[a']$ for all assignments a' differing from a at most in the value of x.

Sometimes I shall also use other connectives which are defined in the usual way for the strong Kleene scheme.

Note that if $\mathrm{S}_1 \cup \mathrm{S}_2$ contains the set of all sentences of $\mathcal{L}_\mathrm{T}$ then $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2)$ is a classical (non-partial) model for $\mathcal{L}_\mathrm{T}$. If $(\mathrm{S}_1, \mathrm{S}_2)$ and $(\mathrm{S}'_1, \mathrm{S}'_2)$ are disjoint pairs of sets such that S_1 and S'_1 are sets of sentences of $\mathcal{L}_\mathrm{T}$ an ordering is defined in the following way:

$$(\mathrm{S}_1, \mathrm{S}_2) \leq (\mathrm{S}'_1, \mathrm{S}'_2) :\Longleftrightarrow \mathrm{S}_1 \subseteq \mathrm{S}'_1 \text{ und } \mathrm{S}_2 \subseteq \mathrm{S}'_2$$

Now the so-called Kripke-jump Φ is defined on such pairs.

$$\Phi(\mathrm{S}_1, \mathrm{S}_2) := (\{\phi\colon \mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \phi\}, \{\phi\colon \mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \neg\phi\})$$

As $(\mathrm{S}_1, \mathrm{S}_2) \leq (\mathrm{S}'_1, \mathrm{S}'_2)$ and $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2) \models \phi[a]$ implies $\mathcal{A}(\mathrm{S}'_1, \mathrm{S}'_2) \models \phi[a]$ it can be concluded that Φ in monotone, that is $(\mathrm{S}_1, \mathrm{S}_2) \leq (\mathrm{S}'_1, \mathrm{S}'_2)$ implies $\Phi(\mathrm{S}_1, \mathrm{S}_2) \leq \Phi(\mathrm{S}'_1, \mathrm{S}'_2)$. Therefore there must be fixed points of Φ, and in particular a smallest fixed point $(\mathrm{S}_1, \mathrm{S}_2)$ satisfying $(\mathrm{S}_1, \mathrm{S}_2) \leq (\mathrm{S}'_1, \mathrm{S}'_2)$ for all other fixed points $(\mathrm{S}'_1, \mathrm{S}'_2)$ of Φ. Hence a sentence is valid in all fixed point models if and only if it is valid in the minimal fixed point model.

Let $\Phi^\alpha(\mathrm{S}_1, \mathrm{S}_2)$ be the α-fold application of Φ to the pair $(\mathrm{S}_1, \mathrm{S}_2)$ (taking unions at limit stages). There is a smallest ordinal κ such that

$$\Phi^\kappa(\emptyset, \emptyset) = \Phi^\alpha(\emptyset, \emptyset)$$

for all ordinals $\alpha > \kappa$. Let κ for rest of the paper be this closure ordinal κ. The model $\Phi^\kappa(\emptyset, \emptyset)$ is of course the minimal fixed point model. Note that the closure ordinal κ cannot be recursive and is hence equal or larger than ω_1^{CK}. For grounded formulas, that is, either true or false at the assignment in the minimal fixed point model, I define the Kripke-degree *kd*.

$$kd(\phi, a) := \min\{\alpha\colon\ \mathcal{A}(\Phi^{\alpha}(\emptyset,\emptyset)) \models \phi[a] \text{ or } \mathcal{A}(\Phi^{\alpha}(\emptyset,\emptyset)) \models \neg\phi[a]\}$$

In the case that there is not such $\alpha < \kappa$ put $kd(\phi, a) = 0$. If ϕ has no free variables $kd(\phi, a) = kd(\phi, b)$ for all assignments a and b. In this case I shall omit the assignment and simply write $kd(\phi)$. In the next two sections I shall show how the Kripke-degrees of formulas and the levels of the Tarskian hierarchy correspond to one another.

4. FROM THE TARSKIAN HIERARCHY TO KRIPKEAN TRUTH

In (1975) Kripke has shown how the finite levels of the Tarskian hierarchy can be embedded into a sublanguage of $\mathcal{L}_{\mathrm{T}}$. Here I shall prove the result for transfinite levels, too. Roughly speaking, one simply replaces all formulas $\mathrm{T}_n t$ (t a term) by a conjunction saying that the translation of t is true and that t is a sentence of the language $L(n)$. More exactly, the recursion theorem is employed to define for any $\alpha < \omega_1^{\mathrm{CK}}$ a recursive function f_α satisfying the following conditions[8].

$$f_\alpha(n) := \begin{cases} \neg f_\alpha(\phi) & \text{if } n = (\neg\phi) \\ f_\alpha(\phi) \wedge f_\alpha(\psi) & \text{if } n = (\phi \wedge \psi) \\ \exists x f_\alpha(\phi) & \text{if } n = (\exists x \phi) \\ \mathrm{T} \underset{\bullet}{f_\alpha}(t) \wedge \mathrm{Sent}_{L(\beta)}(t) & \text{if } n = (\mathrm{T}_{\beta_\prec} t) \text{ and } \beta < \alpha \\ n & \text{else} \end{cases}$$

Here $\mathrm{Sent}_{L(\beta)}(x)$ expresses that x is in the language $\mathcal{L}$ expanded by all predicate symbols T_i such that $i \prec \beta_\prec$. $\mathrm{Sent}_{L(\beta)}(x)$ may be defined as a two-place predicate because the ordering of indices of the truth predicates of the language $L(\alpha)$ is recursive. Hence $\mathrm{Sent}_{L(\beta)}(x)$ depends on α, so I should index $\mathrm{Sent}_{L(\beta)}(x)$ by α, but I have avoided this for sake of a simple notation.

By an easy induction on α it can be shown using the above lemma that the Kripke-degree of a translation of a formula of the hierarchy is bounded by the maximum of the indices occurring in the formula:

LEMMA 4. *If $\phi \in L(\alpha)$ then $kd(f_\alpha(\phi), a) \leq \alpha$ for arbitrary assignments a.*

Again by induction on α it is established that the translation preserves validity.

THEOREM 5. *If ϕ is a sentence of $L(\alpha)$ ($\alpha < \omega_1^{\mathrm{CK}}$), then the following equivalence holds:*

$$\mathcal{A}_{\omega_1^{\mathrm{CK}}} \models \phi[a] \iff \mathcal{A}(\Phi^{\kappa}(\emptyset,\emptyset)) \models f_\alpha(\phi)[a]$$

5. FROM KRIPKEAN TRUTH TO TARSKIAN TRUTH

For this and the final section I shall make an additional assumption on the index system: I suppose that the properties of a notation of an ordinal of being a limit notation or a predecessor of an ordinal α is decidable. The following results concerning the expressive power of the languages of the Tarskian hierarchies are not restricted by this additional assumption, because by Theorem 3 different index systems have no relevance to definability. For other applications a well-known trick (cf. (Rogers, 1967, p. 211)) from recursion theory may be used:

LEMMA 6. *Any linear well-ordering of length α can be effectively transformed into a linear well-ordering of length $\omega \cdot \alpha$ such that the properties of being the predecessor of a given ordinal notation or of being a limit notation is decidable for all notations in the field of the new well-ordering.*

As the ordering is linear there is only one notation for 0 and hence the property of being this number is decidable. Note that by the lemma one gets from any recursive well-ordering a recursive well-ordering such that the mentioned properties a recursive. From the index system $\prec$ we obtain a linear ordering having this pleasant feature, and as $\omega \cdot \omega_1^{\mathrm{CK}} = \omega_1^{\mathrm{CK}}$ the new well-ordering has still the same length.

For the following definition it is essential that all initial segments of the index sytem are recursive. For the initial segment up to γ I shall write $\prec_\gamma$. Again, in the case of Kleene's $\mathcal{O}$ one would not need the $\prec_\gamma$ for the following definition because of a similar observation made before the definition of f_α.

Now I shall show how to translate sentences of the language $\mathcal{L}_{\mathrm{T}}$ into the ramified language $L(\omega_1^{\mathrm{CK}})$. This is done in a way very similar to the tranlation in the preceding section. I use again the recursion theorem to define a two-place function h_γ for each $\gamma < \omega_1^{\mathrm{CK}}$.

$$h_\gamma(k,n) := \begin{cases} \neg h_\gamma(k,\phi) & \text{if } n = (\neg\phi) \\ h_\gamma(k,\phi) \wedge h_\gamma(k,\psi) & \text{if } n = (\phi \wedge \psi) \\ \exists x h_\gamma(k,\phi) & \text{if } n = (\exists x\phi) \\ \bot & \text{if } k = 0_\prec \text{ and } n = \mathrm{T}t \\ \mathrm{T}_k \underset{\cdot}{h}_\gamma(\overline{c}, t) & \text{if } k \text{ is the successor of } c \\ & \text{and } n = \mathrm{T}t \\ \exists c \underset{\cdot}{\prec}_\gamma k \mathrm{T}_k \underset{\cdot}{h}_\gamma(\dot{c}, t) & \text{if } k = \lambda_\prec \text{ is a limit ordinal} \\ & \text{smaller than } \gamma \text{ and } n = \mathrm{T}t \\ n & \text{else} \end{cases}$$

If ϕ is a formula of $\mathcal{L}_{\mathrm{T}}$, then obviously $h_\gamma(\alpha_\prec, \phi)$ is a formula of $L(\alpha{+}1)$. Using this fact one can prove the following result.

THEOREM 7. *Assuming* $kd(\phi, a) \leq \alpha < \gamma$ *the following holds*

$$\mathcal{A}(\Phi^\kappa(\emptyset,\emptyset)) \models \phi[a] \iff \mathcal{A}_{\omega_1^{\mathrm{CK}}} \models h_\gamma(\alpha_\prec, \phi)[a]$$

Proof. By induction on $kd(\phi, a)$. Let γ be fixed throughout the proof. For $\phi \in \mathcal{L}$ the theorem is trivial.

If $kd(\phi, a) = 0$ then either $\mathcal{A}(\emptyset,\emptyset) \models \phi[a]$ or $\mathcal{A}(\emptyset,\emptyset) \models \neg\phi[a]$. I shall prove the theorem only for the first case. Assume $\alpha = 0$. From $(\emptyset,\emptyset) \leq (\emptyset,\omega)$ I conclude by the monotonicity property

$$\mathcal{A}(\emptyset,\omega) \models \phi[a].$$

Now $h_\gamma(0_\prec, \phi)$ is the result of substituting all subformulas $\mathrm{T}t$ by $\perp$. As

$$\mathcal{A}(\emptyset,\omega) \models \neg\mathrm{T}t[a]$$

the formulas $\mathrm{T}t$ are equivalent to $\perp$ for arbitrary assignments. In other words

$$\mathcal{A}_{\omega_1^{\mathrm{CK}}} \models h_\gamma(0_\prec, \phi)[a].$$

In the case $\alpha > 0$ the same method can be used to establish the theorem by putting

$$\begin{aligned} \mathrm{S}_1 &:= \mathcal{A}_{\omega_1^{\mathrm{CK}}}(\mathrm{T}_{\alpha_\prec}) \\ \mathrm{S}_2 &:= \omega - \mathrm{S}_1. \end{aligned}$$

Then $\mathcal{A}(\mathrm{S}_1, \mathrm{S}_2)$ is again a model without truth-value gaps, i.e. a classical model, satisfying again $(\emptyset,\emptyset) \leq (\mathrm{S}_1, \mathrm{S}_2)$, and the theorem can be proved as in the case $\alpha = 0$.

I turn to the successor case $kd(\phi, a) = \beta + 1$. I apply side induction on the inductive definition of validity in partial models with the strong Kleene scheme. Let ϕ be atomic, that is of shape $\mathrm{T}t$. Now

$$\mathcal{A}(\Phi^\kappa(\emptyset,\emptyset)) \models \mathrm{T}t[a]$$

implies the existence of a sentence ψ that is the value of the term t under the assignment a such that $kd(\psi) \leq \beta < \alpha$.

$$\begin{aligned} \mathcal{A}(\Phi^\kappa(\emptyset,\emptyset)) \models \mathrm{T}t[a] &\iff \mathcal{A}(\Phi^\kappa(\emptyset,\emptyset)) \models \psi \\ &\iff \mathcal{A}_{\omega_1^{\mathrm{CK}}} \models h_\gamma(\alpha_\prec, \psi) \text{ for all } \alpha \geq \beta \\ &\iff \mathcal{A}_{\omega_1^{\mathrm{CK}}} \models \mathrm{T}_k \underset{\cdot}{h}_\gamma(\overline{\alpha_\prec}, \overline{\psi}) \\ &\iff \mathcal{A}_{\omega_1^{\mathrm{CK}}} \models h_\gamma((\alpha+1)_\prec, \mathrm{T}t)[a] \\ &\iff \mathcal{A}_{\omega_1^{\mathrm{CK}}} \models h_\gamma(\alpha_\prec, \mathrm{T}t)[a] \text{ for all } \alpha > \beta \end{aligned}$$

If $\phi = \neg \mathrm{T}t$ the proof can be carried out in the same way, except in case $\alpha = 1$ if the value of t is not a $\mathcal{L}_{\mathrm{T}}$-sentence. I skip the easy proofs of the cases $\neg\neg\psi$, $\psi \wedge \chi$, $\neg(\phi \wedge \psi)$, $\exists x\psi$ and $\neg\exists x\psi$.

If $kd(\phi, a) = \lambda$ is a limit, a side induction of the same kind is performed as above.

6. DEFINABILITY

I shall now turn to questions of definability. In the case of partial models two notions of definability have to be distinguished (see, e.g. Burgess (1986)). A set B of members of the domain of the model $\mathcal{A}$ is said to be *weakly definable* if there is a formula $\phi(x)$ with just x free such that

$$\mathcal{A}(\Phi^{\kappa}(\emptyset, \emptyset)) \models \phi(x)[a] \quad \text{iff} \quad a(x) \in B$$

where $a(x)$ is the value of x under the assignment a. In the case that $\mathcal{L}$ is the language of arithmetic and $\mathcal{A}$ the standard model of arithmetic the sets definable in $\mathcal{L}_{\mathrm{T}}$ are exactly the Π^1_1-sets, cf. (Burgess, 1986).

A subset B of the domain of $\mathcal{A}$ is said to be *strongly definable* by ϕ if B is weakly definable by $\phi(x)$ and

$$\mathcal{A}(\Phi^{\kappa}(\emptyset, \emptyset)) \models \neg\phi(x)[a] \quad \text{iff} \quad a(x) \notin B$$

It is known (see again Burgess (1986)) that in the case of arithmetic the strongly definable sets are exactly the hyperarithmetical (Δ^1_1-)sets.

By the two preceding sections the following theorem can be proved.

THEOREM 8. *Assume* $\alpha < \omega_1^{\mathrm{CK}}$. *Then the following three statements are equivalent.*

(i) B *is strongly definable by a formula* $\phi(x)$ *satisfying* $\sup\{kd(\phi(x), a)$: a *an assignment*$\} < \alpha$.
(ii) B *is weakly definable by a formula* $\phi(x)$ *satisfying* $\sup\{kd(\phi(x), a)$: a *an assignment*$\} < \alpha$.
(iii) B *is definable in the language* $L(\alpha)$.

Proof. As (i) to (ii) is trivial I prove (ii) to (iii) first. $kd(\phi, a) \leq \beta < \alpha$ and weak definability imply for $\alpha < \gamma$ by Theorem 7

$$c \in B \iff \mathcal{A}_{\omega_1^{\mathrm{CK}}} \models h_\gamma(\beta_\prec, \phi(x))[a] \text{ and } a(x) = c$$

Because $h_\gamma(\beta_\prec, \phi(x))$ is a formula of the language $L(\alpha)$ it can be therefrom concluded that B is definable in $L(\alpha)$.

(iii) to (i). Assume on the other hand B to be definable in $L(\alpha)$ by a formula $\phi(x)$. By Theorem 5 it follows that

$$c \in B \iff \mathcal{A}(\Phi^{\kappa}(\emptyset,\emptyset)) \models f_{\alpha}(\phi(x))[a] \text{ and } a(x) = c$$

Hence B is weakly definable. To show B to be strongly definable I proceed in the following way:

$$\begin{aligned} c \notin B &\iff \mathcal{A}_{\omega_1^{\mathrm{CK}}} \models \neg\phi(x)[a] \text{ and } a(x) = c \\ &\iff \mathcal{A}(\Phi^{\kappa}(\emptyset,\emptyset)) \models f_{\alpha}(\neg\phi(x))[a] \text{ and } a(x) = c \\ &\iff \mathcal{A}(\Phi^{\kappa}(\emptyset,\emptyset)) \models \neg f_{\alpha}(\phi(x))[a] \text{ and } a(x) = c \end{aligned}$$

Hence B is also strongly definable and

$$\sup\{kd(\phi(x), a)\colon \ a \text{ an assignment}\} < \alpha$$

follows from Lemma 4. This concludes the proof of the theorem.

It is well-known that all strongly definable sets are hyperelementary in $\mathcal{A}$. Hence if $\kappa = \omega_1^{\mathrm{CK}}$, formulas (weakly) defining a non-hyperelementary set have Kripke degree κ.

COROLLARY 9. *If $\kappa = \omega_1^{\mathrm{CK}}$ and $\phi(x)$ weakly defines a non-hyperelementary set*

$$\sup\{\mathit{kd}(\phi(x), a)\colon \ a \text{ an assignment}\} = \omega_1^{\mathrm{CK}}$$

holds.

My next goal is to compare strong definability and definability in the languages in the hierarchy L. Kripke has already pointed out that in certain cases the two notions may coincide and McGee has shown a related result in (1991, p. 124). In order to use the above theorem for such a comparison I have to prove a tiny lemma first.

LEMMA 10. *m If $\phi(x)$ strongly defines a set B, then there is an $\alpha < \kappa$ such that*

$$\sup\{\mathit{kd}(\phi(x), a)\colon \ a \text{ an assignment}\} < \alpha.$$

Proof. By assumption the following holds for all assignments a.

$$\mathcal{A}(\Phi^{\kappa}(\emptyset,\emptyset)) \models \phi \leftrightarrow \mathrm{T}\dot{\overline{\phi}}[a]$$

The dot above $\overline{\phi}$ means that the free variables of ϕ may be bound from outside and occur freely in $\mathrm{T}\dot{\overline{\phi}}$. Hence the universal closure holds.

$$\mathcal{A}(\Phi^{\kappa}(\emptyset,\emptyset)) \models \forall x(\phi \leftrightarrow \mathrm{T}\dot{\overline{\phi}})$$

So there must be $\alpha < \kappa$ such that

$$\mathcal{A}(\Phi^{\alpha}(\emptyset, \emptyset)) \models \forall x(\phi \leftrightarrow \mathrm{T}\dot{\overline{\phi}}).$$

Having established the lemma the comparison theorem follows easily from the above theorem.

THEOREM 11. *If* $\kappa = \omega_1^{\mathrm{CK}}$ *the strongly definable sets and the sets definable in the language* $L(\omega_1^{\mathrm{CK}})$ *coincide.*

In general strong definability by a formula $\phi(x)$ such that

$$\sup\{kd(\phi(x), a)\text{: } a \text{ an assignment}\} < \omega_1^{\mathrm{CK}}$$

and definability in $L(\omega_1^{\mathrm{CK}})$ are the same. In contrast, there may be sets not definable in $L(\omega_1^{\mathrm{CK}})$ but by a formula

$$\sup\{kd(\phi(x), a)\text{: } a \text{ an assignment}\} \leq \omega_1^{\mathrm{CK}}$$

because Lemma 10 cannot be proved with κ replaced by ω_1^{CK}. McGee has shown in (1991) how to extend the Tarskian hierarchy up to κ in order to render the Tarskian hierarchy strong enough to define all strongly definable sets. Here I did not want to drop the restriction that the ordering of the indices of the truth predicates is not recursive. So the Tarskian hierarchy can only handle recursive (constructive) levels while the approach of Kripke allows to go up to the closure ordinal of the structure $\mathcal{A}$.

NOTES

[1] Since then different authors have made contributions to the theory of Tarskian hierarchies, e.g. (Feferman, 1991; Parsons, 1974; Burge, 1979; Church, 1976; Halbach, 1996).

[2] The referee has pointed out that it might be interesting to consider also non-acceptable structures, because the Tarskian hierarchy as well as Kripke's approach can easily be adapted to this more general approach. In general, satisfaction predicates are no longer definable from the truth predicates, because the substitution function is no longer expressible. Some of our arguments below rely on the availability of the substitution function, e.g. in Theorem 7, and it would be nice to see where the assumption of acceptability can be dropped, if the recursive functions are assumed to be still available.

[3] A more detailed treatment of the Tarski hierarchies (in a slightly more general sense) may be found in (Halbach, 1995). In this paper I also gave a proof of Theorem 3.

[4] ω_1^{CK} is the first non-recursive ordinal, that is the least ordinal that is greater than the order type of any well-founded recursive linear ordering. For more details and different characterizations, see, e.g. (Rogers, 1967).

[5] See (Rogers, 1967, p. 208) for a definition of Kleene's $\mathcal{O}$.

[6] Here and in the following I identify a language and the set of its formulas.

[7] Feferman has pointed out this fact in (Feferman, 1991) for the special (and most important) case that $\mathcal{L}$ is the language of arithmetic and $\mathcal{A}$ its standard model. For a more detailed definition of the ramified analytical hierarchy over arbitrary acceptable structures see (Moschovakis, 1974), p. 125.

[8] If there is a recursive function F giving, applied to a $k \in \mathrm{Fld}(\prec)$, (an index of) the index system restricted to the initial segment up to k, a universal function f not depending on α may be defined. This is the case with all versions and paths of Kleene's $\mathcal{O}$.

REFERENCES

Burge, T.: 1979, "Semantical Paradox", *Journal of Philosophy* **76**, 83–118.

Burgess, J.: 1986, "The Truth is Never Simple", *Journal of Symbolic Logic* **51**, 663–681.

Church, A.: 1976, "Comparison of Russel's Resolution of the Semantical Antinomies with that of Tarski", *Journal of Symbolic Logic* **41**, 747–760.

Feferman, S.: 1991, "Reflecting on Incompleteness", *Journal of Symbolic Logic* **56**, 1–49.

Girard J.-Y.: 1987, *Proof Theory and Logical Complexity* vol. 1, Bibliopolis, Napoli.

Halbach, V.: 1995, "Tarski Hierarchies", *Erkenntnis* **43**, 339–367.

Halbach, V.: 1996, *Axiomatische Wahrheitstheorien*, Akademie Verlag, Berlin.

Jokusch, C. G.: 1975, "Recursiveness of Initial Segments of Kleene's $\mathcal{O}$", *Fundamenta Mathematicae* **LXXXVII**, 161–167.

Kripke, S.: 1975, "Outline of a Theory of Truth", *Journal of Philosophy* **72**, 690–716.

Martin, R. L.: 1975, "On Representing 'True-in-L' in L", *Philosophia* **5**, 213–217.

McGee, V.: 1991, *Truth, Vagueness and Paradox*, Hackett Publishing, Indianapolis.

Moschovakis, Y. N.: 1974, *Elementary Induction on Abstract Structures*, North-Holland, Amsterdam.

Parsons, C.: 1974, "The Liar Paradox", *Journal of Philosophical Logic* **3**, 381–412.

Rogers, H.: 1967, *Theory of Recursive Functions and Effective Computability*, McGraw-Hill Book Company, New York.

Ochsenfelder Straße 12,
85072 Eichstaett,
Germany,
e-mail: halbach@cis.uni-muenchen.de