

## Low-Frequency Instability of Solid Rocket Motors. Influence of the Mache Effect and Charge Geometry

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Nonacoustic (low-frequency) instability of combustion of a solid propellant in a jet motor is considered. A model of unsteady combustion in motors with channel charges is proposed. The model takes into account the change in the temperature distribution in combustion products induced by the change in the gas pressure (Mache effect). The unsteady gas-release rate and the temperature of combustion products are determined on the basis of the Zel'dovich phenomenological approach, taking into account the change in the propellant-surface temperature (Novozhilov model) and the flame temperature (Gostintsev–Sukhanov model). The dependence of the stability limit of the solid rocket motor (SRM) on the charge-channel length and pre-nozzle volume of the motor chamber is determined. It is shown that the Mache effect leads to a significant constriction (by a factor of 1.5 to 2) of the range of stable combustion parameters in the SRM. It is found that an SRM with a channel charge has a narrower range of stability in terms of the Zel'dovich parameter  $k$  than an SRM with a butt-end charge and an identical combustion-chamber volume. For channel charges, the stability limit depends mainly on the volume of the pre-nozzle part of the combustion chamber.

### INTRODUCTION

Low-frequency (nonacoustic) instability of combustion in SRMs is usually manifested as subsidence of propellant burning with its subsequent self-ignition (“sneezing” of the motor) or in the form of periodic pressure fluctuations with a frequency lower than 10 Hz. The knowledge of the stability limit and its dependence on the propellant-combustion and combustion-chamber characteristics is necessary for motor design. Therefore, the mechanism of emergence of instability in SRMs, beginning from the pioneering works of Zel'dovich and Leipunskii [1–3], is of traditional interest for the solid propellant combustion theory (see, for example, [4–17]).

Usually, low-frequency instability is called the  $L^*$ -instability, which is related to the existence of the critical value of the reduced length of the combustion chamber

$$L^* = W/\sigma,$$

where  $W$  is the free volume of the combustor and  $\sigma$  is the nozzle-throat area.

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$L^*$ -instability arises if the free volume of the combustor is not sufficiently large. Such a situation is typical of the beginning of combustion, when combustion may be affected by many factors (ignition, erosion, unsteady burning, incompleteness of burning, etc.); that is why the experimental and theoretical studies are so complicated.

Despite the important applied significance of the problem, there is no commonly accepted quantitative theory of instability in the literature. The first qualitative explanations were proposed by Pobedonostsev (see [12]). He related instability to poor ignition and also to flameout in long channels. The first physically grounded theory was proposed by Zel'dovich in [1, 3], where he showed that oscillations are related to the thermal inertia of the combustion wave propagating in a solid propellant and to the flow-rate characteristics of the combustion chamber. Low-frequency instability arises when the characteristic time of thermal relaxation of the combustion wave  $\tau_T = \kappa/u^2$  ( $\kappa$  is the temperature diffusivity and  $u$  is the burning rate of the propellant) becomes comparable with the characteristic time of pressure relaxation in the combustor

$$\tau_r = W/ART_b\sigma,$$

where  $A$  is the discharge coefficient,  $R$  is the gas constant, and  $T_b$  is the temperature of the propellant-burning products.

Most theories of low-frequency instability use the assumption of a uniform distribution of the temperature of combustion products over the combustor volume (and even a constant gas temperature). This assumption simplifies the mathematical formulation of the problem but ignores the gas-temperature dependence in compression or expansion on the initial pressure at which this gas was formed (Mache effect [18]). It was noted in some papers that this fact is principally important (see, e.g., [6, 7, 12]). T'ien et al. [7] were the first to propose the theory of  $L^*$ -instability of SRM, taking into account the temperature (entropy) waves in the combustor for motors with butt-end charges. An essential assumption of the theory of [7] limiting its application is the assumption of the constant residence time of gas particles in the combustor, which is considered as a parameter of the problem. The analysis of  $L^*$ -instability performed in the present work takes into account the significant variation of the particle residence time in the combustor in the course of burning and also its dependence on the place of gas formation (burning-surface geometry). This can be done using the Lagrangian description for the gas flow in the combustion chamber [19].

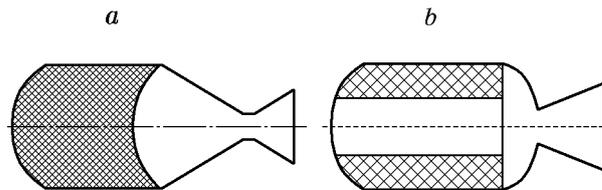
The present work is the further development of the theory of  $L^*$ -instability, which was suggested by the authors previously for motors with butt-end burning charges [17]. The general phenomenological approach [1, 4, 5] is used to describe unsteady burning of the propellant, which allows one to avoid the known limitations [9, 12] arising in choosing a particular model of the propellant flame [7].

## 1. MODEL OF THE PROCESS FOR BUTT-END BURNING CHARGES

For convenience of consideration, we first consider a motor with a butt-end charge, namely, a charge whose burning surface is equidistant from the motor nozzle (see Fig. 1a). We briefly describe the main postulates and relations of the theory suggested (see [17] for more detail).

Our consideration is based on the following assumptions.

1. The gas motion from the burning surface toward the nozzle is laminar and one-dimensional; gas portions formed at different times do not mix during the entire time of their residence in the combustor.



**Fig. 1.** SRM schemes: (a) butt-end charge with an equidistant surface from the nozzle; (b) channel charge.

2. The gas pressure in the combustor ( $p$ ) is identical over the volume and changes only in time:  $p = p(t)$ . This assumption makes the principal difference from the problem of acoustic instability of combustion, where the pressure in the combustion chamber is nonuniform, but the mean pressure is constant in time. The change in the mean pressure is a second-order effect in the acoustic wave amplitude. In the limiting case of quasi-steady disturbances, both formulations of the problem coincide.

3. The mass velocity of gas outflow through the nozzle ( $Ap\sigma$ ) changes in accordance with the current values of pressure  $p(t)$  and temperature  $T(t)$  of the gas particle that has reached the nozzle throat by the time  $t$ ; the nozzle-throat area  $\sigma$  is constant in time.

4. The propellant is homogeneous; the density, heat capacity, and thermal conductivity of the condensed phase are constant.

5. The composition and heat capacity of combustion products are constant. This assumption corresponds to two limiting cases for the relation between the chemical reaction time and the particle residence time in the combustion chamber. In the first case, the chemical reaction zone width and, hence, the reaction time are negligibly small as compared to other spatial and time scales of the problem. The composition of combustion products and the flame temperature correspond to conditions of thermodynamic equilibrium. In the second case (flameless burning), the relationship between the scales is the opposite; therefore, the chemical reaction has not enough time to produce a noticeable effect on the gas state in the combustion chamber. The conditions of realization of these regimes were analyzed in [14, 15].

Despite the uniform pressure, the temperatures of gas portions formed at different times may be significantly different due to the Mache effect. As a result, the gas flow along the chamber has nonuniform distributions of temperature and density. The characteristic spatial scale  $l$  of temperature nonuniformity in the combustion chamber, which coincides with the scale of dif-

ferent gas portions, is determined by the characteristic time of pressure relaxation ( $\tau_r$ ) and gas velocity in the combustor ( $V$ ):  $l = V\tau_r$ .

Heat exchange between gas particles is ignored due to the rather low frequency of pressure oscillations and the small residence time of the gas in the combustor [17].

### Basic Relations

We use the Lagrangian approach [19] to describe the gas motion in the combustor and nozzle. This method (in contrast to the traditional Eulerian approach) allows us to make the formulation of the problem independent of the particular charge geometry.

Let a gas portion of mass  $dm$  at a pressure  $p(t)$  be formed at a time  $t$  near the burning surface during a small period  $dt$ . The temperature of the gas portion formed is equal to the burning temperature  $T_b(t)$  determined by burning conditions at a given time. Within the framework of the phenomenological theory of unsteady burning of a solid propellant [1, 4, 5] (assuming that the gas-phase reactions are completed and the gas-phase combustion wave is inertialess), the temperature  $T_b$  is a function of instantaneous values of pressure  $p(t)$  and temperature gradient in the propellant near the burning surface  $\Phi(t)$ :  $T_b = T_b(p, \Phi)$ .

We denote the time of motion of the gas portion to the nozzle as  $\tau_n$ . Then, at the moment of outflow  $t + \tau_n$ , the pressure in the combustion chamber is  $p(t + \tau_n)$ , and the temperature of this gas portion  $T(t + \tau_n)$  depends on  $T_b(t)$ ,  $p(t)$ , and  $p(t + \tau_n)$ . The residence time  $\tau_n$  in an unsteady process is a function of the time  $t$ .

For an adiabatic (in the general case, polytropic) process of Lagrangian particle motion, the temperature of the portion formed at the time  $t_*$  is

$$T(t) = T_b(t_*) [p(t)/p(t_*)]^{(N-1)/N},$$

where  $N$  is the ratio of specific heats.

Taking into account that the discharge coefficient  $A$  depends on the gas temperature at the nozzle entrance as  $T^{-1/2}$ , we obtain

$$A(t) = A_b(t - \tau_n) [p(t - \tau_n)/p(t)]^{(N-1)/2N}, \quad (1)$$

where  $A_b(t)$  is the discharge coefficient at the temperature  $T_b(t)$ .

The free volume of the combustion chamber occupied by the gas is

$$W = \int_{t-\tau_n}^t \left[ \frac{p(t_*)}{p(t)} \right]^{1/N} \frac{\gamma S u(t_*)}{\rho [p(t_*)]} dt_*, \quad (2)$$

where  $\gamma$  is the propellant density,  $\rho$  is the gas density,  $S$  is the burning surface area, and  $u(t)$  is the instantaneous burning rate of the propellant. For a given

combustor volume, formula (2) should be considered as an integral equation with respect to  $p(t)$ .

Taking into account that, by definition, the entire gas mass formed during the time between  $t = 0$  and  $t - \tau_n$  is exhausted through the nozzle by the time  $t$ , we may write

$$\int_0^{t-\tau_n} u(t_*) \gamma S dt_* = \int_0^l A(t_*) p(t_*) \sigma dt_*.$$

Differentiating this equation with respect to  $t$ , we obtain

$$\frac{d\tau_n}{dt} = 1 - \frac{A(t)p(t)\sigma}{u(t - \tau_n)\gamma S}. \quad (3)$$

System (1)–(3) describes the dynamics of entropy perturbations in the SRM combustion chamber (Mache effect).

### Instability of Steady Burning

We consider unsteady burning in an SRM with a small perturbation of the steady regime:

$$u_0 \gamma S = A_0 p_0 \sigma.$$

It follows from Eq. (2) that the steady value of the residence time of the gas in the combustion chamber equals the time of pressure relaxation in the SRM combustor ( $\tau_r$ ):

$$\tau_n^0 = \frac{W \rho_0}{\gamma S u_0} = \frac{W}{A_0 R T_{b,0} \sigma} \equiv \tau_r.$$

We pass to dimensionless variables

$$\tau = \frac{u_0^2}{\varkappa} t, \quad v = \frac{u}{u_0} - 1, \quad \varphi = \frac{\Phi}{\Phi_0} - 1,$$

$$\eta = \frac{p}{p_0} - 1, \quad \psi = \frac{\tau_n}{\tau_n^0} - 1, \quad \theta = \frac{T_s - T_{s,0}}{T_{s,0} - T_0},$$

where  $T_s$  is the burning surface temperature and  $T_0$  is the initial temperature of the propellant. We rewrite system (1)–(3) in these variables:

$$\begin{aligned} & \chi \left[ \frac{1}{N} \eta(\tau) - \psi(\tau) \right] \\ & = \int_{\tau-\chi}^{\tau} \left[ \left( \alpha - \frac{N-1}{N} \right) \eta(\tau_*) + \varepsilon \varphi(\tau_*) + v(\tau_*) \right] d\tau_*, \quad (4) \end{aligned}$$

$$\begin{aligned} \chi \frac{d\psi}{d\tau} = & - \left[ \frac{N+1}{2N} \eta(\tau) - \frac{1}{2} \left( \alpha - \frac{N-1}{N} \right) \eta(\tau - \chi) \right. \\ & \left. - \frac{1}{2} \varepsilon \varphi(\tau - \chi) - v(\tau - \chi) \right]. \quad (5) \end{aligned}$$

Here  $\chi = \tau_r u_0^2 / \nu$ ,  $\alpha$  is the sensitivity of the flame temperature to the gas pressure, and  $\nu$  is the relative fluctuation of the burning rate.

Substituting perturbations in the form

$$\begin{aligned}\eta(\tau) &= \eta_1 \exp(\Omega\tau), & v &= v_1 \exp(\Omega\tau), \\ \varphi &= \varphi_1 \exp(\Omega\tau), & \psi &= \psi_1 \exp(\Omega\tau)\end{aligned}$$

into (4) and (5) and eliminating  $\psi_1$ , we obtain the relation between perturbation amplitudes:

$$\begin{aligned}\left[ \frac{1}{N} \Omega\chi + \frac{N+1}{2N} - \left( \alpha - \frac{N-1}{N} \right) \left( 1 - \frac{1}{2} \exp(-\Omega\chi) \right) \right] \eta_1 \\ - v_1 - \varepsilon \left( 1 - \frac{1}{2} \exp(-\Omega\chi) \right) \varphi_1 = 0.\end{aligned}\quad (6)$$

Relation (6) closes the system of equations of the theory of unsteady burning of the propellant [4, 5].

The characteristic equation derived from the condition of solvability of the system of equations of unsteady burning and from Eq. (6) has the form

$$\begin{aligned}\frac{\sqrt{1+4\Omega} - 1}{2\Omega} \left[ ky + \varepsilon(\delta - \nu) \left( 1 - \frac{1}{2} \exp(-\Omega\chi) \right) \right] \\ - (k+r-1)y + \delta - \nu - \frac{\sqrt{1+4\Omega} + 1}{2\Omega} \\ \times \left[ \delta - ry + \varepsilon(\delta + \mu) \left( 1 - \frac{1}{2} \exp(-\Omega\chi) \right) \right] = 0,\end{aligned}\quad (7)$$

$$y = \frac{\Omega\chi}{N} + \frac{N+1}{2N} - \left( \alpha - \frac{N-1}{N} \right) \left( 1 - \frac{1}{2} \exp(-\Omega\chi) \right).$$

Here

$$\begin{aligned}k &= (T_{s,0} - T_0) \frac{\partial \ln u_0}{\partial T_0}, & r &= \frac{\partial T_{s,0}}{\partial T_0}, \\ \mu &= \frac{1}{T_{s,0} - T_0} \frac{\partial T_{s,0}}{\partial \ln p}, & \nu &= \frac{\partial \ln u_0}{\partial \ln p}, & \delta &= \nu r - \mu k\end{aligned}$$

are the characteristics of sensitivity of the steady burning rate of the propellant  $u_0$  and the surface temperature  $T_{s,0}$  to changes in the initial temperature and pressure [4, 5].

### Stability Limits of Steady Combustion

The boundary of parameters of stable combustion of the SRM propellant is determined by solving the characteristic equation (7) with imaginary values of the frequency  $\Omega$ . First, from (7), we can easily obtain the critical relation between the propellant parameters at which quasi-steady oscillations are formed ( $\Omega \rightarrow 0$ ):

$$\delta - r(1 - \alpha/2) + \varepsilon(\delta + \mu)/2 = 0.$$

From here, for  $\varepsilon = \mu = 0$ , we obtain the critical relation  $\nu = 1 - \alpha/2$ , which is in agreement with the results of [11], where the stability of combustion to quasi-steady acoustic oscillations in the motor was studied.

Some results of the numerical solution of Eq. (7) are plotted in Fig. 2 (see also [17]). Curve 1 corresponds to isothermal combustion products ( $\alpha = \varepsilon = 0$ ) and an isothermal process in the SRM combustor ( $N = 1$ ) — there is no Mache effect. In the case of a nonisothermal polytropic process in the combustor (e.g., for  $N = 1.2$ ), even for isothermal combustion products ( $\alpha = \varepsilon = 0$ ), the region of stable combustion in the SRM becomes narrower (curves 2). Nonisothermal combustion products additionally destabilize SRM combustion. The stability limit for this case is shown by curves 3 ( $\alpha = 0.25$  and  $\varepsilon = 0$ ) and 4 ( $\alpha = 0.25$  and  $\varepsilon = 0.25$ ). It should be noted that all curves converge rather rapidly ( $\chi \geq 2$ ) to a common vertical asymptote ( $k = k^*$ ) corresponding to the boundary of stable combustion of the propellant at constant pressure [9]:  $r = (k^* - 1)^2 / (k^* + 1)$ .

It follows from the results obtained that the stability limit does not depend on the combustor length and is determined only by its volume (more exactly, by the reduced length of the combustion chamber  $L_* = W/\sigma$ ). At first glance, this result contradicts the model proposed, since  $L^*$ -instability may be interpreted (by analogy with acoustic instability) as the resonance between entropy waves in the combustion chamber and the oscillatory system, which is the burning propellant [9]. However, as is shown in [17], this contradiction is apparent; the reason is that, in butt-end combustion, the residence time of the gas in the combustor equals the time of pressure relaxation and depends only on the combustor volume (and not on its length).

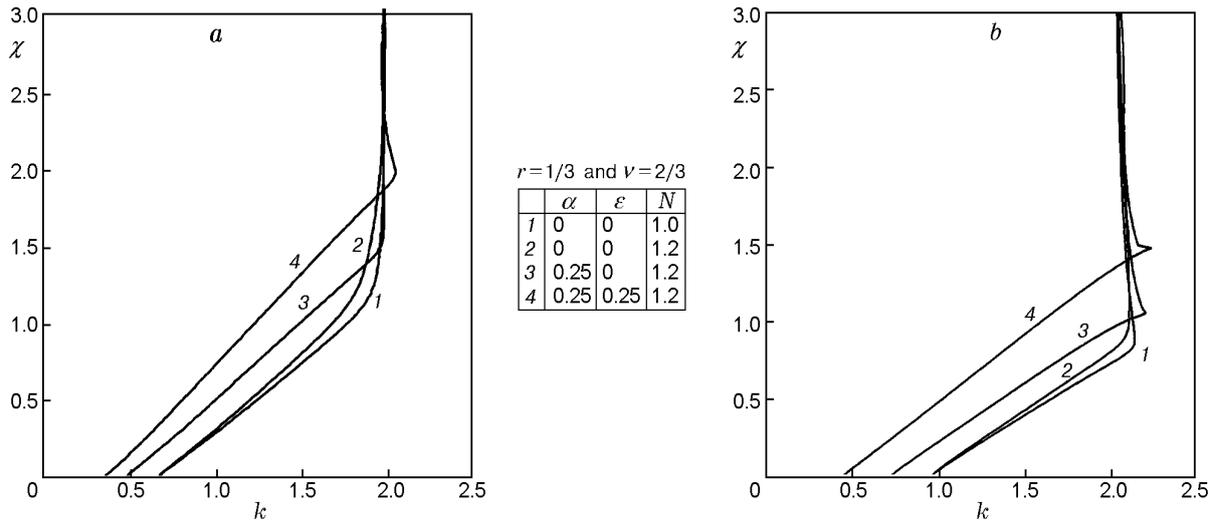
In the general case (for instance, for channel charges), the residence time of the gas in the combustor depends significantly on the fact at which point of the burning surface this or that portion of the gas was formed. Therefore, the condition of stability of an SRM with a channel charge may be expected to depend on the charge-channel length.

## 2. MODEL OF THE PROCESS FOR CHANNEL CHARGES

We consider the scheme of a motor with an axisymmetric channel charge (see Fig. 1b). Propellant combustion occurs only over the inner surface of the channel: the butt-end faces and the external surface of the charge are reinforced. There is a pre-nozzle volume between the back end face of the charge and the chamber nozzle.

The following basic assumptions are used.

1. The pressure in the combustor (including the channel) is uniform; the effect of gas-dynamic pressure differences on the combustion and exhaustion processes



**Fig. 2.** Stability limit versus the characteristics of steady combustion of a butt-end charge for  $\mu = 0$  (a) and 0.2 (b): 1) isothermal gas in the combustor; 2) isothermal gas release from the burning surface; 3, 4) nonisothermal combustion products.

is ignored (this corresponds to subsonic flows in the combustor).

2. All portions of the charge surface burn simultaneously, i.e., with an identical unsteady burning rate and flame temperature.

3. Mixing of different portions of combustion products in the channel, combustor, and nozzle and heat transfer between them are negligibly small (the motion of each gas portion is adiabatic).

Each of the above assumptions imposes certain restrictions on the channel length. For example, it follows from assumption 1 that the channel length should be smaller than the critical value [20] at which sonic choking of the channel occurs. Assumption 3 limits the channel length by a value corresponding to the beginning of flow turbulization (the case of turbulent mixing of combustion products was considered in [7, 9, 14, 15]).

**Basic Relations**

In the case considered, each elementary portion of combustion products is characterized not only by the time of its formation  $t_*$  but also by the point of the burning surface, where it was formed. To characterize the position of the point on the burning surface, we introduce the variable  $S$ , which is numerically equal to the surface area enclosed between the channel cross section considered and a certain “initial” cross section. The initial cross section is chosen in such a way that

the time  $\tau_n$  is a monotonically increasing function of the parameter  $S$ :  $0 \leq S \leq S_0$ , where  $S_0$  is the total burning surface of the charge, which coincides with the channel surface.

Taking into account that  $\tau_n = \tau_n(t, S)$ , the discharge coefficient for each gas portion described by formula (1) is also a function of the parameters  $t$  and  $S$ :  $A = A(t, S)$ .

Owing to the assumption of simultaneous combustion of all points of the charge surface, the quantities  $u$ ,  $T_b$ ,  $A_b$ , and  $\Phi$  are functions of the time  $t_*$  only.

Passing through the nozzle, each portion of combustion products forms a stream tube with the minimum cross section  $d\sigma$ .

The law of conservation of mass for combustion products formed at the section  $dS$  of the charge surface has the form

$$dS \int_0^{t-\tau_n(t,S)} u(t_*)\gamma dt_* = d\sigma \int_0^t A(t_*, S)p(t_*) dt_*$$

Differentiating this relation with respect to  $t$ , we obtain the equation

$$\left[ 1 - \frac{\partial \tau_n(t, S)}{\partial t} \right] \frac{u(t - \tau_n)\gamma}{A(t, S)p(t)} = \frac{\partial \sigma}{\partial S}. \tag{8}$$

Obviously, the sum of instantaneous areas of the minimum cross sections of all stream tubes corresponding to different portions of combustion products is equal to the area of the minimum cross section of the nozzle  $\sigma$ . Then, integrating Eq. (8), we obtain

$$\int_0^{S_0} \left[ 1 - \frac{\partial \tau_n(t, S)}{\partial t} \right] \frac{u(t - \tau_n) \gamma}{A(t, S) p(t)} dS = \sigma. \quad (9)$$

Equation (9) is the extension of Eq. (3) to the case of channel charges.

To close the system of equations, it is necessary to determine the dependence  $\tau_n(t, S)$ . For this purpose, we assume that the velocity distribution of combustion products over the channel cross section is uniform. This assumption is widely used in flow calculations in SRM combustors.

The combustor volume (see Fig. 1b) may be divided into two parts: channel volume and pre-nozzle volume  $W_0$ . Between an arbitrary cross section of the channel  $S$  and the “initial” cross section, there is a volume  $W(S)$  that contains only combustion products formed at the section  $[0, S]$  of the burning surface. For a given charge geometry, the function  $W(S)$  is always known.

By analogy with (2), for the volume  $W(S)$ , we obtain

$$W(S) = \int_0^S \int_{t-\tau_n(t, S_*)}^{t-\tau_n(t, S)} \left[ \frac{p(t_*)}{p(t)} \right]^{1/N} \frac{\gamma u(t_*)}{\rho[p(t_*)]} dt_* dS_*. \quad (10)$$

From the same reasoning, for the total volume of the combustion chamber  $W_\Sigma = W(S_0) + W_0$ , we can write

$$W(S_0) + W_0 = \int_0^{S_0} \int_{t-\tau_n(t, S_*)}^t \left[ \frac{p(t_*)}{p(t)} \right]^{1/N} \frac{\gamma u(t_*)}{\rho[p(t_*)]} dt_* dS_*. \quad (11)$$

Subtracting Eq. (10) for  $S = S_0$  from Eq. (11), after simple transformations, we obtain

$$W_0 = \int_{t-\tau_n(t, S_0)}^t \left[ \frac{p(t_*)}{p(t)} \right]^{1/N} \frac{\gamma u(t_*) S_0}{\rho[p(t_*)]} dt_*, \quad (12)$$

which exactly coincides with Eq. (2).

System (1), (9), (10), (12) describes the dynamics of entropy perturbations in the SRM combustor with a channel charge.

Differentiating (10) with respect to  $S$ , we obtain

$$\frac{\partial W(S)}{\partial S} = -S \left[ \frac{p(t - \tau_n)}{p(t)} \right]^{1/N} \times \frac{\gamma u(t - \tau_n)}{\rho[p(t - \tau_n)]} \frac{\partial \tau_n(t, S)}{\partial S}. \quad (13)$$

We confine ourselves to the case  $\partial W(S)/\partial S = \text{const}$ . This corresponds to a constant-area channel (without the inflation effect [2, 12]).

It follows from Eqs. (12) and (13) that the following formula is valid for the steady regime of combustion:

$$\tau_n^0(S) = \tau_n^0(S_0) - a \frac{\rho_0}{\gamma u_0} \ln \frac{S}{S_0} \quad (14)$$

$$\left[ \tau_n^0(S_0) = \frac{W_0 \rho_0}{\gamma S_0 u_0} = \frac{W_0}{A_0 R T_{b,0} \sigma} \equiv \tau_r \right].$$

We transform Eq. (13) to a more convenient form for further analysis:

$$\int_S^{S_0} \frac{1}{S} \frac{dW}{dS} dS = \int_{t-\tau_n(t, S)}^{t-\tau_n(t, S_0)} \left[ \frac{p(t_*)}{p(t)} \right]^{1/N} \frac{\gamma u(t_*)}{\rho[p(t_*)]} dt_*. \quad (15)$$

## Stability of Steady Combustion

We consider small perturbations of the parameters relative to their steady values:

$$\psi_0 = \frac{\tau_n(t, S_0)}{\tau_r} - 1, \quad \psi_s(t, S) = [\tau_n(t, S) - \tau_s^0(S)] \frac{u_0^2}{\varkappa}.$$

We define the instrumental constant  $\chi_0$  as  $\chi_0 = \tau_r u_0^2 / \varkappa$ . Note that the constant  $\chi_0$  depends only on the pre-nozzle volume of the SRM combustor.

We introduce the function

$$\chi_s(S) = \frac{\tau_n^0(S) u_0^2}{\varkappa} = \chi_0 + \beta \ln \frac{S_0}{S}, \quad (16)$$

which may be considered as an “instrumental” function of the motor, where the parameter

$$\beta = a \frac{\rho_0 u_0}{\gamma \varkappa} = \frac{a S_0}{W_0} \chi_0$$

reflects the effect of the channel geometry.

For small perturbations, Eq. (12) takes the form similar to Eq. (8):

$$\chi_0 \left[ \frac{1}{n} \eta(\tau) - \psi_0(\tau) \right] = \int_{\tau - \chi_0}^{\tau} \left[ \left( \alpha - \frac{N-1}{N} \right) \eta(\tau_*) + \varepsilon \varphi(\tau_*) + v(\tau_*) \right] d\tau_*, \quad (17)$$

and Eq. (15) acquires the form

$$\psi_s(\tau, S_0) - \psi_s(\tau, S) = \int_{\tau - \chi_s(S)}^{\tau - \chi_s(S_0)} \left[ \left( \alpha - \frac{N-1}{N} \right) \eta_*(\tau_*) + \varepsilon \varphi_*(\tau_*) + v_*(\tau_*) \right] d\tau_*. \quad (18)$$

From Eq. (9), we can readily obtain

$$\frac{N+1}{2N} \eta_*(\tau) S_0 = \int_0^{S_0} \left[ v_*(\tau - \chi_s) + \frac{1}{2} \left( \alpha - \frac{N-1}{N} \right) \eta_* \times (\tau - \chi_s) + \frac{1}{2} \varepsilon \varphi_*(\tau - \chi_s) - \frac{\partial \psi_s(\tau, S)}{\partial \tau} \right] dS. \quad (19)$$

Substituting perturbations in an exponential form and also the perturbation  $\psi_s(\tau, S)$  in the form  $\psi_s(\tau, S) = \psi_{s,1}(S) \exp(\Omega\tau)$  into Eq. (18), after simple transformations, we obtain

$$\psi_{s,1} = \chi_0 \psi_1 - \frac{1}{\Omega} \left[ \left( \alpha - \frac{N-1}{N} \right) \eta_1 + \varepsilon \varphi_1 + v_1 \right] \times \left[ 1 - \left( \frac{S}{S_0} \right)^{\Omega\beta} \right] \exp(-\Omega\chi_0). \quad (20)$$

In deriving this formula, we took into account the obvious condition  $\psi_s(\tau, S_0) = \chi_0 \psi_*$  or  $\psi_{s,1}(S_0) = \chi_0 \psi_1$ .

Substituting Eq. (20) into (17) and (19) and eliminating  $\psi_1$ , we obtain

$$\left[ \frac{1}{n} \Omega \chi_0 + \frac{N+1}{2N} - \left( \alpha - \frac{N-1}{N} \right) \times \left( 1 - \frac{1}{2(1+\beta\Omega)} \exp(-\Omega\chi_0) \right) \right] \eta_1 - v_1 - \left[ 1 - \frac{1}{2(1+\beta\Omega)} \exp(-\Omega\chi_0) \right] \varepsilon \varphi_1 = 0. \quad (21)$$

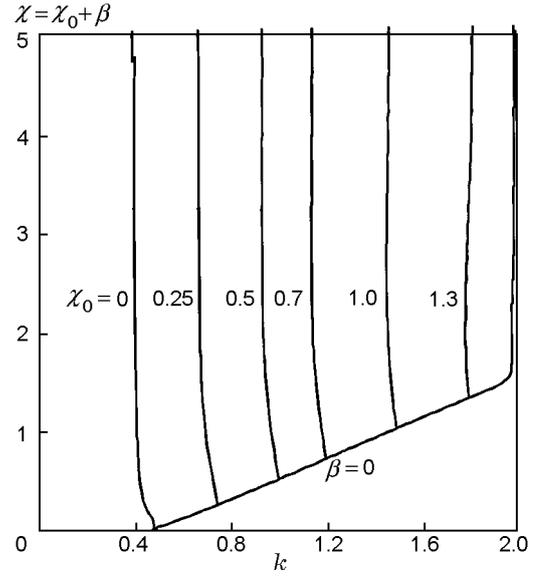
The characteristic equation obtained from the condition of solvability of equations of unsteady combustion and Eq. (21) has the form

$$\frac{\sqrt{1+4\Omega} - 1}{2\Omega} \left[ ky + \varepsilon(\delta - \nu) \times \left( 1 - \frac{1}{2(1+\beta\Omega)} \exp(-\Omega\chi_0) \right) \right] - (k+r-1)y + \delta - \nu - \frac{\sqrt{1+4\Omega} + 1}{2} \left[ \delta - ry + \varepsilon(\delta + \mu) \times \left( 1 - \frac{1}{2(1+\beta\Omega)} \exp(-\Omega\chi_0) \right) \right] = 0, \quad (22)$$

where

$$y = \frac{1}{N} \Omega \chi_0 + \frac{N+1}{2N} - \left( \alpha - \frac{N-1}{N} \right) \times \left( 1 - \frac{1}{2(1+\beta\Omega)} \exp(-\Omega\chi_0) \right).$$

For  $\beta = 0$ , Eq. (22) is transformed into Eq. (7) for an SRM with a butt-end charge. It can be easily seen that the parameter  $\beta$ , which reflects the channel effect, enters Eq. (22) in the form of the product  $\beta\Omega$ . Therefore, in the limiting case of quasi-steady perturbations ( $\Omega \rightarrow 0$ ), the critical conditions for channel and butt-end charges coincide. In the general case ( $\Omega > 0$ ), the critical condition and the corresponding frequency of oscillations depend on the parameter  $\beta$ .



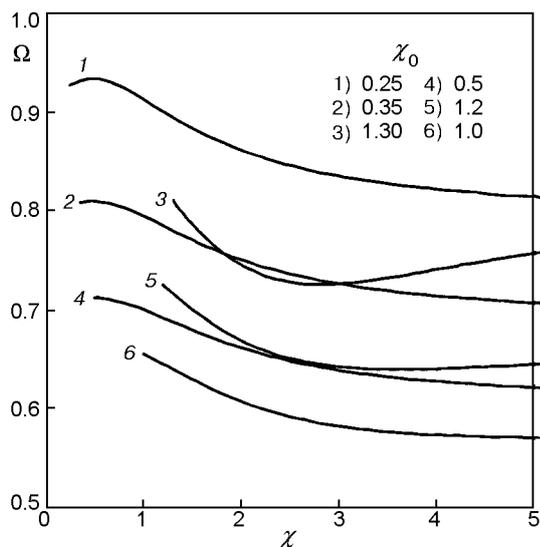
**Fig. 3.** Limits stability for different values of the pre-nozzle volume of the combustion chamber with a channel charge.

### Stability Limits of Steady Combustion

We consider the effect of the parameters  $\beta$  and  $\chi_0$  on the stability limits in the plane  $(\chi, k)$ . Note that the instrumental constant for the free volume of the combustion chamber is  $\chi = \beta + \chi_0$ , where  $\chi_0$  and  $\beta$  correspond to the pre-nozzle volume and channel volume (see Fig. 1b). The results of the numerical solution of Eq. (22) at the limit of stability for  $r = 1/3$ ,  $\nu = 2/3$ ,  $\mu = 0$ ,  $\alpha = 0.25$ ,  $\varepsilon = 0$ ,  $n = 1.2$ , and different values of the parameter  $\chi_0$  are shown in Figs. 3–6. The region of instability in Fig. 3 is located to the right of the curves. It is seen from the figure that the stability limit depends not only on the total volume of the combustion chamber  $\chi$  but also on the fraction of this volume  $\chi_0$  that refers to the pre-nozzle volume.

The curves presented are limited from below by the line ( $\beta = 0$ ,  $\chi = \chi_0$ ), which corresponds to the limit of stability for a butt-end charge. It is seen from Figs. 3 and 2a that the charge shape has a significant effect on low-frequency instability of combustion in SRM. Thus, for an identical volume of the combustion chamber, an SRM with a channel charge has a significantly narrower region of stability in terms of the parameter  $k$  than an SRM with a butt-end charge, the region of stability becoming narrower as the pre-nozzle volume decreases.

As it follows from Fig. 3, the stability limit may be represented as the dependence  $k^* = k^*(\chi, \chi_0)$ . It can be easily seen that the dependence on  $\chi$  is weak, and we may approximately write  $k^* = k^*(\chi_0)$ . With a sufficient



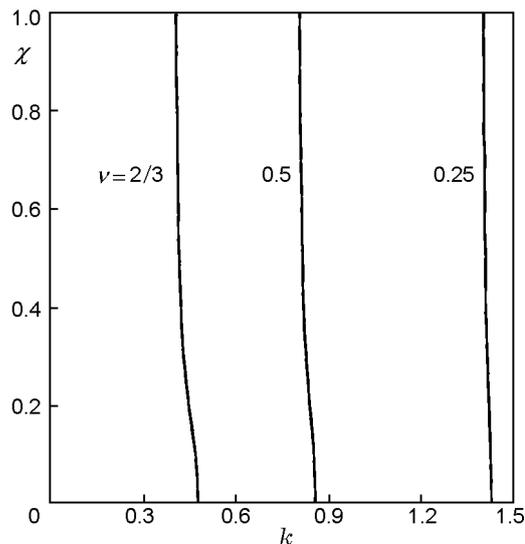
**Fig. 4.** Dimensionless frequency of oscillations at the limit of stability versus the pre-nozzle volume of the combustion chamber with a channel charge.

degree of accuracy, we may assume that the process in the SRM is unstable if  $k > k^*(\chi_0)$  or  $\chi_0 < \chi^*(k)$ ; the functions  $k = k^*(\chi_0)$  and  $\chi_0^* = \chi^*(k)$  are reciprocal and describe the boundary of low-frequency instability for butt-end charges. The dependence  $\chi = \chi^*(k)$  is shown by the lower curve in Fig. 3.

According to the data of Fig. 3, we may argue that the limit of low-frequency instability for channel charges does not depend on the total volume of the combustion chamber but is mainly determined by the pre-nozzle volume. Therefore, if the combustor volume of an SRM with a butt-end charge coincides with the pre-nozzle volume of an SRM with a channel charge, their limits of low-frequency instability coincide. It follows from here that it is necessary to increase the pre-nozzle volume to stabilize the process in the SRM with a channel charge, since an increase in the channel volume has almost no effect on the stability of the process.

We also note a practically important feature of combustion of channel charges: with increasing free volume of the combustion chamber due to increasing channel volume, the limit of stability in the SRM does not approach the limit of stability of propellant combustion at constant pressure (which is observed in butt-end combustion).

Figure 4 shows the eigenfrequencies (in a dimensionless form) for an SRM with a channel charge at the limit of stability. Depending on the pre-nozzle volume, the eigenfrequency may change severalfold, and the dependence of the dimensionless frequency on  $\chi_0$  is not monotonic.



**Fig. 5.** Boundaries of the "minimum" stability of an SRM with a channel charge for different values of the parameter  $\nu$  in the burning rate law.

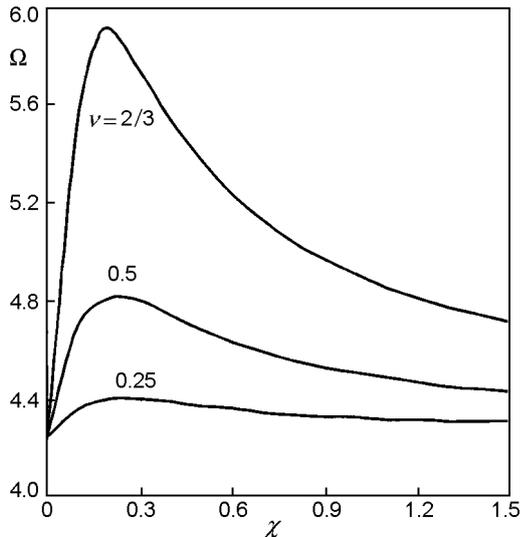
As it follows from the results in Fig. 3, the smaller the pre-nozzle volume, the narrower the region of stability of the SRM. For  $\chi_0 = 0$ , other conditions being equal, the region of stability is minimum.

Figure 5 shows the boundary of the "minimum" region of stability of channel-charge combustion, which corresponds to a zero pre-nozzle volume. In calculations, we used the same values of parameters as for Fig. 3. It is seen from Fig. 5 that an increase in the value of  $\nu$  in the burning rate law makes the region of SRM stability narrower in terms of the parameter  $k$ .

Figure 6 illustrates the effect of the parameter  $\nu$  on the frequency of oscillations in an SRM with a channel charge at the limit of minimum stability. It is seen from Figs. 5 and 6 that the dimensionless frequency of oscillations increases with decreasing volume of the combustion chamber (curves 1, 2, 4, and 6 in Fig. 4) and may be significantly greater than unity if there is no pre-nozzle volume (see Fig. 6).

It is of interest to compare the effect of the gas in the combustion chamber with the influence of thermal inertia of the gas phase of the combustion wave at constant pressure [20]. In contrast to the gas phase of the combustion wave, the gas in the channel destabilizes the combustion process. Nevertheless, the damping action of the gas in the pre-nozzle volume of the motor is similar to the effect of thermal inertia of the gas phase of the combustion wave.

We may assume that the Mache effect is one of the important reasons for the existence of a critical channel length after which channel ignition and subsequent



**Fig. 6.** Dimensionless frequency of oscillations at the limit of the “minimum” stability of an SRM with a channel charge versus the channel volume and the parameter  $\nu$ .

combustion are accompanied by low-frequency oscillations of high amplitude [21].

Thus, the model proposed allows one to analyze the mechanism of the stabilizing effect of the pre-nozzle volume on the combustion process in the SRM, which was first observed experimentally by Leipunskii [2].

Finally, we note that the proposed model of unsteady combustion in the SRM may be useful in analyzing the Mache effect in motors with more complicated shapes of the charge and combustion chamber (for instance, for multi-cartridge and multi-channel charges, and also for channel charges of uniform combustion).

## CONCLUSIONS

It is shown that the frequency characteristics and critical conditions of nonacoustic instability of combustion in SRMs with channel charges depend on the size ratio of the charge channel and pre-nozzle volume of the combustion chamber. The nonequilibrium distribution of temperature in combustion products along the combustor is one of the most important factors that influence combustion stability in the SRM, which is comparable with the effect of the dependence of the burning rate on the initial temperature.

For  $\chi < 2$ , the Mache effect leads to an increase in the critical value of the instrumental constant  $\chi$  by a factor of 1.5 to 2, with an unchanged value of the Zel’dovich parameter  $k$ . This effect necessitates an in-

crease in the minimum allowable free volume of the combustion chamber by a factor of 1.5 to 2, other conditions being equal. The main stabilizing effect is exerted by the increase in the pre-nozzle volume of the combustion chamber.

With increasing free volume of the combustor due to the increase in the channel volume, the combustion-stability limit in the motor does not approach the combustion-stability limit at constant pressure (which is observed in butt-end combustion).

This work was devoted to the 90th anniversary of O. I. Leipunskii.

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