Workbook on Aspects of Dynamical Meteorology
A Self Discovery Mathematical Journey for Inquisitive Minds

First Edition

Jan D. Gertenbach

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ISBN 0-620-27229-5

This book was typeset in $\mathcal{A}\mathcal{M}\mathcal{S}$TEX.

Photographs were obtained from http://www.photolib.noaa.gov

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Preface

In this Workbook[1] the reader is invited to activate an eager inquisitive mind, to go on a self discovery journey through the world of Dynamical Meteorology, by the ready and efficient vehicle of Mathematics. The reader thus becomes a learner, an integrator of knowledge, capable of bringing together previous experiences and applying new skills to problems. The attitude to the learning process should move from an information gathering process to an active outcome-based life-relevant experience. It is the writer's sincere hope that the problem-solving approach—to both mathematical theory and meteorological applications—will enhance the learner's intellectual development and self esteem. The writer believes that a sound self esteem and courage towards self discovery are vital prerequisites for success in Mathematics and Dynamical Meteorology.

The reader is encouraged and challenged to develop a problem-solving attitude. As may be expected, some effort is needed to embark on such a journey. The satisfaction after success, the establishment of a long-term securely rooted foundation, is worth it.

The learner is invited to develop a life-long learning attitude, grounded in life-long assessment of progress. Group work and evaluation by peers may be part of the learning process. Not only successes and correct answers are important—the lessons learnt from failures should not be forgotten.

For background theory, the reader is referred to the literature—no claim to completeness or self sufficiency is made. The purpose of the book is to help the reader discover through exercise.

The classroom, lecturer and library no longer remain the sole basis of information. The reader is encouraged to test, exercise and evaluate newly gained experience, using self developed computer programmes and graphical displays, as well as the Internet and multi media information systems. The life-long learner should use all appropriate senses to secure newly gained competence—competence encompassing knowledge, skills and attitudes.

The levels of competence the learner should reach include the ability to *do* things, to be able to *demonstrate* what was learnt and to *reflect and apply to new problems*. The vision of the learner should be to

> know *that*,
> know *why*,
> *understand*,
> know *how*,
> know *how and why not differently*.

These aspects should constantly be kept in mind when doing the Workbook exercises.

[1] The ambiguity in the subtitle of the book is intentional: the book is intended to be both a discovery of the *self* through mathematical achievement and a discovery of *mathematics* by self involvement and exercise.

The workbook should be used in conjunction with other books on Dynamical Meteorology, e.g. the book by Holton (1992). Books on Calculus e.g. Apostol (1967) and (1969) can be used to revise and supplement fundamental Mathematics. The References include several works on Continuum Mechanics and the rational foundation thereof. The learner is challenged to first exercise self discovery, then to do self assessment against the literature and finally (if relevant) to participate in a group assessment (peer review).

Students frequently experience problems with interpreting and understanding what they have read. Moreover, knowledge gained (in other courses) but not used tend to be forgotten and shelved very soon. This should not be the norm: integration of knowledge, whereby the learners create their own cognitive structure, linking related topics from different parts of the text, is of utmost importance. The author thus sets an example by frequently referring to previous statements of a specific topic, e.g. the idea of an Eulerian or Lagrangian description of fluid flow (see Section 1.2, Example 1.3.5, Section 1.4.3, Section 1.5.2, Exercises 1.5.3.1 etc.). It is the author's wish that the learners should constantly develop their ability to, on the one hand, read accurately and, on the other, formulate their thoughts precisely.

I would like to thank my colleagues for their interest and suggestions for improvement of the book.

I greatly appreciate Dr. A. P. Burger's thorough reading of the manuscript and valuable comments regarding content and preciseness.

Keywords to note: Workbook: Dynamical Meteorology; Learning: self discovering, involvement, multimedia, integration of knowledge, evaluation by peers and by self; Outcome based life-long assessment; Modelling of the atmosphere: mathematics, computing, graphical displays.

Orientation for a good study programme: the learner devise a curriculum, compile examination papers and memoranda, compile a portfolio containing successes, dead-ends and wrong efforts together with an evaluation why things did not work.

We see,

measure,

and model

$$\begin{aligned}
\frac{Du}{Dt} - 2\Omega v \sin\phi + 2\Omega w \cos\phi - \frac{uv\tan\phi}{a} + \frac{uw}{a} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_{rx} \\
\frac{Dv}{Dt} + 2\Omega u \sin\phi + \frac{u^2\tan\phi}{a} + \frac{vw}{a} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_{ry} \\
\frac{Dw}{Dt} - 2\Omega u \cos\phi - \frac{u^2+v^2}{a} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_{rz} \\
\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{U}) &= 0 \\
c_p\frac{D\ln T}{Dt} - R\frac{D\ln p}{Dt} - \frac{J}{T} &= 0 \\
\frac{D\boldsymbol{V}}{Dt} + \nabla_p\Phi + f\boldsymbol{k}\times\boldsymbol{V} &= 0 \\
\nabla_p\cdot\boldsymbol{V} + \frac{\partial\omega}{\partial p} &= 0 \\
\frac{\partial\Phi}{\partial p} + \frac{RT}{p} &= 0.
\end{aligned}$$

using the Greek alphabet

alpha	α	beta	β	gamma	γ
delta	δ	epsilon	ϵ	zeta	ζ
eta	η	theta	θ	iota	ι
kappa	κ	lambda	λ	mu	μ
nu	ν	xi	ξ	pi	π
rho	ρ	sigma	σ	tau	τ
upsilon	υ	phi	ϕ	chi	χ
psi	ψ	omega	ω		

Alpha	-	Beta	-	Gamma	Γ
Delta	Δ	Epsilon	-	Zeta	-
Eta	-	Theta	Θ	Iota	-
Kappa	-	Lambda	Λ	Mu	-
Nu	-	Xi	Ξ	Pi	Π
Rho	-	Sigma	Σ	Tau	-
Upsilon	Υ	Phi	Φ	Chi	-
Psi	Ψ	Omega	Ω		

and mathematical symbols

$$\frac{d}{dt} \quad \frac{\partial}{\partial t} \quad \partial_t \quad \partial_x \quad \nabla \quad \int_V \quad \oint \quad \times \quad \rightarrow \quad \infty \quad \cdots$$

Chapter 1

Fundamental Mathematical Aspects

A true story. The South African Weather Bureau (SAWB) uses mathematical models for the prediction of the state of the atmosphere. Due to the complexity of the atmosphere and the consequential cost of model development, models that were developed elsewhere are used. An upgraded version of such a model, configured at an 80km horizontal resolution, was received during 1997 from overseas. The SAWB decided that an increase in horizontal resolution is vital. A research project, addressing amongst others the extent of the horizontal domain, the limitation of errors at the lateral boundaries, a feasible number of vertical layers and the optimum configuration, keeping limited computer resources in mind, was approved.

The SAWB implemented the model with only limited help from overseas. One particular issue was the determination of parameters related to the chosen grid resolution. A mathematical transformation is used to avoid the effect of the convergence of the meridians at the earth's poles. The transformation was given in the model documentation, but contained a typing error. After hours of dedicated reasoning, marred by several dead-ends, a good sketch and sound vector calculus were instrumental in obtaining the correct formula. Imagine the emotion when, after all this effort, the correct formula was found in older documentation of the model! A computer programme for the calculation of the transformed grid was written, whereby a complete understanding of the workings of the transformation was thought to be obtained. However, when a very large grid, was calculated, strange kinks occurred in the southeastern and southwestern corners. Investigation of the computer source code revealed that a formula that differs from the one in the manual was used. Further examination of the mathematical derivation gave an equivalent but different formula, and on using this no unexpected kinks occurred.

With this story the reader is motivated to pursue a problem based approach to the learning process. The identification of a problem will determine the tools needed for its solution. Which problems can be identified from this story? Think about good up-to-date documentation and the background needed by an employee, responsible for the maintenance of a model, to be able to retrace the steps of the original modeller and programmer.

Introduction. The author's aim is to help the learner obtain a complete understanding of the mathematical tools available for making Dynamical Meteorology easier to understand and to provide an investment for the learner's future. Chapter 1 is the launching site for the self discovery journey the learner is about to embark on. Instead of a mere mathematical introduction to Dynamical Meteorology, the learners are lead to enhance their mathematical ability through relevant meteorological (or physical) examples and exercises. By early inclusion of topics like geopotential, potential temperature, temperature advection, reference and current configurations, etc. learners are motivated to *always strive* for integration of meteorological knowledge and experience.

1.1. Vector algebra.

1.1.1. Vector space.

Exercise 1.1.1. Revise the vector space concept. Use and expand on Fig. 1.1 to motivate why the sums of the components of two vectors determine the sum vector. Repeat with the concept of scalar multiple of a vector (Fig. 1.2).

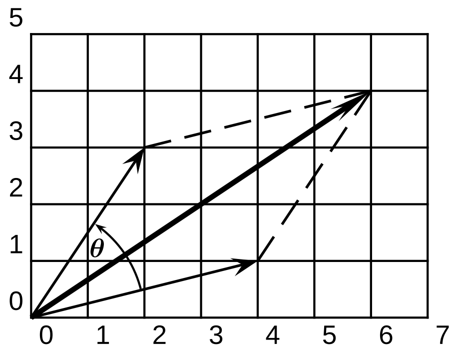

Figure 1.1

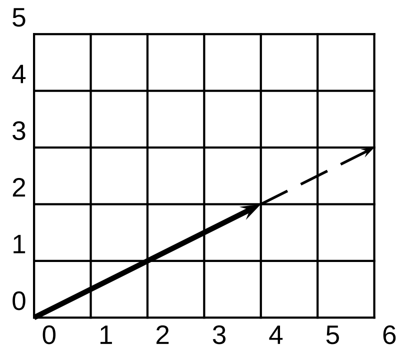

Figure 1.2

1.1.2. Basis and co-ordinate system.

With every basis $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\}$ of a vector space $\boldsymbol{X}$, a system of co-ordinates is associated as follows: to each vector $\boldsymbol{x} \in \boldsymbol{X}$ we assign the unique n-tuple of real numbers $(x_1, \cdots, x_n)$ such that

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \cdots + x_n\boldsymbol{e}_n = \Sigma_{i=1}^{n} x_i\boldsymbol{e}_i.$$

The numbers $(x_1, \cdots, x_n)$ are called co-ordinates or components of the vector $\boldsymbol{x}$ and depend on the basis vectors we have chosen. A basis determines a frame of reference for the description of the physical (meteorological) variable. Whereas the physical variable is co-ordinate free, it is convenient to choose a frame of reference for manipulation and closer description.

Exercises 1.1.2.

(a) Revise the following basic concepts: every vector space has a basis, and every basis of a specific vector space has the same number of elements.

(b) Check the following and use the result to show that the vector $\boldsymbol{x} = (1, -2, 3)$ has different components relative to different frames of reference:

$$\begin{aligned}\boldsymbol{x} &= 1(1,0,0) + (-2)(0,1,0) + 3(0,0,1)\\ &= (-12)(1,1,1) + 18(1,0,0) + 5(-1,2,3).\end{aligned}$$

1.1.3. Scalar product. Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$ be vectors in R^n $(n \geq 2)$. Define the scalar product (or dot product) $\boldsymbol{x} \cdot \boldsymbol{y}$ and norm (length of a vector) $\|\boldsymbol{x}\|$ by

$$\begin{aligned}\boldsymbol{x} \cdot \boldsymbol{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \Sigma_{i=1}^{n} x_i y_i\\ \|\boldsymbol{x}\| &= \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}.\end{aligned}$$

Exercises 1.1.3.

(a) Revise the symmetric property

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$$

and the linear property

$$\begin{aligned}\boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) &= \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \boldsymbol{z}\\ \boldsymbol{x} \cdot \lambda \boldsymbol{y} &= \lambda(\boldsymbol{x} \cdot \boldsymbol{y})\end{aligned}$$

of the scalar product.

(b) Prove that

$$\boldsymbol{x} \cdot \boldsymbol{y} = \|\boldsymbol{x}\| \; \|\boldsymbol{y}\| \cos\theta$$

with θ the angle between the two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ (see Fig. 1.1). For simplicity let $\boldsymbol{x} = (x,\ 0)$ and $\boldsymbol{y} = (y_1,\ y_2)$. Is this a severe restriction? Motivate your answer. Revise the concept of projection of a vector along a given line.

(c) **Spherical co-ordinates.** Prove that the vectors

$$\begin{aligned}\boldsymbol{i} &= (-\sin\lambda, \cos\lambda, 0)\\ \boldsymbol{j} &= (-\sin\phi\cos\lambda, -\sin\phi\sin\lambda, \cos\phi))\\ \boldsymbol{k} &= (\cos\phi\cos\lambda, \cos\phi\sin\lambda, \sin\phi)\end{aligned}$$

are linearly independent unit vectors. Next, expand and use Fig. 1.3 to enlighten your findings.

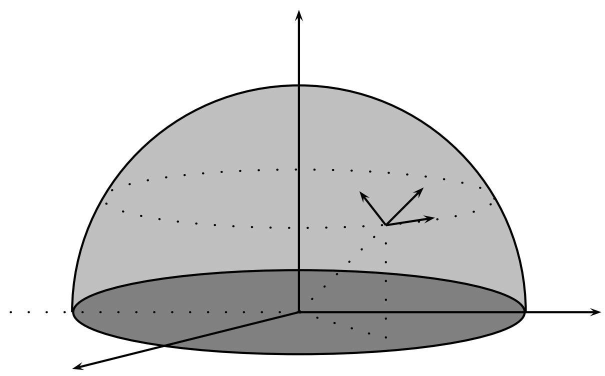

Figure 1.3

1.1.4. Vector product (in R^3). The following mnemonic form of the cross product is well known:

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

Exercises 1.1.4.

(a) Revise the concept of cross product. Next, for simplicity, let $\boldsymbol{x} = (x,\ 0,\ 0)$ and $\boldsymbol{y} = (y_1,\ y_2,\ 0)$. Prove that

$$\boldsymbol{x} \times \boldsymbol{y} = (0,\ 0,\ \|\boldsymbol{x}\|\ \|\boldsymbol{y}\| \sin\theta)$$

with θ the angle between the two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Show that the norm of $\boldsymbol{x} \times \boldsymbol{y}$ equals the area of the parallelogram determined by $\boldsymbol{x}$ and $\boldsymbol{y}$.

(b) Prove the following:

$$(1,\ 0,\ 0) \times (0,\ 1,\ 0) = (0,\ 0,\ 1)$$
$$(0,\ 1,\ 0) \times (0,\ 0,\ 1) = (1,\ 0,\ 0)$$
$$(0,\ 0,\ 1) \times (1,\ 0,\ 0) = (0,\ 1,\ 0).$$

(c) Let $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ be as in Exercise 1.1.3(c). Prove the following:

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$$
$$\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$$
$$\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}.$$

1.1.5. Orthogonal vectors. Definition: $\boldsymbol{x} \perp \boldsymbol{y}$ if and only if $\boldsymbol{x} \cdot \boldsymbol{y} = 0$.

Exercises 1.1.5.

(a) Show that

$$\begin{aligned}(a,\ b) &\perp (-b,\ a)\\ (1,\ 0,\ 0) &\perp (0,\ 1,\ 0)\\ (1,\ 0,\ 0) &\perp (0,\ 0,\ 1)\\ (0,\ 1,\ 0) &\perp (0,\ 0,\ 1).\end{aligned}$$

(b) Let $\boldsymbol{a} = (1, 2, 3)$, $\boldsymbol{b} = (3, 0, -1)$ and $\boldsymbol{c} = (5, -1, -1)$. Show that

$$\begin{aligned}&\boldsymbol{a} \perp \boldsymbol{b}\\ &\boldsymbol{a} \perp \boldsymbol{c}, \text{ but}\\ &\boldsymbol{b} \not\perp \boldsymbol{c}.\end{aligned}$$

(c) **Integration of knowledge.** Apply integration of knowledge to enhance your understanding of vector algebra. Amongst others, write a computer program that calculates vector sums, products of vectors with scalars, scalar products and cross products. Test your program with mathematically worked out examples. Use a computer graphics package to plot your results.

1.2. Functions. A function f, defined on domain $\mathfrak{D}(f)$ in a region $G \subset R^n$, is denoted by $f : x \mapsto f(x)$. If the context is clear, the abbreviation, $x \mapsto f(x)$, will be used. The range of f is the set $\mathfrak{R}(f)$ consisting of images of the elements of $\mathfrak{D}(f)$ under f. If $\mathfrak{D}(f) = X$, we will also write $\mathfrak{R}(f) = f(X)$.

To describe the mechanics of bodies or the motion of fluids, we disregard the microscopic structure and consider the body or fluid to be composed of a set of particles, distributed throughout some region of space. The set $\mathfrak{C}_t$ in three-dimensional Euclidian space, associated with the particles of the body at a given instant of time, is called the *configuration* or *state* of the body at time t. The co-ordinates $(x_1, x_2, x_3) \in \mathfrak{C}_t$ are known as *spatial* or *Eulerian* co-ordinates. A function f, defined on the configuration at time t is called a field function. We may also choose co-ordinates (X_1, X_2, X_3) from a *reference configuration* $\mathfrak{C}$ to describe individual particles. These co-ordinates are called *material (or Lagrangian)* co-ordinates.

Example 1.2. Atmospheric pressure in a vertical column may be considered to be a function of height $z \mapsto p(z)$. Conversely, the height z of an air parcel may be considered to depend on pressure $p \mapsto z(p)$ in a region where p is monotone (i.e. either increasing or decreasing).

Exercises 1.2.

(a) Revise the following concepts: a function, the domain and range of a function, the inverse of a function (if it exists) and continuity of a function.

(b) In Example 1.2 the symbol z is used to indicate geometric height both as independent variable and as a function of pressure, p. Similarly, the symbol p may be used to indicate atmospheric pressure both as independent variable and as a function of height. A combination of the definitions of the two functions gives a relationship $p(z(p)) = p$, showing that they are inverses of each other. However, p then becomes both independent and dependent variable in one single equation. To resolve this undesirable matter, define new functions $\tilde{p}$ and $\tilde{z}$, for example, and make the above discussion precise and mathematically sound.

(c) **The geopotential.** Define the geopotential Φ as $\Phi(z) = \int_0^z g\,dz'$. Use Newton's law of universal gravitation to prove that, in the absence of centripetal acceleration,

$$\Phi(z) = g_0 \frac{az}{a+z}.$$

(d) Let $Z = \Phi(z)/g_0$ and $a = 6000$km. Calculate Z for $z = 10, 20, \cdots 150$ km. Use a computer graphics package to plot your result.

(e) Use Taylor's theorem to prove that for small z/a

$$Z \approx z - a\left(\frac{z}{a}\right)^2.$$

Calculate $a\left(\frac{z}{a}\right)^2$ for $z = 1, 10$ and 100 km.

(f) **Potential temperature** Define the *potential temperature* θ as

$$\theta = T\left(\frac{p_s}{p}\right)^{R/c_p}.$$

Let $R = 287\,\mathrm{J\,K^{-1}\,kg^{-1}}$, $c_p = 1004\,\mathrm{J\,K^{-1}\,kg^{-1}}$ and $p_s = 1013.25\,\mathrm{hPa}$. For $\theta =$ 293 K, calculate $T : p \mapsto T(p)$, in °C, for pressure levels $p = 1000, 900, \cdots, 100$ hPa. Use a computer graphics package to plot your result. Repeat with $\theta = 303$ K and $\theta = 313$ K. Plot your results on the same graph.

1.3. Differentiation.

1.3.1. Scalar valued functions of one variable. Revise the concept of the derivative of a function:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

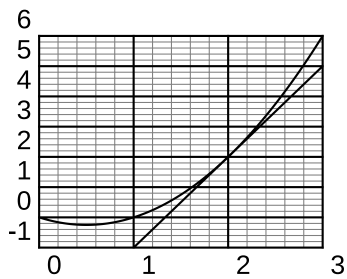

Figure 1.4. The derivative as slope of the tangent line

Carefully note that the derivative is a function in its own right and that the notation f' (or $\frac{df}{dx}$) refers to a single object, namely the derivative. Thus the argument (the element from the domain of f') has nothing in common with the x in the notation $\frac{df}{dx}$, see for example the use of the chain rule of differentiation in Equations (3.15) and (3.16) in Section 3.2.3 of Chapter 3.

Exercise 1.3.1. Use the limit definition to determine the derivative function f' of the following functions:

$$f(x) = x$$
$$f(x) = x^2$$
$$f(x) = e^x.$$

1.3.2. Vector valued functions of one variable.

Exercises 1.3.2. Let $\boldsymbol{F}(t) = (x(t),\ y(t))$.

(a) Explain how $\boldsymbol{F}$ can be used to describe a curve in the two dimensional plane. Next, let x and y be differentiable. To investigate the differentiability of the vector valued function $\boldsymbol{F}$, prove the following:

$$\frac{\boldsymbol{F}(t+h) - \boldsymbol{F}(t)}{h} = \left(\frac{x(t+h) - x(t)}{h},\ \frac{y(t+h) - y(t)}{h}\right)$$

$$\frac{\boldsymbol{F}(t+h) - \boldsymbol{F}(t)}{h} - (x'(t),\ y'(t)) = (\frac{x(t+h) - x(t)}{h} - x'(t),\ \frac{y(t+h) - y(t)}{h} - y'(t))$$

so that

$$\left\|\frac{\boldsymbol{F}(t+h) - \boldsymbol{F}(t)}{h} - (x'(t), y'(t))\right\|$$

$$= \sqrt{\left[\frac{x(t+h) - x(t)}{h} - x'(t)\right]^2 + \left[\frac{y(t+h) - y(t)}{h} - y'(t)\right]^2}$$

$$\to 0 \text{ as } h \to 0.$$

(b) Use the limit result above to motivate

$$\boldsymbol{F}'(t) = (x'(t),\ y'(t)).$$

(c) Let a be constant. The function $t \mapsto \boldsymbol{F}(t)$ defined by

$$\boldsymbol{F}(t) = (x(t),\ y(t)) = (a\cos\theta(t),\ a\sin\theta(t))$$

describes circular motion with radius a around a fixed point. Calculate $\boldsymbol{F}'(t)$.

(d) Let $\boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t))$. Prove that

$$\frac{d}{dt}\|\boldsymbol{x}(t)\|^2 = 2\boldsymbol{x}(t) \cdot \boldsymbol{x}'(t).$$

1.3.3. Scalar valued functions of several variables.

Example 1.3.3. Heat conduction and Newton's cooling law. Consider heat conduction in a material in the absence of motion. Consider the temperature $T(\boldsymbol{x}, t)$ at position $\boldsymbol{x}$ and time t. If h is a fixed real number and $\boldsymbol{n}$ an arbitrary unit vector, then $T(\boldsymbol{x} + h\boldsymbol{n}, t)$ gives the temperature on a sphere with radius h about the point $\boldsymbol{x}$. In general, the flux $\boldsymbol{\Phi}$ of energy (rate of heat flow) at $\boldsymbol{x}$ in the direction $\boldsymbol{n}$ will depend on the direction $\boldsymbol{n}$. We may imagine it to be proportional to the temperature difference, $T(\boldsymbol{x}+h\boldsymbol{n}, t) - T(\boldsymbol{x}, t)$, between nearby points and inversely proportional to the distance between them, i.e. proportional to $\frac{T(\boldsymbol{x}+h\boldsymbol{n},t)-T(\boldsymbol{x},t)}{h}$. Assuming heat flow is from hot to cold, we look for a relation of the form $\boldsymbol{\Phi} \cdot \boldsymbol{n} \propto -\frac{T(\boldsymbol{x}+h\boldsymbol{n},t)-T(\boldsymbol{x},t)}{h}$. For simplicity, we suppress the dependency of temperature on time t, so that the flux becomes proportional to $\frac{T(\boldsymbol{x}+h\boldsymbol{n})-T(\boldsymbol{x})}{h}$, which reminds of the difference quotients in Sections 1.3.1 or 1.3.2. We thus define

The directional derivative. Let $f : \boldsymbol{x} \mapsto f(\boldsymbol{x})$ be a scalar valued function of the variables $x_1,\ x_2,\ \cdots x_k$, where $(x_1, x_2, \cdots, x_k) = \boldsymbol{x}$. Let $\boldsymbol{n} = (n_1, n_2, \cdots n_k)$ be a unit vector denoting a fixed direction in space. The directional derivative at the point $\boldsymbol{x}$ in the direction of $\boldsymbol{n}$ is defined by

$$f'(\boldsymbol{x}; \boldsymbol{n}) = \lim_{h \to 0} \frac{f(\boldsymbol{x} + h\boldsymbol{n}) - f(\boldsymbol{x})}{h},$$

see for example Apostol (1969), p. 252.

If $h \to 0$, the difference quotient $\frac{T(\boldsymbol{x}+h\boldsymbol{n})-T(\boldsymbol{x})}{h}$ thus tends to $T'(\boldsymbol{x}; \boldsymbol{n})$, so that the temperature flux in the direction $\boldsymbol{n}$ becomes $\boldsymbol{\Phi}(\boldsymbol{x}) \cdot \boldsymbol{n} = -\kappa T'(\boldsymbol{x}; \boldsymbol{n})$, with κ a proportionality constant.

Exercise 1.3.3. Let $f(x, y) = x^2 + y^2$. Sketch a few curves on which f is constant. Next, let $\boldsymbol{x} = (x_1, x_2)$. Show that $g(\boldsymbol{x}) = {x_1}^2 + {x_2}^2$ defines the same function, that is, $f = g$. Use the limit definition to calculate $f'(\boldsymbol{x}; \boldsymbol{n})$, for arbitrary directions $\boldsymbol{n} = (n_1, n_2)$. In which direction is the derivative a maximum and in which a minimum? What happens in the other directions?

1.3.4. Partial derivatives.

Define

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$
$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Notation 1.3.4. We also use the economical and sufficiently clear notation,

$$\partial_x = \frac{\partial}{\partial x}, \quad \partial_y = \frac{\partial}{\partial y}, \quad \partial_p = \frac{\partial}{\partial p}, \quad etc.$$

for partial derivatives. In Cartesian co-ordinates $x_1, x_2, \cdots x_n$ we write

$$\partial_i = \frac{\partial}{\partial x_i}, \quad \text{for} \quad i = 1, 2, \cdots n.$$

The del or nabla operator is denoted by

$$\nabla = (\partial_1, \partial_2, \cdots, \partial_n),$$

or in three dimensions

$$\nabla = \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z},$$

with $\boldsymbol{e}_x, \boldsymbol{e}_y$ and $\boldsymbol{e}_z$ any three orthogonal unit vectors. Again, as mentioned at the beginning of Section 1.3.1, note (and revise) the use of the chain rule of differentiation.

Remember that a vector represents a co-ordinate free quantity and that you have the freedom to choose a frame of reference suitable for your application. The gradient vector $\nabla f = (\partial_1 f, \partial_2 f, \cdots, \partial_n f)$ or $\nabla f = \boldsymbol{e}_x \frac{\partial f}{\partial x} + \boldsymbol{e}_y \frac{\partial f}{\partial y} + \boldsymbol{e}_z \frac{\partial f}{\partial z}$ is a co-ordinate dependent representation of a vector field. Its relation to the directional derivative is given by

Theorem 1.3.4. The directional derivative and partial differentiation. Once a frame of reference is chosen, the directional derivative can be represented by the gradient as follows:

$$f'(\boldsymbol{x}; \boldsymbol{n}) = \nabla f(\boldsymbol{x}) \cdot \boldsymbol{n},$$

see Apostol (1969), p. 259.

Exercises 1.3.4.

(a) Let $f(x, y) = x^2 + y^2$ as in Exercise 1.3.3. Use differentiation to show that

$$\frac{\partial f}{\partial x}(x, y) = 2x$$
$$\frac{\partial f}{\partial y}(x, y) = 2y.$$

(b) Combine the result in (a) with Exercise 1.3.3 to show that

$$f'((x, y); \boldsymbol{n}) = \left(\frac{\partial f}{\partial x}(x, y),\ \frac{\partial f}{\partial y}(x, y) \right) \cdot \boldsymbol{n}.$$

(c) Use theorem 1.3.4 to prove

$$\frac{\partial f}{\partial x_i}(\boldsymbol{x}) = f'(\boldsymbol{x}; \boldsymbol{e}_i), \text{for arbitrary } i$$

and give an interpretation of partial derivatives in terms of the concept of directional derivatives. Enlighten your answer with a sketch.

(d) Show that the flux vector in Example 1.3.3 satisfies

$$\boldsymbol{\Phi}(\boldsymbol{x}) \cdot \boldsymbol{n} = -\kappa \nabla T(\boldsymbol{x}) \cdot \boldsymbol{n}.$$

Use the fact that $\boldsymbol{n}$ is arbitrary to show that

$$\boldsymbol{\Phi}(\boldsymbol{x}) = -\kappa \nabla T(\boldsymbol{x})$$
$$\boldsymbol{\Phi}(\boldsymbol{x}, t) = -\kappa \nabla T(\boldsymbol{x}, t).$$

1.3.5. The total derivative and temperature advection. A field function describes a physical quantity from the viewpoint of a fixed observer while a material description refers to an observer that moves with the fluid (Section 1.2). The total derivative of a function of space and time reflects the possible influence of fluid movement together with local changes with time. Heuristically, for an arbitrary chosen function f, it may be represented by:

$$\begin{aligned}\frac{Df}{Dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v + \frac{\partial f}{\partial z}w,\end{aligned}$$

see Holton (1992), p. 29. For a formal definition, let $(\boldsymbol{x}, t) \mapsto F(\boldsymbol{x}, t)$ be any differentiable function, $T(\boldsymbol{x}, t)$ a temperature field and

$$\boldsymbol{U}(\boldsymbol{x}, t) = (u(\boldsymbol{x}, t),\ v(\boldsymbol{x}, t),\ w(\boldsymbol{x}, t))$$

a three dimensional velocity field.

Definitions 1.3.5.

(a) **The total derivative.** The *material*, the *total* or the *substantial* derivative of F is defined by

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \boldsymbol{U} \cdot \nabla F = \frac{\partial F}{\partial t} + \left(u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} \right)$$

and represents the rate of change, following the motion, of the field variable F.

(b) **Temperature advection.** The *temperature advection* is defined as $-\boldsymbol{U} \cdot \nabla T$.

Note that definition 1.3.5(a) implies $\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \boldsymbol{U} \cdot \nabla T$, so that the local time rate of change of temperature, $\frac{\partial T}{\partial t}$, is the sum of the total derivative (following the motion) and the temperature advection.

Example 1.3.5. Temperature advection. Let $T(x, t)$ denote temperature in a spatial one dimensional region, at position x and at time t. T is a field function and an Eulerian description (see Section 1.2) is used. Suppose the wind is blowing with constant speed u and that the particle, initially at position x, is transported downstream a distance $u\delta t$ while retaining its temperature. From Fig. 1.5 follows $T(x + u\delta t, t) = T(x, t - \delta t)$, and we find that

$$\frac{T(x + u\delta t, t) - T(x, t)}{\delta t} = \frac{T(x, t - \delta t) - T(x, t)}{\delta t}.$$

In the limit $\delta t \to 0$, using L'Hospital's rule, we deduce

$$u\frac{\partial T}{\partial x}(x, t) = -\frac{\partial T}{\partial t}(x, t).$$

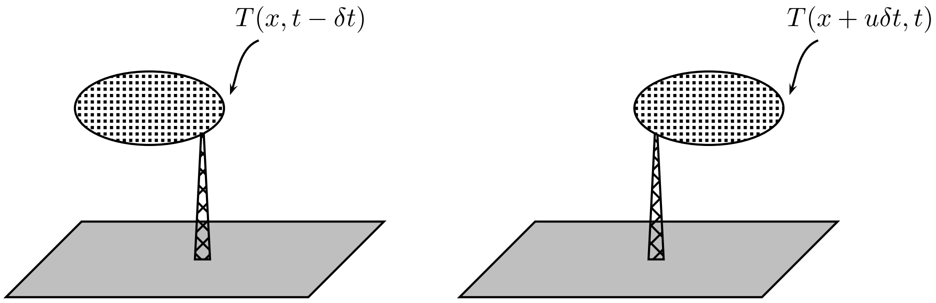

Figure 1.5

Exercises 1.3.5.
(a) As may be expected, the material derivative of the temperature function T in Example 1.3.5 vanishes. Prove this statement and discuss the local change in temperature caused by an easterly wind, blowing from a colder area towards the observer.
(b) Suppose $T(x, t + \delta t) = T(x - u\delta t, t)$. Interpret this statement and repeat the previous argument to once more show that $\frac{\partial T}{\partial t}(x, t) + u\frac{\partial T}{\partial x}(x, t) = 0$.

1.3.6. Vector valued functions of several variables.
See for example Section 8.18 in Apostol (1969).
Example 1.3.6. Velocity advection of a fluid in rigid body rotation. The operator $\boldsymbol{U}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ in Definition 1.3.5 is simply a linear combination of partial derivatives. It may thus be applied to a vector valued function. In two dimensions we define $\boldsymbol{V} = (u, v)$ and we may write $\boldsymbol{V}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$.

Using Notation 1.3.4, we put $\boldsymbol{x} = (x_1, x_2)$ and

$$\boldsymbol{V}(\boldsymbol{x}) = (-\omega x_2, \omega x_1) = (V_1(\boldsymbol{x}), V_2(\boldsymbol{x})),$$

where ω represents a constant angular velocity. By direct differentiation

$$\begin{aligned}(\boldsymbol{V}\cdot\nabla)\boldsymbol{V} &= (\partial_1\boldsymbol{V})V_1 + (\partial_2\boldsymbol{V})V_2\\ &= (0, \omega)V_1 + (-\omega, 0)V_2.\end{aligned}$$

Using $V_1 = -\omega x_2$ and $V_2 = \omega x_1$, the advection of velocity at $\boldsymbol{x}$ becomes

$$\begin{aligned}((\boldsymbol{V}\cdot\nabla)\boldsymbol{V})(\boldsymbol{x}) &= -x_2\omega(0, \omega) + x_1\omega(-\omega, 0)\\ &= -\omega^2(x_1, x_2)\\ &= -\omega^2\boldsymbol{x}.\end{aligned}$$

The general case. Let $\boldsymbol{F}(\boldsymbol{x}) = (F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \cdots, F_n(\boldsymbol{x}))$ and suppose each component F_i is differentiable.

Exercises 1.3.6.
(a) Show that

$$\frac{\boldsymbol{F}(\boldsymbol{x}+h\boldsymbol{n})-\boldsymbol{F}(\boldsymbol{x})}{h} =$$

$$\left(\frac{F_1(\boldsymbol{x}+h\boldsymbol{n})-F_1(\boldsymbol{x})}{h}, \frac{F_2(\boldsymbol{x}+h\boldsymbol{n})-F_2(\boldsymbol{x})}{h}, \cdots, \frac{F_n(\boldsymbol{x}+h\boldsymbol{n})-F_n(\boldsymbol{x})}{h}\right).$$

(b) Complete the following:

$$\left\|\frac{\boldsymbol{F}(\boldsymbol{x}+h\boldsymbol{n})-\boldsymbol{F}(\boldsymbol{x})}{h} - \left(F_1{}'(\boldsymbol{x};\boldsymbol{n}),\cdots,F_n{}'(\boldsymbol{x};\boldsymbol{n})\right)\right\| =$$

$$\sqrt{\left(\frac{F_1(\boldsymbol{x}+h\boldsymbol{n})-F_1(\boldsymbol{x})}{h} - F_1{}'(\boldsymbol{x};\boldsymbol{n})\right)^2 + \cdots}$$

$$\to 0 \text{ as } h \to 0.$$

(c) Define the *Jacobian matrix*

$$\tilde{A} = \begin{bmatrix} \nabla F_1 \\ \nabla F_2 \\ \vdots \\ \nabla F_n \end{bmatrix} = \begin{bmatrix} \partial_1 F_1 & \partial_2 F_1 \cdots \partial_n F_1 \\ \partial_1 F_2 & \partial_2 F_2 \cdots \partial_n F_2 \\ & \vdots \\ \partial_1 F_n & \partial_2 F_n \cdots \partial_n F_n \end{bmatrix}.$$

Write $\tilde{A} = \boldsymbol{DF}(\boldsymbol{x})$ as on p. 270 of Apostol (1969) and show that the result of (b) for $\boldsymbol{F}'(\boldsymbol{x};\boldsymbol{n})$ can be written in terms of the product of the Jacobian matrix $\boldsymbol{DF}(\boldsymbol{x})$ and the vector $\boldsymbol{n}$. Compare the result with Theorem 1.3.4.

Remark 1.3.6. Note that relative to your frame of reference, the gradient vector ∇F or the matrix $\boldsymbol{DF}$ forms part of a co-ordinate dependent representation of the co-ordinate free directional derivative $\boldsymbol{F}'(\boldsymbol{x};\boldsymbol{n})$.

1.3.7. Integration of knowledge. Regarding the concept of functions and the derivatives of functions, make a summary of what was learnt, what mistakes and misconceptions existed (and why) and other relevant ideas. Look for life-relevant aspects touched upon and applications to meteorology. Evaluate your work and put it in a portfolio for future reference.

1.4. Integration.

1.4.1. Fundamental theorems of the Calculus. Revise the following:

$$F(x) = \int_a^x f(x')\,dx' \Rightarrow F'(x) = f(x) \text{ if } f \text{ is continuous at } x, x \in [a,b] \quad (1.1)$$

$$\int_a^b F'(x)\,dx = F(b) - F(a) \text{ if } F \text{ is smooth enough.} \quad (1.2)$$

Exercises 1.4.1.

(a) Differentiation of integrals. Define $F(x) = \int_a^x f(x')\,dx'$. Use the chain rule of differentiation $\frac{d}{dt}[F(r(t))] = F'(r(t))r'(t)$ and (1.1) to show that

$$\frac{d}{dt}\int_a^{r(t)} f(x')\,dx' = f(r(t))r'(t). \quad (1.3)$$

(b) Define $g(t) = \int_a^b f(x,t)\,dx$. Use the limit definition of $g'(t)$ to show that

$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx. \quad (1.4)$$

You may assume the function f is such that you may interchange the order of limit taking and integration.

(c) Define $F(x,t) = \int_a^x f(x',t)\,dx'$. Use

$$\frac{d}{dt}(F(r(t),t)) = \frac{\partial F}{\partial x}(r(t),t)r'(t) + \frac{\partial F}{\partial t}(r(t),t)$$

in combination with (1.3) and (1.4) to show that

$$\frac{d}{dt}\int_a^{r(t)} f(x,t)\,dx = f(r(t),t)r'(t) + \int_a^{r(t)} \frac{\partial f}{\partial t}(x,t)\,dx. \quad (1.5)$$

(d) Use (1.5) to show that

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)\,dx. \quad (1.6)$$

1.4.2. Transformation of integrals. Revise the general rule

$$\int_{f(a)}^{f(b)} g(u)\,du = \int_a^b g(f(x))f'(x)\,dx. \quad (1.7)$$

Example 1.4.2.

$$\int_0^t \frac{1}{\sqrt{1-x^2}}\,dx = \int_0^{\sin^{-1} t} \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta$$
$$= \int_0^{\sin^{-1} t} 1\,d\theta$$
$$= \sin^{-1} t.$$

Exercises 1.4.2.

(a) To test the result in Example 1.4.2, we need to prove

$$\frac{d}{dt}\int_0^t \frac{1}{\sqrt{1-x^2}}\,dx = \frac{d}{dt}\sin^{-1} t = \frac{1}{\sqrt{1-t^2}}.$$

Explain this statement and prove it.

(b) Show that

$$\int_0^1 2xe^{x^2}\,dx = e - 1$$

by using the substitution $x = \sqrt{s}$.

(c) Show that

$$\int_0^x \frac{1}{1+t^2}\,dt = \tan^{-1} x$$

by using the substitution $t = \tan\theta$.

1.4.3. Interchanging total differentiation and integration. Consider the motion of a part $\mathfrak{P}$ of the reference configuration $\mathfrak{C}$ in a Lagrangian description of fluid flow. The fluid associated with the given set $\mathfrak{P}$ of particles is called a *material volume*. As time progresses the material volume always consists of the same particles and moves with the fluid. To describe such a material volume we need a mapping from a reference configuration (e.g. the configuration at time $t = 0$) to the current configuration.

The reference and current configurations. Consider a one dimensional continuum with reference configuration the interval $(0, L)$. The transformation $x = r(X), X \in (0, L)$ maps the interval $(0, L)$ to the current configuration $(0, l)$ where $0 = r(0)$ and $l = r(L)$ (Fig. 1.6). Let $[A, B] \subset (0, L)$ and put $a = r(A)$ and $b = r(B)$.

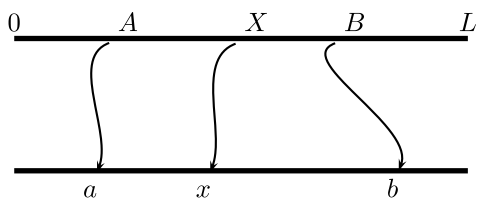

Figure 1.6

The following exercises show how total differentiation can arise from the differentiation of an integral defined on a material volume. For simplicity a one dimensional approach is followed. Let ρ denote fluid density and $f : (x,t) \mapsto f(x,t)$ a field function. The main result of this Section is

$$\frac{d}{dt}\int_{r(A,t)}^{r(B,t)} \rho(x,t)f(x,t)\,dx = \int_{r(A,t)}^{r(B,t)} \rho(x,t)\frac{Df}{Dt}(x,t)\,dx \tag{1.8}$$

and it is proven in Exercises 1.4.3(d).

Exercises 1.4.3. Suppose a function σ exists such that

$$\int_{r(A)}^{r(B)} \rho(x)\,dx = \int_A^B \sigma(X)\,dX \text{ for every } [A,B] \subset (0,L).$$

(a) Use the transformation $x = r(X)$ and the general rule (1.7) to prove that

$$\rho(r(X))r'(X) = \sigma(X).$$

(b) Use the transformation $x = r(X)$ and the result $\rho(r(X))r'(X) = \sigma(X)$ to prove that

$$\int_{r(A)}^{r(B)} \rho(x)f(x)\,dx = \int_A^B \sigma(X)f(r(X))\,dX.$$

(c) Let t be fixed and

$$\int_{r(A,t)}^{r(B,t)} \rho(x,t)\,dx = \int_A^B \sigma(X)\,dX \text{ for every } A, B.$$

Consider the mapping r in Fig. 1.6 that maps the reference configuration $[0,L]$ to the configuration $[0,l]$ at time t. We write $x = r(X,t), X \in (0,L)$. Use reasoning similar to that in Exercise (b) to show that

$$\rho(r(X,t),t)\frac{\partial r}{\partial X}(X,t) = \sigma(X)$$

and that

$$\int_{r(A,t)}^{r(B,t)} \rho(x,t)f(x,t)\,dx = \int_A^B \sigma(X)f(r(X,t),t)\,dX.$$

(d) Prove (1.8) by completing the following:

$$\begin{aligned}\frac{d}{dt}\int_{r(A,t)}^{r(B,t)} \rho(x,t)f(x,t)\,dx &= \frac{d}{dt}\int_A^B \sigma(X)f(r(X,t),t)\,dX \\ &= \int_A^B \sigma(X)\left(\frac{\partial f}{\partial t}(r(X,t),t) + \frac{\partial r}{\partial t}(X,t)\frac{\partial f}{\partial x}(r(X,t),t)\right)\,dX.\end{aligned}$$

Hint: replace $\sigma(X)$ by the left hand side of the result in (c), replace $\frac{\partial r}{\partial t}$ by the velocity field v, given by

$$v(r(X,t),t) = \frac{\partial r}{\partial t}(X,t), \tag{1.9}$$

use the definition of the material derivative $\frac{Df}{Dt}$

$$\frac{Df}{Dt}(x,t) = \frac{\partial f}{\partial t}(x,t) + v(x,t)\frac{\partial f}{\partial x}(x,t),$$

and transform back to the variable x, using $x = r(X,t)$.

1.5. More examples with meteorological applications. To proceed, the student should have a thorough understanding of the concepts of vectors, functions and differentiation as described in the previous paragraphs and mathematical textbooks. To test his understanding and simultaneously to introduce new mathematical, physical and meteorological concepts, the concepts of circular motion of a particle, circulation and vorticity of a fluid, divergence and convergence, and advection are presented. Chapter 1 ends with some useful vector algebraic results. The reader may expand on this by adding results on triple products, containing the nabla operator and vector valued functions.

1.5.1. Motion of a particle in a circle. The motion of a particle in a circle is discussed to provide an example for introducing the concepts of velocity, acceleration, circulation, divergence and advection. Moreover, it serves to prepare the way for the derivation of a formula for the acceleration of a small fluid element in a rotating frame of reference (Chapter 2). A solid body rotation in two dimensions can be generated from a collection of circular particle motions, each with the same angular velocity. To understand rotation of a fluid (or air mass) an understanding of solid body rotation is beneficial.

Consider the motion of a particle in a circle with radius a. Choose a frame of reference and position vector

$$\boldsymbol{x}(t) = (a\cos\theta(t),\ a\sin\theta(t)).$$

Example 1.5.1. Velocity and acceleration. Differentiation gives

$$\begin{aligned}\boldsymbol{x}'(t) &= a\theta'(t)(-\sin\theta(t),\ \cos\theta(t))\\ \boldsymbol{x}''(t) &= a\theta''(t)(-\sin\theta(t),\ \cos\theta(t)) - a\theta'(t)^2(\cos\theta(t),\ \sin\theta(t)).\end{aligned}$$

so that the definitions of the following **vector valued functions** are evident:

- Position

$$\boldsymbol{x} = (a\cos\theta,\ a\sin\theta)$$

- Velocity

$$\boldsymbol{V} = \boldsymbol{x}' = a\omega(-\sin\theta,\ \cos\theta)$$

• Acceleration

$$\boldsymbol{a} = \boldsymbol{x}'' = a\omega'(-\sin\theta,\ \cos\theta) - a\omega^2(\cos\theta,\ \sin\theta),$$

with θ the angle represented by the scalar valued function $t \to \theta(t)$ and ω the angular velocity represented by the function $t \to \theta'(t)$.

Definition 1.5.1. Define (see Fig. 1.7)

$$\begin{aligned} \boldsymbol{e}_r &= (\cos\theta,\ \sin\theta) \\ \boldsymbol{e}_\theta &= (-\sin\theta,\ \cos\theta). \end{aligned}$$

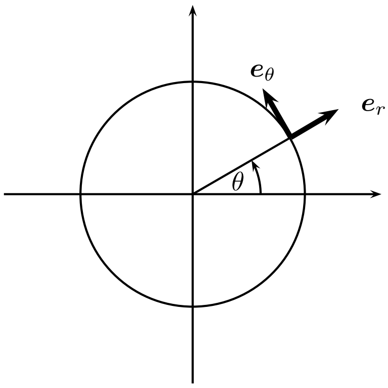

Figure 1.7

Exercises 1.5.1. Co-ordinates or components with respect to a chosen frame of reference.

(a) Check the differentiations in Example 1.5.1.

(b) Show that

$$\begin{aligned} \boldsymbol{e}_r \cdot \boldsymbol{e}_\theta &= 0 \\ \boldsymbol{V} \cdot \boldsymbol{e}_r &= 0, \text{ the component in the direction of } \boldsymbol{e}_r \\ \boldsymbol{V} \cdot \boldsymbol{e}_\theta &= a\omega, \text{ the component in the direction of } \boldsymbol{e}_\theta \\ \boldsymbol{V} &= a\omega\boldsymbol{e}_\theta \\ \boldsymbol{a} \cdot \boldsymbol{e}_r &= -a\omega^2, \text{ the component in the direction of } \boldsymbol{e}_r \\ \boldsymbol{a} \cdot \boldsymbol{e}_\theta &= a\omega', \text{ the component in the direction of } \boldsymbol{e}_\theta \\ \boldsymbol{a} &= a\omega'\boldsymbol{e}_\theta - a\omega^2\boldsymbol{e}_r. \end{aligned}$$

(c) **Vector representation and the scalar product.** Let $\boldsymbol{x}, \boldsymbol{V}$ and $\boldsymbol{a}$ be as in Example 1.5.1. Simplify the scalar products in the following and interpret the results:

$$\begin{aligned}
\boldsymbol{x} &= (\boldsymbol{x}\cdot\boldsymbol{e}_r)\boldsymbol{e}_r + (\boldsymbol{x}\cdot\boldsymbol{e}_\theta)\boldsymbol{e}_\theta\\
\boldsymbol{V} &= (\boldsymbol{V}\cdot\boldsymbol{e}_r)\boldsymbol{e}_r + (\boldsymbol{V}\cdot\boldsymbol{e}_\theta)\boldsymbol{e}_\theta\\
\boldsymbol{a} &= (\boldsymbol{a}\cdot\boldsymbol{e}_r)\boldsymbol{e}_r + (\boldsymbol{a}\cdot\boldsymbol{e}_\theta)\boldsymbol{e}_\theta\\
\boldsymbol{V} &= (\boldsymbol{V}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1 + (\boldsymbol{V}\cdot\boldsymbol{e}_2)\boldsymbol{e}_2.
\end{aligned}$$

Note that a vector has different components with respect to different bases. Compare, for example, the components of $\boldsymbol{V}$ in the second and fourth result above. Also, note the concept of projections (Exercise 1.1.3(b)). See Exercises 1.1.2 and 1.6 (later on) for the concept of basis of a vector space.

(d) **Vector product, angular velocity vector.** Define an angular velocity vector $\boldsymbol{\omega} = \omega(0,0,1) = \omega\boldsymbol{k}$. Let $\boldsymbol{r} = (\boldsymbol{x},\ 0)$ and $\boldsymbol{U} = (\boldsymbol{V},\ 0)$, with $\boldsymbol{x}$ and $\boldsymbol{V}$ as in Example 1.5.1. Show that

$$\begin{aligned}
\boldsymbol{\omega}\times\boldsymbol{r} &= a\omega(-\sin\theta,\ \cos\theta,\ 0)\\
&= (\boldsymbol{V},\ 0)\\
&= \boldsymbol{U}.
\end{aligned}$$

Notation 1.5.1. As in Holton (1992), we use the symbol $\boldsymbol{V}$ to indicate a two dimensional velocity field and the symbol $\boldsymbol{U}$ to indicate a three dimensional velocity field.

1.5.2. Circulation and vorticity. Let C_a be the circle with radius a and area $A = \pi a^2$. Consider an Eulerian description of the velocity $\boldsymbol{V}$ given in Exercise 1.5.1(b) as $\boldsymbol{V} = a\omega\boldsymbol{e}_\theta$. Then $C_a = \{\boldsymbol{x}|\ \|\boldsymbol{x}\| = a\}$ and the circulation C about the closed contour C_a is (see Holton (1992), p. 88)

$$\begin{aligned}
C &= \oint \boldsymbol{V}\cdot d\boldsymbol{l}\\
&= \int_{C_a} \boldsymbol{V}\cdot d\boldsymbol{l}\\
&= \int_0^{2\pi} (a\omega\boldsymbol{e}_\theta)\cdot(ad\theta)\boldsymbol{e}_\theta\\
&= \int_0^{2\pi} (a\omega\boldsymbol{e}_\theta\cdot a\boldsymbol{e}_\theta)\,d\theta\\
&= \int_0^{2\pi} a^2\omega d\theta\\
&= a^2\omega 2\pi\\
&= 2\omega A.
\end{aligned}$$

A mnemonic form of the rotation (curl) of a vector $\boldsymbol{A}$ is

$$curl\ \boldsymbol{A} = \nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}.$$

Exercise 1.5.2.1. Let $\boldsymbol{U} = (u(x,y),\ v(x,y),\ 0)$. Prove

$$curl\ \boldsymbol{U} = (0,\ 0,\ \partial_x v - \partial_y u).$$

Exercises 1.5.2.2. In contrast with single particle motion, we now consider different fluid particles, each rotating in a circle with its own radius r, around the same fixed point. Consider circular motion, with radius $r = \sqrt{x^2 + y^2}$ and velocity

$$\boldsymbol{U}(x,y) = (-r\omega \sin\theta, r\omega\cos\theta, 0) = \omega(-y, x, 0).$$

Let ω be constant so that a rigid body motion ensues.

(a) Compare the velocity field with $\boldsymbol{V}$ in Example 1.3.6 and discuss the direction of the velocity vector.

(b) Show that the *vorticity* vector $\nabla \times \boldsymbol{U}$ satisfies

$$\nabla \times \boldsymbol{U} = 2\omega \boldsymbol{k}$$

and that

$$\iint_A \nabla \times \boldsymbol{U} \cdot \boldsymbol{k}\, dA = 2\omega A = \oint \boldsymbol{U} \cdot d\boldsymbol{l}.$$

1.5.2.3. Stokes' theorem. Since $\nabla \times \boldsymbol{U}$ is constant, **Stokes' theorem**

$$\iint_A \nabla \times \boldsymbol{U} \cdot\ d\boldsymbol{A} = \oint \boldsymbol{U} \cdot d\boldsymbol{l}$$

follows trivially for this special case.

Exercise 1.5.2.3. Stokes' theorem on a rectangle. Consider the rectangle $A = \{\boldsymbol{x} |\ 0 < x_1 < a, 0 < x_2 < b\}$, with boundary denoted by ∂A. Check each step in the following:

$$\begin{aligned} \iint_A curl\ \boldsymbol{U} \cdot d\boldsymbol{A} &= \iint_A curl\ \boldsymbol{U} \cdot \boldsymbol{k} dA \\ &= \iint_A (\partial_x v - \partial_y u) dA \\ &= \int_0^b (\int_0^a \partial_x v dx) dy - \int_0^a (\int_0^b \partial_y u dy) dx \\ &= \int_0^b (v(a,y) - v(0,y)) dy - \int_0^a (u(x,b) - u(x,0)) dx \\ &= \int_{\partial A} \boldsymbol{U} \cdot d\boldsymbol{l}. \end{aligned}$$

Exercise 1.5.2.4. Polar co-ordinates for the rigid body motion. Let $\boldsymbol{U}(r,\theta,z) = r\omega\boldsymbol{e_\theta}$ and $\boldsymbol{k} = \boldsymbol{e}_z$. For this example, a simplification of the general rule in Calculus suffices, viz.

$$\nabla \times \boldsymbol{A} \;=\; \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ A_r & rA_\theta & A_z \end{vmatrix}.$$

Use this result to show that $\nabla \times \boldsymbol{U} = \nabla \times r\omega\boldsymbol{e_\theta} = 2\omega\boldsymbol{k}$ and compare with Exercise 1.5.2.2(b).

1.5.3. Divergence and convergence. Define the divergence of a vector $\boldsymbol{A}$ in Cartesian co-ordinates by

$$\begin{aligned}\nabla \cdot \boldsymbol{A} &= \partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 \\ &= \partial_x A_x + \partial_y A_y + \partial_z A_z.\end{aligned}$$

Exercises 1.5.3.1.
(a) Let $\boldsymbol{U} = (u(x,y),\ v(x,y),\ 0)$. Prove that $\nabla \cdot \boldsymbol{U} = \partial_x u + \partial_y v$.
(b) Let ω be constant. Show that the circular motion, with velocity field given by $\boldsymbol{U}(x,y) = \omega(-y,x,0)$, is divergence free, i.e. $\nabla \cdot \boldsymbol{U} = 0$.
(c) Explain how the vector field in (b) can be considered a field theoretical (Eulerian) description and find its relation to a material (Lagrangian) description. Hint: sketch the velocity vector at several points around the centre, define $\boldsymbol{x}(t) = a(\cos(\omega t), \sin(\omega t), 0)$ and find a relation between $\boldsymbol{U}(\boldsymbol{x}(t))$ and $\boldsymbol{x}'(t)$.
(d) **The material derivative.** Find a relation between $\boldsymbol{x}''(t)$ and $\frac{D\boldsymbol{U}}{Dt}$ in Exercise (c).
(e) **The geostrophic wind.** Denote the two dimensional Laplace operator by $\nabla^2 = {\partial_x}^2 + {\partial_y}^2$. Let f_0 be a constant. Define the geostrophic wind by

$$\boldsymbol{V}_g = \left(-\frac{1}{f_0}\partial_y \Phi\right)\boldsymbol{i} + \left(\frac{1}{f_0}\partial_x \Phi\right)\boldsymbol{j}.$$

Show that the geostrophic wind is divergence free:

$$\nabla \cdot \boldsymbol{V}_g = 0$$

but has non zero vorticity:

$$\boldsymbol{k} \cdot \nabla \times \boldsymbol{V}_g = \frac{1}{f_0}\nabla^2\Phi.$$

(f) Prove that $\boldsymbol{V}_g \perp \nabla\Phi$ and $\nabla\Phi$ is perpendicular to all the curves where $\Phi = constant$. What does this imply regarding the direction of $\boldsymbol{V}_g$ and surfaces where $\Phi = constant$?

Exercises 1.5.3.2. Gauss's theorem. Let $\boldsymbol{B} = (x_1, x_2, x_3)$. Let V be a sphere with radius a and surface A.

(a) Show that $\nabla \cdot \boldsymbol{B} = 3$.
(b) Since $\nabla \cdot \boldsymbol{B}$ is constant, **Gauss's theorem**

$$\iiint_V \nabla \cdot \boldsymbol{B}\, dV = \iint_A \boldsymbol{B} \cdot \boldsymbol{n}\, dA$$

follows trivially for this special case. Prove this statement.

Exercise 1.5.3.3. Gauss's theorem on a rectangular block. Consider the block $R = \{(x, y, z) | 0 < x < a, 0 < y < b, 0 < z < c\} = (0, a) \times (0, b) \times (0, c)$. Let A denote the surface of R and $\boldsymbol{n}$ the outward directed normal vector. Check each step and complete the omitted parts in the following:

$$\begin{aligned}
\iiint_R \nabla \cdot \boldsymbol{B}\, dV &= \iiint_R (\partial_x B_x + \partial_y B_y + \partial_z B_z)\, dx\, dy\, dz \\
&= \int_0^c (\int_0^b (\int_0^a \partial_x B_x\, dx)\, dy)\, dz + \cdots \\
&= \int_0^c (\int_0^b (B_x(a, y, z) - \cdots)\, dy)\, dz + \cdots \\
&= \int_0^c (\int_0^b B_x(a, y, z)\, dy)\, dz - \int_0^c (\int_0^b B_x(0, y, z)\, dy)\, dz + \cdots \\
&= \int_0^c (\int_0^b \boldsymbol{B}(a, y, z) \cdot \boldsymbol{n}(a, y, z)\, dy)\, dz \\
&\qquad - \int_0^c (\int_0^b \boldsymbol{B}(0, y, z) \cdot \boldsymbol{n}(0, y, z)\, dy)\, dz + \cdots \\
&= \iint_A \boldsymbol{B}(x, y, z) \cdot \boldsymbol{n}(x, y, z)\, dA.
\end{aligned}$$

1.5.4. Advection and the material derivative. Let ω be a constant angular velocity. To bring the rigid body fluid motion of Example 1.3.6 in line with the particle motion of Section 1.5.1, put $\theta(t) = \omega t$. We thus aim to relate the velocity field $\boldsymbol{V}$, as defined in Example 1.3.6, to the particle motion described by the following vector valued function $\boldsymbol{r}$ of a real variable t:

$$\boldsymbol{r} : t \mapsto \boldsymbol{r}(t) = (a \cos(\omega t),\ a \sin(\omega t)).$$

The velocity $\boldsymbol{V}(\boldsymbol{r}(t))$ experienced by the particle equals

$$\boldsymbol{V}(a \cos(\omega t),\ a \sin(\omega t)) = (-\omega a \sin(\omega t),\ \omega a \cos(\omega t)) = a\omega(-\sin(\omega t),\ \cos(\omega t)).$$

The acceleration can be obtained by differentiation, as follows:

$$\frac{d}{dt}[\boldsymbol{V}(\boldsymbol{r}(t))] = a\omega^2(-\cos(\omega t),\ -\sin(\omega t)) = -a\omega^2 \boldsymbol{e}_r.$$

Substituting $\boldsymbol{r}(t)$ in the place of $\boldsymbol{x}$ in Example 1.3.6, we get the velocity advection

$$((\boldsymbol{V}\cdot\nabla)\boldsymbol{V})(\boldsymbol{r}(t)) = -\omega^2\boldsymbol{r}(t) = -a\omega^2\boldsymbol{e}_r,$$

which equals the derivative, following the motion. Thus the material derivative equals the advection:

$$\frac{d}{dt}\left[\boldsymbol{V}(\boldsymbol{r}(t))\right] = ((\boldsymbol{V}\cdot\nabla)\boldsymbol{V})(\boldsymbol{r}(t)).$$

Note that in this example $\boldsymbol{V}$ is not explicitly a function of time t, so that $\frac{\partial V}{\partial t} = 0$ and the above result conforms to the definition of the material derivative,

$$\frac{d}{dt}\left[\boldsymbol{V}(\boldsymbol{r}(t))\right] = \frac{\partial V}{\partial t}(\boldsymbol{r}(t)) + ((\boldsymbol{V}\cdot\nabla)\boldsymbol{V})(\boldsymbol{r}(t)) = \frac{D\boldsymbol{V}}{Dt}(\boldsymbol{r}(t)),$$

given in Definition 1.3.5.

Exercise 1.5.4. A direct way of obtaining this result is to use the chain rule of differentiation, to give

$$\begin{aligned}\frac{d}{dt}\left[\boldsymbol{V}(r_1(t),\ r_2(t))\right] &= (\partial_1\boldsymbol{V})r_1{}'(t) + (\partial_2\boldsymbol{V})r_2{}'(t)\\ &= (\partial_1\boldsymbol{V})V_1 + (\partial_2\boldsymbol{V})V_2.\end{aligned}$$

Make the derivation complete by proving amongst others that $r_1{}'(t) = V_1$ and $r_2{}'(t) = V_2$.

1.5.5. Integration of knowledge. Develop your own integrated summary of relevant meteorological ideas, used as examples in Sections 1.4 an 1.5. Update your portfolio of successes, mistakes and evaluations.

1.6. Further vector algebra. Define unit vectors in R^n

$$\begin{aligned}\boldsymbol{e}_1 &= (1, 0, \cdots, 0)\\ \boldsymbol{e}_2 &= (0, 1, \cdots, 0)\\ &\ \vdots\\ \boldsymbol{e}_n &= (0, 0, \cdots, 1).\end{aligned}$$

Exercise 1.6. Show that $B = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n\}$ is a basis for R^n, by testing and interpretation of the following:

$$\begin{aligned}\boldsymbol{x} &= x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \cdots x_n\boldsymbol{e}_n\\ &= \Sigma_{i=1}^n x_i\boldsymbol{e}_i\\ &= \Sigma_{i=1}^n (\boldsymbol{x}\cdot\boldsymbol{e}_i)\boldsymbol{e}_i\\ &= (\boldsymbol{x}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1 + (\boldsymbol{x}\cdot\boldsymbol{e}_2)\boldsymbol{e}_2 + \cdots + (\boldsymbol{x}\cdot\boldsymbol{e}_n)\boldsymbol{e}_n.\end{aligned}$$

Compare this result with Exercise 1.5.1(c).

Remark 1.6. Let δ_{ij} denote the Kronecker delta,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j\\ 0, & \text{if } i \neq j.\end{cases}$$

Then $\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij}$ and B is an orthonormal basis.

1.6.1. Vector product (in R^3).

$$
\begin{aligned}
\boldsymbol{x} \times \boldsymbol{y} &= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\
&= \Sigma_{i=1}^3 \left((\boldsymbol{x} \times \boldsymbol{y}) \cdot \boldsymbol{e}_i \right) \boldsymbol{e}_i \\
&= \Sigma_{i=1}^3 \left(\Sigma_{j=1}^3 \Sigma_{k=1}^3 \epsilon_{ijk} x_j y_k \right) \boldsymbol{e}_i
\end{aligned}
$$

with the permutation symbol $\epsilon_{ijk} = 0$ if any two indices are equal, otherwise $|\epsilon_{ijk}| = 1$. A positive value, $\epsilon_{ijk} = 1$, is chosen if $(i, j, k) \in \{(1,2,3); (2,3,1); (3,1,2)\}$, else the value is -1.

1.6.2. The scalar triple product.
Exercise 1.6.2. Combine the definitions for the scalar and cross product to show that

$$\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} = \Sigma_{i=1}^3 \Sigma_{j=1}^3 \Sigma_{k=1}^3 \epsilon_{ijk} a_i b_j c_k.$$

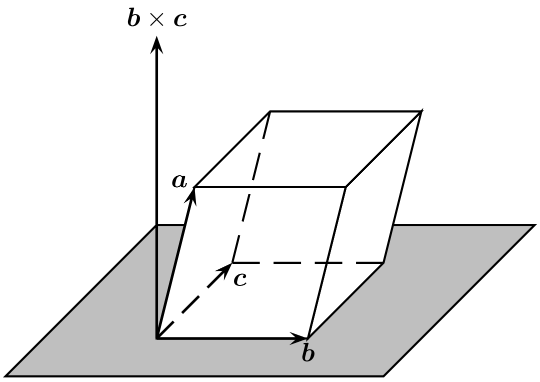

Figure 1.8

Show that the volume of the parallelepiped in the sketch is $\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}$. (See Apostol (1967).)

1.6.3. The vector triple product. Combine the definitions for the cross product to show that

$$
\begin{aligned}
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= \Sigma_{i=1}^3 \left(\Sigma_{j=1}^3 \Sigma_{k=1}^3 \epsilon_{ijk} a_j \Sigma_{l=1}^3 \Sigma_{m=1}^3 \epsilon_{klm} b_l c_m \right) \boldsymbol{e}_i \\
&= \Sigma_{i=1}^3 \left(\Sigma_{j=1}^3 \Sigma_{k=1}^3 \Sigma_{l=1}^3 \Sigma_{m=1}^3 \epsilon_{kij} \epsilon_{klm} a_j b_l c_m \right) \boldsymbol{e}_i \\
&= \Sigma_{i=1}^3 \left(\Sigma_{j=1}^3 \Sigma_{l=1}^3 \Sigma_{m=1}^3 (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj}) a_j b_l c_m \right) \boldsymbol{e}_i \\
&= \Sigma_{i=1}^3 \Sigma_{j=1}^3 \left((a_j c_j) b_i - (a_j b_j) c_i \right) \boldsymbol{e}_i \\
&= (\boldsymbol{a} \cdot \boldsymbol{c}) \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b}) \boldsymbol{c}.
\end{aligned}
$$

Exercise 1.6.3. Check each step in the previous derivation and make it more complete where necessary.

1.6.4. Integration of knowledge.

Exercise 1.6.4. Expand the differentiations in expressions like

$$\begin{aligned}
&\nabla \cdot \boldsymbol{a} \times \boldsymbol{b} \\
&\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) \\
&\nabla \cdot c\boldsymbol{a} \\
&\nabla \times c\boldsymbol{a}, \quad etc.
\end{aligned}$$

and compare with the literature.

Chapter 2

The momentum equation in rotating spherical co-ordinates

A true story about Coriolis. Persson (2000) writes: *The early nineteenth century was a time of change, with the Industrial Revolution in full swing and French industry lagging behind the British. A radical and patriotic movement developed within l'Ecole Polytechnique to promote technical development by educating workers, craftsmen and engineers in 'mechanique rationelle' to make them understand the functioning of machines in order to improve them. In his 1829 book* Calcul de l'effet des machines *Coriolis presented mechanics in a way that could be used by the industry. The book was also a milestone in the general development of physics since it established for the first time the correct relation between potential and kinetic energy, and showed that their sum remained constant in the absence of any external force.*

Persson (2000) also writes: *Meteorological textbooks go to great lengths to impress on the reader that the Coriolis force due to its inertial nature is 'fictitious', 'artificial', a 'pseudo force' or even a 'mental construct'. The centrifugal force, which of course is equally 'fictitious', is rarely talked about in this way. This might easily mislead an innocent reader into believing that some 'fictitious' forces are more fictitious than others.*

Introduction. The purpose of Chapter 2 is to obtain the acceleration terms in the momentum equations in the form (2.19) – (2.21) of Holton (1992), p. 37. The benefit to the student should be

a clear understanding of the movement of an air parcel in an inertial reference frame

- from the viewpoint of an observer on the rotating earth
- in spherical co-ordinates (as in Holton (1992) paragraph 2.3, p.33)

by the mathematical approach of direct differentiation of the position and velocity vectors in the inertial frame of reference.

2.1. The inertial reference frame. Suppose the movement of the earth around the sun contributes in a short time period to a constant velocity of an air parcel only—thus not contributing to the acceleration of the parcel. We thus choose our inertial frame of reference to be a frame moving with the earth around the sun, but not rotating. We choose the origin of this inertial frame to be at the centre of the earth and the x–y plane to intersect the earth at the equator. A point in the inertial frame will be denoted by a Cartesian 3-tuple, e.g.

$$\boldsymbol{x} = (x_1, x_2, x_3). \tag{2.1}$$

2.1.1. Spherical co-ordinates. The spherical co-ordinates, λ_a (longitude in inertial or *absolute* frame), ϕ (latitude) and r (radius) of the point $\boldsymbol{x}$ in (2.1) can be obtained from

$$\begin{aligned} \lambda_a &= \arctan(x_2/x_1) \\ r &= \sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2} \\ \phi &= \arcsin(x_3/r). \end{aligned}$$

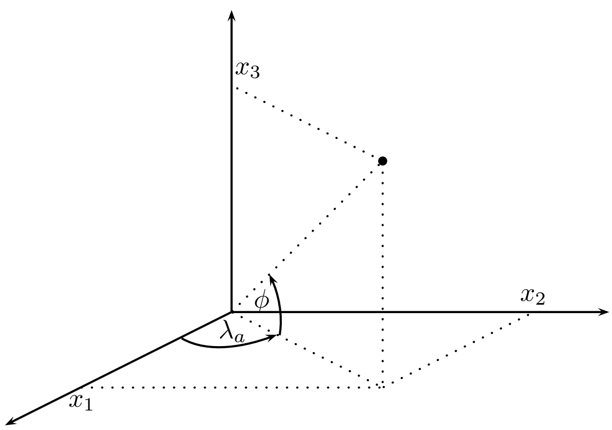

Figure 2.1

Exercises 2.1.1.
(a) Show that

$$\boldsymbol{x} = r(\cos\phi\cos\lambda_a, \cos\phi\sin\lambda_a, \sin\phi). \tag{2.2}$$

(b) Let $\boldsymbol{x} = (1.732, 1.732, 1.414)$. Calculate λ_a, ϕ, r.

2.2. The rotating frame of reference. Let λ denote the longitude of a parcel in the rotating frame. Then $\lambda_a = \Omega t + \lambda$ and (2.2) becomes

$$\boldsymbol{x} = r(\cos\phi\cos(\Omega t + \lambda), \cos\phi\sin(\Omega t + \lambda), \sin\phi). \tag{2.3}$$

2.2.1. Spherical co-ordinates in the rotating frame of reference

Exercises 2.2.1.

(a) Let t be fixed in equation (2.3) and discuss the dependency of $\boldsymbol{x}$ on λ (longitude), ϕ (latitude) and r (radius) in the rotating frame. Hint: consider big and small circles on the earth.

(b) Calculate $\frac{\partial \boldsymbol{x}}{\partial \lambda}$, $\frac{\partial \boldsymbol{x}}{\partial \phi}$ and $\frac{\partial \boldsymbol{x}}{\partial r}$.

(c) **Integration of knowledge.** Interpret and expand Exercise 1.1.3(c) in the light of new knowledge.

2.2.2. The motion of a particle (parcel of air) in the rotating frame. Let $t \mapsto \boldsymbol{r}(t)$ be a function describing the motion of the particle. Since its longitude, latitude and height z ($z = r - a$, $a =$ earth radius) depend on time we may write

$$\boldsymbol{r}(t) = r(t)(\cos\phi(t)\cos(\Omega t + \lambda(t)), \cos\phi(t)\sin(\Omega t + \lambda(t)), \sin\phi(t)) \tag{2.4}$$

or suppressing the variable t

$$\boldsymbol{r} = r(\cos\phi\cos\lambda_a, \cos\phi\sin\lambda_a, \sin\phi). \tag{2.5}$$

Exercises 2.2.2.

(a) Show that

$$\frac{d\boldsymbol{r}}{dt} = (r\cos\phi)(\Omega + \frac{d\lambda}{dt})\boldsymbol{i} + r\frac{d\phi}{dt}\boldsymbol{j} + \frac{dr}{dt}\boldsymbol{k} \tag{2.6}$$

where

$$\boldsymbol{i} = (-\sin\lambda_a, \cos\lambda_a, 0) \tag{2.7}$$

$$\boldsymbol{j} = (-\sin\phi\cos\lambda_a, -\sin\phi\sin\lambda_a, \cos\phi) \tag{2.8}$$

$$\boldsymbol{k} = (\cos\phi\cos\lambda_a, \cos\phi\sin\lambda_a, \sin\phi) \tag{2.9}$$

are orthogonal unit vectors rotating with the earth (see Exercise 1.1.3(c)). Note that $\boldsymbol{r} = r\boldsymbol{k}$.

(b) Define $\boldsymbol{\Omega} = (0, 0, \Omega)$. Let $\boldsymbol{r}$ be as in (2.5). Show that

$$\boldsymbol{\Omega} \times \boldsymbol{r} = (r\Omega\cos\phi)\boldsymbol{i}. \tag{2.10}$$

(c) Define

$$u = r\cos\phi\frac{d\lambda}{dt} \tag{2.11}$$

$$v = r\frac{d\phi}{dt} \tag{2.12}$$

$$w = \frac{dr}{dt} = \frac{dz}{dt}. \tag{2.13}$$

Show that (2.6) implies

$$\frac{d\boldsymbol{r}}{dt} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k} + \boldsymbol{\Omega} \times \boldsymbol{r}. \tag{2.14}$$

2.3. Differentiation of an arbitrary vector *A*. We prefer to combine the reasoning of Holton (1992) Sections 2.1.1 and 2.2. Thus we choose A_x, A_y and A_z to be the components of the vector $\boldsymbol{A}$ in the rotating frame of reference. We have

$$\boldsymbol{A} = A_x \boldsymbol{i} + A_y \boldsymbol{j} + A_z \boldsymbol{k}. \tag{2.15}$$

(Note that Holton (1992) uses the notation A'_x, A'_y and A'_z and $\boldsymbol{i}', \boldsymbol{j}'$ and $\boldsymbol{k}'$ instead.)
Exercises 2.3. Let λ, ϕ and r be functions of time t in (2.7) – (2.9). Let u, v and w be as in (2.11) – (2.13).
(a) Show that

$$\frac{d\boldsymbol{i}}{dt} = (\frac{u}{r}\tan\phi + \Omega\sin\phi)\boldsymbol{j} - (\frac{u}{r} + \Omega\cos\phi)\boldsymbol{k} \tag{2.16}$$

$$\frac{d\boldsymbol{j}}{dt} = -(\frac{u}{r}\tan\phi + \Omega\sin\phi)\boldsymbol{i} - \frac{v}{r}\boldsymbol{k} \tag{2.17}$$

$$\frac{d\boldsymbol{k}}{dt} = (\frac{u}{r} + \Omega\cos\phi)\boldsymbol{i} + \frac{v}{r}\boldsymbol{j}. \tag{2.18}$$

(b) Show that

$$\begin{aligned}\boldsymbol{\Omega} &= (\boldsymbol{\Omega}\cdot\boldsymbol{i})\boldsymbol{i} + (\boldsymbol{\Omega}\cdot\boldsymbol{j})\boldsymbol{j} + (\boldsymbol{\Omega}\cdot\boldsymbol{k})\boldsymbol{k} \\ &= (\Omega\cos\phi)\boldsymbol{j} + (\Omega\sin\phi)\boldsymbol{k}.\end{aligned} \tag{2.19}$$

(c) Define the *Coriolis* parameter f by $f = 2\Omega\sin\phi$. Show that $\boldsymbol{\Omega}$ can be written in terms of horizontal and vertical components as follows:

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_h + \frac{f}{2}\boldsymbol{k}.$$

Explain your result by calculating the components from a sketch.
(d) Show that

$$\boldsymbol{\Omega} \times \boldsymbol{i} = -(\Omega\cos\phi)\boldsymbol{k} + (\Omega\sin\phi)\boldsymbol{j} \tag{2.20}$$

$$\boldsymbol{\Omega} \times \boldsymbol{j} = -(\Omega\sin\phi)\boldsymbol{i} \tag{2.21}$$

$$\boldsymbol{\Omega} \times \boldsymbol{k} = (\Omega\cos\phi)\boldsymbol{i}. \tag{2.22}$$

(e) Use the results of (d) to show that (a) implies

$$\frac{d\boldsymbol{i}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{i} + (\frac{u}{r}\tan\phi)\boldsymbol{j} - \frac{u}{r}\boldsymbol{k} \tag{2.23}$$

$$\frac{d\boldsymbol{j}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{j} - (\frac{u}{r}\tan\phi)\boldsymbol{i} - \frac{v}{r}\boldsymbol{k} \tag{2.24}$$

$$\frac{d\boldsymbol{k}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{k} + \frac{u}{r}\boldsymbol{i} + \frac{v}{r}\boldsymbol{j}. \tag{2.25}$$

(f) Show that

$$\begin{aligned}\frac{d\boldsymbol{A}}{dt} = \frac{dA_x}{dt}\,\boldsymbol{i} + \frac{dA_y}{dt}\,\boldsymbol{j} + \frac{dA_z}{dt}\,\boldsymbol{k} + \boldsymbol{\Omega}\times\boldsymbol{A} \\ + \frac{1}{r}(-A_y u\tan\phi + A_z u)\boldsymbol{i} \\ + \frac{1}{r}(A_x u\tan\phi + A_z v)\boldsymbol{j} \\ + \frac{1}{r}(-A_x u - A_y v)\boldsymbol{k}.\end{aligned} \tag{2.26}$$

2.3.1. Notation. For convenience, but always keeping in mind the meaning of each term in (2.26) we define

$$\frac{D_a\boldsymbol{A}}{Dt} = \frac{d\boldsymbol{A}}{dt} \tag{2.27}$$

and

$$\begin{aligned}\frac{D\boldsymbol{A}}{Dt} = (\frac{dA_x}{dt} - \frac{uA_y\tan\phi}{r} + \frac{uA_z}{r})\boldsymbol{i} \\ + (\frac{dA_y}{dt} + \frac{uA_x\tan\phi}{r} + \frac{vA_z}{r})\boldsymbol{j} \\ + (\frac{dA_z}{dt} - \frac{uA_x + vA_y}{r})\boldsymbol{k}\end{aligned} \tag{2.28}$$

so that (2.26) becomes

$$\frac{D_a\boldsymbol{A}}{Dt} = \frac{D\boldsymbol{A}}{Dt} + \boldsymbol{\Omega}\times\boldsymbol{A}. \tag{2.29}$$

Exercise 2.3.1. Compare (2.28) with equation (2.14) in Holton (1992), Section 2.3, p. 36.

2.4. Velocity and acceleration.

2.4.1. The velocity vector.

Exercise 2.4.1. Apply (2.28) to the position vector $\boldsymbol{r}(t)$ to obtain

$$\frac{D\boldsymbol{r}}{Dt} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}. \tag{2.30}$$

Hint: Note that if $\boldsymbol{A} = \boldsymbol{r}$ then $A_x = 0, A_y = 0, A_z = r$ in (2.28). Note also (2.13) which is (2.9) in Holton (1992), p.33.
Definition 2.4.1. Define the (absolute) velocity vector as

$$\boldsymbol{U}_a = \frac{D_a\boldsymbol{r}}{Dt} \tag{2.31}$$

and the velocity vector as

$$\boldsymbol{U} = \frac{D\boldsymbol{r}}{Dt} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k} \tag{2.32}$$

so that (2.29) gives

$$\boldsymbol{U}_a = \frac{D_a\boldsymbol{r}}{Dt} = \boldsymbol{U} + \boldsymbol{\Omega} \times \boldsymbol{r}. \tag{2.33}$$

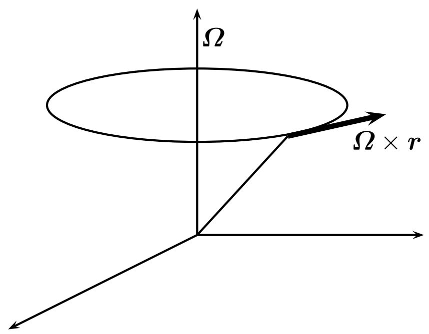

Figure 2.2

2.4.2. The acceleration vector. We now apply (2.29) to the absolute velocity vector $\boldsymbol{U}_a$ to obtain

$$\begin{aligned}
\frac{D_a\boldsymbol{U}_a}{Dt} &= \frac{D_a}{Dt}(\boldsymbol{U} + \boldsymbol{\Omega} \times \boldsymbol{r}) \\
&= \frac{D_a\boldsymbol{U}}{Dt} + \boldsymbol{\Omega} \times \frac{D_a\boldsymbol{r}}{Dt} \\
&= \frac{D\boldsymbol{U}}{Dt} + \boldsymbol{\Omega} \times \boldsymbol{U} + \boldsymbol{\Omega} \times (\boldsymbol{U} + \boldsymbol{\Omega} \times \boldsymbol{r}) \quad \text{by use of (2.29) and (2.33)} \\
&= \frac{D\boldsymbol{U}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{U} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}). \qquad (2.34)
\end{aligned}$$

Exercise 2.4.2.
(a) Use (2.28) to prove the following

$$\begin{aligned}
\frac{D\boldsymbol{U}}{Dt} &= (\frac{du}{dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r})\boldsymbol{i} \\
&+ (\frac{dv}{dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r})\boldsymbol{j} \\
&+ (\frac{dw}{dt} - \frac{u^2+v^2}{r})\boldsymbol{k}. \qquad (2.35)
\end{aligned}$$

(b) Compare (2.34) with Holton (1992) (2.7) and (2.35) with Holton (1992) (2.14).

(c) Use (2.19) and (2.32) to show that

$$\boldsymbol{\Omega} \times \boldsymbol{U} = (w\Omega\cos\phi - v\Omega\sin\phi)\boldsymbol{i} + (u\Omega\sin\phi)\boldsymbol{j} - (u\Omega\cos\phi)\boldsymbol{k}. \tag{2.36}$$

(d) Show that

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = (r\Omega^2\cos\phi\sin\phi)\boldsymbol{j} - (r\Omega^2\cos^2\phi)\boldsymbol{k}. \tag{2.37}$$

(e) Substitute (2.35) – (2.37) in (2.34) and write your result in the form (2.19) – (2.21), Holton (1992), p.37.

(f) In (2.34) we have used $\frac{D_a\boldsymbol{\Omega}}{Dt} = \mathbf{O}$. Prove $\frac{D\boldsymbol{\Omega}}{Dt} = \mathbf{O}$, using the differentiation rule (2.28) and the expression (2.19) for $\boldsymbol{\Omega}$. Next, prove $\frac{D_a\boldsymbol{\Omega}}{Dt} = \mathbf{O}$.

(g) Let f be the Coriolis parameter as in Exercise 2.3(c). Show that

$$2\boldsymbol{\Omega} \times \boldsymbol{U} = 2\Omega\cos\phi(w\boldsymbol{i} - u\boldsymbol{k}) + f\boldsymbol{k} \times \boldsymbol{V}, \tag{2.38}$$

with

$$\boldsymbol{V} = u\boldsymbol{i} + v\boldsymbol{j} \tag{2.39}$$

and

$$\boldsymbol{U} = \boldsymbol{V} + w\boldsymbol{k} \tag{2.40}$$

in accordance with Notation 1.5.1 for the two and three dimensional velocity field respectively.

Chapter 3

Balance laws in physics

Cauchy wrote: *The geometers who have investigated the equations of equilibrium or motion of thin plates or of surfaces, either elastic or inelastic, have distinguished two kinds of forces, the one produced by dilatation or contraction, the other by the bending of these surfaces ... It has seemed to me that these two kinds of forces could be reduced to a single one, which ought to be called always* tension *or* pressure, *a force which acts upon each element of a section at will, not only in a flexible surface but also in a solid, whether elastic or inelastic, and which is of the same kind as the hydrostatic pressure exerted by a fluid at rest upon the exterior surface of a body, except that the new pressure does not always remain perpendicular to the faces subject to it, nor is it the same in all directions at a given point.* Cauchy, On the pressure or tension in a solid body, *Exercises de Mathématiques, Seconde Année* (1827). (See Truesdell (1977), p. 118)

In many otherwise good textbooks a standing confusion reigns between three groups of forces: 1. Internal and external forces. 2. Volume and surface forces—a distinction which the mechanics of points is altogether incapable of perceiving. 3. Applied forces and forces of reaction. Hamel, On the foundations of Mechanics, *Mathematische Annalen* **66**, 350–397 (1909). (See Truesdell (1977), p. 118)

Introduction. The purpose of this chapter is to

- give a thorough background of the physics and mathematics involved in the laws of mass conservation, momentum balance and energy conservation
- show that these laws can be considered as special cases of a general or abstract balance law
- show how the constitutive equations make the general law applicable to a special situation and specific material
- transform differential expressions to isobaric (pressure-based) co-ordinates.

The reader, familiar with Holton's (1992) approach to conservation laws, may find it illuminating to know that these laws are treated from a fundamental point of view in various books on continuum mechanics, e.g. Spencer (1980) and Truesdell

(1977), as well as in other works, e.g. Serrin (1959) in Handbuch der Physik. This chapter is based on ideas from Fung (1969), Serrin (1959), Atkin & Fox (1980) and Spencer (1980). Themes that will not be discussed in this chapter are the conservation of angular momentum, resulting in the symmetry property of the stress tensor; the Cauchy formula ($\boldsymbol{T} = \tilde{\tau}\boldsymbol{n}$), giving the stress vector $\boldsymbol{T}$ in terms of the stress tensor $\tilde{\tau}$ and the orientation, specified by $\boldsymbol{n}$, of the surface on which the stress vector is desired; and the stress-strain-rate relationships (Fung, 1969). Instead, the interested reader is referred to the literature.

Note the symmetry between the Cauchy formula $\boldsymbol{T} = \tilde{\tau}\boldsymbol{n}$ and the result $\boldsymbol{F}'(\boldsymbol{x};\boldsymbol{n}) = D\boldsymbol{F}(\boldsymbol{x})\boldsymbol{n}$ of Exercise 1.3.6(c): the stress vector is equal to the product of a matrix and a unit vector, and similarly, the directional derivative is equal to the product of the Jacobian matrix and a unit vector.

Let ρ be a density field (function) and note that it has dimension $[\rho] = ML^{-3}$, where M denotes mass (kg) and L denotes length (m).

3.1. One dimensional derivation of the mass conservation principle. To prepare the reader for the three dimensional situation and to clearly convey the basic concepts, we first consider one dimensional flow in a pipe with constant cross section A. We model the situation by considering a density function $\rho : (x,t) \mapsto \rho(x,t), x \in [0,l], t \geq 0$. The interval $[a,b]$ will always be an arbitrary subinterval of $[0,l]$.

3.1.1. Eulerian approach. We will use the powerful tools of the Calculus to convert the balance of mass in an arbitrary interval $[a,b]$ from a statement in integral form to a differential equation. The mass in $[a,b]$ is $\int_a^b \rho(x,t)A\,dx$ and the time rate of change of this mass is $\frac{d}{dt}\int_a^b \rho(x,t)A\,dx$. The flow of mass (dimension MT^{-1}) is caused by inflow of material at a and outflow at b. If we denote the mass flux by ϕ, where $[\phi] = MT^{-1}L^{-2}$, we have the basic conservation law

$$\frac{d}{dt}\int_a^b \rho(x,t)A\,dx = A\phi(a,t) - A\phi(b,t). \tag{3.1}$$

By (3.1) an increase (or convergence) of mass in $[a,b]$ takes place if $\phi(a,t) > \phi(b,t)$, i.e. more material enters at a than leaves at b. Decrease (or divergence) takes place if $\phi(a,t) < \phi(b,t)$. Dividing (3.1) by A and using the fundamental theorem of the Calculus (1.2), we obtain

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = -\int_a^b \frac{\partial\phi}{\partial x}(x,t)\,dx.$$

Using (1.4) we get

$$\int_a^b \left[\frac{\partial\rho}{\partial t}(x,t) + \frac{\partial\phi}{\partial x}(x,t)\right] dx = 0 \text{ for every } [a,b] \subset [0,l].$$

Since $[a, b]$ is arbitrary, we may write the above balance law in the differential equation form

$$\frac{\partial \rho}{\partial t}(x,t) + \frac{\partial \phi}{\partial x}(x,t) = 0 \text{ for every } x \in [0,l]. \tag{3.2}$$

3.1.2. The constitutive equation. The mass flux is the amount of mass that flows out per unit time per unit area. A volume $A(v\delta t)$, having mass $\rho A(v\delta t)$ flows out during time δt. The constitutive equation for the mass flux is $\phi = \frac{\rho A(v\delta t)}{A(\delta t)} = \rho v$.

The continuity equation. Substituting $\phi = \rho v$ into (3.2) yields the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0. \tag{3.3}$$

3.1.3. Lagrangian approach. We still consider fluid motion in a one dimensional continuum. It is convenient to consider the fluid at any time instant t to be a mapping from a reference configuration to its current configuration (see Section 1.4.3). Thus, consider a mapping $r : (X,t) \mapsto r(X,t)$ of the reference interval $[0, L]$ to the configuration $[0, l]$ at time t. Let $[A, B]$ be an arbitrary interval in the reference configuration, so that $[r(A,t), r(B,t)]$ always contains the same material. In Section 1.4.3 such an interval was called a *material volume*. In (1.9) we defined the fluid velocity by

$$v(r(X,t),t) = \frac{\partial r}{\partial t}(X,t).$$

It follows that

$$v(x,t) = \frac{\partial r}{\partial t}(X,t) \tag{3.4}$$

with

$$x = r(X,t). \tag{3.5}$$

Consider the conservation of mass, expressed by

$$\frac{d}{dt}\int_{r(A,t)}^{r(B,t)} \rho(x,t)\,dx = 0. \tag{3.6}$$

Exercises 3.1.3

(a) Use (1.6) to write the mass conservation principle (3.6) as

$$\rho(r(B,t),t)\frac{\partial r}{\partial t}(B,t) - \rho(r(A,t),t)\frac{\partial r}{\partial t}(A,t) + \int_{r(A,t)}^{r(B,t)} \frac{\partial \rho}{\partial t}(x,t)\,dx = 0. \tag{3.7}$$

(b) Use (3.7) and the above definition of the velocity v to prove

$$\begin{aligned}\int_{r(A,t)}^{r(B,t)} &\frac{\partial}{\partial x}[\rho(x,t)v(x,t)]\,dx + \int_{r(A,t)}^{r(B,t)} \frac{\partial \rho}{\partial t}(x,t)\,dx \\ &= \rho(r(B,t),t)v(r(B,t),t) - \rho(r(A,t),t)v(r(A,t),t) + \int_{r(A,t)}^{r(B,t)} \frac{\partial \rho}{\partial t}(x,t)\,dx \\ &= 0.\end{aligned}$$

Show that this again implies (3.3).

3.2. Three dimensional derivation of the mass conservation principle.

3.2.1. Eulerian approach. In Section 3.1.1 the fundamental theorem of the Calculus (1.2) was used to write the net flux over the end points of the interval $[a, b]$ as an integral. In the three dimensional situation the divergence theorem (Gauss' divergence theorem)

$$\int_A \boldsymbol{B} \cdot \boldsymbol{n} \, dA = \int_V \nabla \cdot \boldsymbol{B} \, dV \tag{3.8}$$

serves a similar purpose. Let V be an arbitrary fixed volume in space. Let V be bounded by its surface A, having outward directed unit normal vector $\boldsymbol{n}$ at $\boldsymbol{x}$. (Note that $\boldsymbol{n} : \boldsymbol{x} \mapsto \boldsymbol{n}(\boldsymbol{x})$ changes in direction as $\boldsymbol{x}$ moves on the surface A of V.) Let ρ be the fluid density and $\boldsymbol{\Phi}$ the outward flux vector. The basic mass balance law is

$$\frac{d}{dt} \int_V \rho \, dV = - \int_A \boldsymbol{\Phi} \cdot \boldsymbol{n} \, dA. \tag{3.9}$$

Exercises 3.2.1. Assume interchanging the time derivative with integration, as in (1.4), also holds for three dimensional integrals:

$$\frac{d}{dt} \int_V f(\boldsymbol{x}, t) \, dV = \int_V \frac{\partial f}{\partial t}(\boldsymbol{x}, t) \, dV. \tag{3.10}$$

(a) Discuss the direction and sign of the flux $\boldsymbol{\Phi}$ and the integral of its normal component, using an adaptation of the arguments in Section 3.1.1.

(b) Use (3.8) and (3.10) to write the basic mass balance law (3.9) as

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{\Phi}\right) dV = 0$$

and motivate the ensuing balance law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{\Phi} = 0. \tag{3.11}$$

Compare (3.11) with (3.2).

(c) Let $\boldsymbol{U}$ be the fluid velocity field. Motivate the assumption $\boldsymbol{\Phi} \cdot \boldsymbol{n} = \rho \boldsymbol{U} \cdot \boldsymbol{n}$, by considering flow out of an area element δA during a time interval δt. Then, use the fact that $\boldsymbol{n}$ is arbitrary (it was associated with the orientation, at the point $\boldsymbol{x}$, of the surface area A of the arbitrary volume V) together with the result of Exercise 1.6 to prove that $\boldsymbol{\Phi} = \rho \boldsymbol{U}$.

3.2.2. The continuity equation. Substitution of $\boldsymbol{\Phi} = \rho \boldsymbol{U}$ in (3.11) gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{U}) = 0. \tag{3.12}$$

Exercise 3.2.2. Compare (3.12) with (3.3). Next, show that (3.12) implies

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{U} = 0, \tag{3.13}$$

where the material derivative $\frac{D}{Dt}$ is defined in Definition 1.3.5.

3.2.3. Lagrangian approach. We first give an alternative derivation for the one dimensional case of Section 3.1.3. Put $J(X,t) = \frac{\partial r}{\partial X}(X,t)$. By Exercise 1.4.3(c)

$$\begin{aligned}
\int_{r(A,t)}^{r(B,t)} \rho(x,t)\,dx &= \int_A^B \sigma(X)\,dX \\
&= \int_A^B \rho(r(X,t),t)\frac{\partial r}{\partial X}(X,t)\,dX \\
&= \int_A^B \rho(r(X,t),t)J(X,t)\,dX. \qquad (3.14)
\end{aligned}$$

Since time differentiation of (3.14) requires knowledge of $\frac{\partial}{\partial t}[\rho(r(X,t),t)]$ and $\frac{\partial J}{\partial t}(X,t)$, consider

$$\begin{aligned}
\frac{\partial}{\partial t}\Big[\rho(r(X,t),t)\Big] &= \frac{\partial \rho}{\partial t}(r(X,t),t) + \frac{\partial \rho}{\partial x}(r(X,t),t)\frac{\partial r}{\partial t}(X,t) \\
&= \frac{\partial \rho}{\partial t}(r(X,t),t) + \frac{\partial \rho}{\partial x}(r(X,t),t)v(r(X,t),t) \\
&= \frac{D\rho}{Dt}(r(X,t),t) \qquad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial J}{\partial t}(X,t) &= \frac{\partial}{\partial t}\left[\frac{\partial r}{\partial X}(X,t)\right] \\
&= \frac{\partial^2 r}{\partial X \partial t}(X,t) \\
&= \frac{\partial}{\partial X}\left[v(r(X,t),t)\right] \\
&= \frac{\partial v}{\partial x}(r(X,t),t)\frac{\partial r}{\partial X}(X,t) \\
&= \frac{\partial v}{\partial x}(r(X,t),t)J(X,t) \\
&= J(X,t)\frac{\partial v}{\partial x}(r(X,t),t). \qquad (3.16)
\end{aligned}$$

Thus, differentiation of (3.14) yields

$$\begin{aligned}
\frac{d}{dt}\int_{r(A,t)}^{r(B,t)} \rho(x,t)\,dx &= \frac{d}{dt}\int_A^B \rho(r(X,t),t)J(X,t)\,dX \\
&= \int_A^B \frac{\partial}{\partial t}\Big[\rho(r(X,t),t)J(X,t)\Big]\,dX
\end{aligned}$$

$$
\begin{aligned}
&= \int_A^B \left[J(X,t)\frac{\partial}{\partial t}\Big\{ \rho(r(X,t),t)\Big\} + \rho(r(X,t),t)\frac{\partial J}{\partial t}(X,t))\right] dX \\
&= \int_A^B \left[J(X,t)\frac{D\rho}{Dt}(r(X,t),t) + \rho(r(X,t),t)J(X,t)\frac{\partial v}{\partial x}(r(X,t),t)\right] dX \\
&= \int_A^B J(X,t) \left[\frac{D\rho}{Dt}(r(X,t),t) + \rho(r(X,t),t)\frac{\partial v}{\partial x}(r(X,t),t)\right] dX \\
&= \int_{r(A,t)}^{r(B,t)} \left(\frac{D\rho}{Dt}(x,t) + \rho(x,t)\frac{\partial v}{\partial x}(x,t)\right) dx. \qquad (3.17)
\end{aligned}
$$

We are now ready to proceed with the three dimensional version of (3.16) and (3.17). Let $\boldsymbol{f} : \boldsymbol{x} \in V_0 \mapsto \boldsymbol{f}(\boldsymbol{x}) \in V$ be a one-to-one and continuously differentiable vector mapping from V_0 to V. Then $D\boldsymbol{f}(\boldsymbol{x})$ is the Jacobian matrix defined in Exercise 1.3.6(c). Let J be the *Jacobian determinant*: $J(\boldsymbol{x}) = \det D\boldsymbol{f}(\boldsymbol{x})$. The following three dimensional version of (1.7) is adapted from p. 408 in Apostol (1969):

$$\int_V g(\boldsymbol{u})\, d\boldsymbol{u} = \int_{V_0} g(\boldsymbol{f}(\boldsymbol{x}))|J(\boldsymbol{x})|\, d\boldsymbol{x}. \qquad (3.18)$$

Let $\chi : (\boldsymbol{X},t) \mapsto \chi(\boldsymbol{X},t)$ be a mapping from the three dimensional reference configuration $\mathfrak{C}$ into the current configuration $\mathfrak{C}_t$. We choose $\boldsymbol{f}$ in (3.18) to be the function $\chi(\cdot,t) : \boldsymbol{X} \mapsto \chi(\boldsymbol{X},t)$, with t kept fixed. Thus $\boldsymbol{f}(\boldsymbol{X}) = \chi(\boldsymbol{X},t)$, for every $\boldsymbol{X} \in \mathfrak{C}$. The derivative of the Jacobian determinant is proved by Serrin (1959) and Atkin & Fox (1980) to satisfy

$$\frac{DJ}{Dt} = J\nabla\cdot\boldsymbol{U}. \qquad (3.19)$$

In continuum mechanics $J > 0$, since a body can not penetrate itself (Atkin & Fox (1980)).

Exercise 3.2.3.

(a) Compare (3.16) with (3.19) and compare (1.7) with (3.18).

(b) Let $V(t) = \chi(\mathfrak{C},t)$ be the material volume obtained when the reference configuration $\mathfrak{C}$ is mapped by χ to its current state. Use (3.18) to transform $\int_{V(t)} \rho(\boldsymbol{u},t)\, d\boldsymbol{u}$ to an integral over the reference state $\mathfrak{C}$. Differentiate your result, using (3.10) to differentiate the integral over the reference state $\mathfrak{C}$. Use (3.15) and (3.19) to obtain

$$\int_{\mathfrak{C}} (\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{U})J\, dV_0 = 0, \qquad (3.20)$$

and thus

$$\int_{V(t)} (\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{U})\, dV = 0, \qquad (3.21)$$

so that

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{U} = 0,$$

which again proves (3.13).

3.3. Abstract balance law.

3.3.1. Eulerian derivation.

(a) **One dimensional case.** Let $f : (x,t) \mapsto f(x,t)$ be any quantity that is conserved or for which a balance law is sought. Let F be the external forcing (or sources) per unit length and ϕ the flux of f. The balance of quantities in the one dimensional interval $(0,l)$ is

$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \phi(a,t) - \phi(b,t) + \int_a^b F(x,t)\,dx \text{ for every } (a,b) \subset (0,l). \tag{3.22}$$

The abstract balance law is described by the following partial differential equation:

$$\frac{\partial f}{\partial t}(x,t) + \frac{\partial \phi}{\partial x}(x,t) = F \text{ for every } x \in (0,l) \text{ and } t > 0. \tag{3.23}$$

(b) **Three dimensional case.** Let $f : (\boldsymbol{x},t) \mapsto f(\boldsymbol{x},t)$ be a field variable that is conserved (or for which a balance law is sought) in a region $G \subset R^3$ and $\boldsymbol{\Phi} : (\boldsymbol{x},t) \mapsto \boldsymbol{\Phi}(\boldsymbol{x},t)$ be the flux of f. Let F be the external forcing or sources per unit volume and $\boldsymbol{S}$ the external forcing or sources per unit area. Let V be any arbitrary subset of G. The integral form of the balance law for f in the three dimensional region V is

$$\frac{d}{dt}\int_V f(\boldsymbol{x},t)\,dV = -\int_A \boldsymbol{\Phi}(x,t)\cdot\boldsymbol{n}\,dA + \int_V F(\boldsymbol{x},t)\,dV + \int_A \boldsymbol{S}\cdot\boldsymbol{n}\,dA \tag{3.24}$$

for every $V \subset G$

and implies

$$\frac{\partial f}{\partial t} + \nabla\cdot\boldsymbol{\Phi} = F + \nabla\cdot\boldsymbol{S}. \tag{3.25}$$

Exercises 3.3.1.

(a) **The heat equation.** Let $f/\rho = e = c_v T$ be the internal energy per unit mass (Holton (1992), Section 2.6, p. 51), $\boldsymbol{\Phi} = -\kappa\nabla T$, the heat flux (Example 1.3.3 and Exercise 1.3.4(d)) and put $k = \frac{\kappa}{c_v\rho}$. Let $F = 0$ and $\boldsymbol{S} = \mathbf{O}$. Show that (3.25) implies

$$\frac{\partial T}{\partial t} = k\nabla^2 T.$$

What is the one dimensional version (spatial) of this equation? Use (3.22) and (3.23).

(b) **The momentum equations.** Let $f = \rho U_i$ be the momentum per unit volume in the i-th direction and $\boldsymbol{\Phi} = f\boldsymbol{U}$ the momentum flux. Let $g_i = \boldsymbol{g}\cdot\boldsymbol{e}_i$ be the component of gravitational acceleration in the i-th direction. Put $F_i = \rho(g_i - 2\boldsymbol{\Omega}\times\boldsymbol{U}\cdot\boldsymbol{e}_i)$ and $\boldsymbol{S} = -p\boldsymbol{e}_i$. Let V be any arbitrary set and A its boundary. Show how the following *momentum balance* law can be considered to be a generalization for fluid motion based on Newton's second law:

$$\frac{d}{dt}\int_V f(\boldsymbol{x},t)\,dV = -\int_A \boldsymbol{\Phi}(x,t)\cdot\boldsymbol{n}\,dA + \int_V \rho(g_i - 2\boldsymbol{\Omega}\times\boldsymbol{U}\cdot\boldsymbol{e}_i)\,dV - \int_A p\boldsymbol{e}_i\cdot\boldsymbol{n}\,dA.$$

Show that the balance law implies

$$\frac{\partial(\rho U_i)}{\partial t} + \frac{\partial p}{\partial x_i} + \nabla \cdot (\rho U_i \boldsymbol{U}) - \rho(g_i - 2\boldsymbol{\Omega} \times \boldsymbol{U} \cdot \boldsymbol{e}_i) = 0.$$

Next, show that a combination with the continuity equation results in

$$\frac{DU_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + (\boldsymbol{g} - 2\boldsymbol{\Omega} \times \boldsymbol{U}) \cdot \boldsymbol{e}_i$$

and consequently

$$\frac{D\boldsymbol{U}}{Dt} = -\frac{1}{\rho}\nabla p + \boldsymbol{g} - 2\boldsymbol{\Omega} \times \boldsymbol{U}.$$

3.3.2. Lagrangian derivation of the thermodynamic equation. Choose f in (3.24) to be the total thermodynamic energy per unit volume, $\rho[e + \frac{1}{2}\|\boldsymbol{U}\|^2]$. Let $\boldsymbol{\Phi} = \mathbf{O}$ and let $\boldsymbol{S} = -p\boldsymbol{U}$, with $-\boldsymbol{S}$ the pressure flux. Let $\boldsymbol{g} \cdot \boldsymbol{U}$ be the rate at which body forces do work per unit mass, and let J denote the rate of heating per unit mass owing to radiation, conduction, and latent heat release and combine these two contributions into $F = \rho(\boldsymbol{g} \cdot \boldsymbol{U} + J)$.
Exercise 3.3.2. Formulate a three dimensional version of (1.8) and adapt (3.24) for the material description to show that

$$\rho(\frac{De}{Dt} + \frac{D}{Dt}\frac{1}{2}\|\boldsymbol{U}\|^2) = -\nabla \cdot (p\boldsymbol{U}) + \rho(\boldsymbol{g} \cdot \boldsymbol{U} + J).$$

Note that the same symbol J is also used for the totally different concept of the Jacobian determinant in Section 3.2.3.

3.4. Transformation of conservation equations. The pressure/height interdependency was already discussed in Section 1.2. The main result in this section is the transformation of the continuity equation (3.13) from $(\boldsymbol{x}, z)$ co-ordinates to the so called *isobaric* co-ordinates $(\boldsymbol{x}, p)$. The transformed equation, (3.50) does not contain density ρ and does not involve time derivatives. Holton (1992) sees the simplicity of (3.50) as one of the main advantages of the isobaric co-ordinate system. In Chapter 4 density is also removed from the momentum equations, after introducing the geopotential (Exercise 1.2(c)) and transforming to isobaric co-ordinates.

3.4.1. A choice: atmospheric pressure *or* height as independent variable. First consider atmospheric pressure, restricted to a vertical plane. Choose the x–z co-ordinate surface to coincide with this plane. Thus for each point (x, z) on the vertical plane there is defined a pressure $p(x, z)$. Suppose p is constant on a curve C in this plane and that C can be described by a function $z : x \mapsto z(x)$. In Holton (1992), p. 22 a part of z is sketched where it is assumed to be a linear increasing function. Along the curve C we have

$$p(x, z(x)) = constant \tag{3.26}$$

$$\frac{d}{dx}[p(x, z(x))] = 0 \tag{3.27}$$

$$\frac{\partial p}{\partial x}(x, z(x)) + \frac{\partial p}{\partial z}(x, z(x))z'(x) = 0. \tag{3.28}$$

Exercise 3.4.1. Compare (3.28) with

$$\left(\frac{\partial p}{\partial x}\right)_z + \left(\frac{\partial p}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_p = 0 \tag{3.29}$$

which is given on p. 22 of Holton (1992) and note carefully the subscripts z, x and p. Look how (3.29) is derived in Holton (1992) and note the curves along which z, x or p are being kept fixed.

Three dimensional situation. Next, consider a given point $\boldsymbol{x} = (x_1, x_2)$ on a horizontal plane. Exercise 1.2(b) suggests the following possibilities:

- *Pressure as a function of three independent variables:* during a vertical sounding a pressure, say p_1, is measured at height z_1, giving

$$p(\boldsymbol{x}, z_1) = p_1. \tag{3.30}$$

- *Height as a function of three independent variables:* while the vertical sounding, implied by (3.30), gives pressure p_1 at height z_1, it can equivalently be said that the height z_1 is reached when the pressure is p_1. Thus

$$z(\boldsymbol{x}, p_1) = z_1. \tag{3.31}$$

3.4.2. Pressure and height as inverses.

Example 3.4.2. A function notation for height and pressure. A combination of (3.30) – (3.31) necessitates the definition of two symbols for pressure and two symbols for height: the pressure p at a height z and the functions $\tilde{p}$ and $\tilde{z}$ describing their interdependency. Thus (3.30) and (3.31) can be written as

$$\tilde{p}(\boldsymbol{x}, z) = p \tag{3.32}$$

$$\tilde{z}(\boldsymbol{x}, p) = z. \tag{3.33}$$

Exercises 3.4.2.

(a) Substitute (3.33) into (3.32), (thereby eliminating z) and write your result down here, as equation (3.34):

$$\tilde{p}(\boldsymbol{x}, \ldots\ldots\ldots\ldots) = p \tag{3.34}$$

Differentiate your result (3.34) with respect to x_1. Next, differentiate your (3.34) with respect to x_2. Combine the result in vector form as

$$\nabla_z \tilde{p} + \left(\frac{\partial \tilde{p}}{\partial z}\right)_{\boldsymbol{x}} \nabla_p \tilde{z} = 0. \tag{3.35}$$

(b) Differentiate your equation (3.34) with respect to p to give

$$\frac{\partial \tilde{p}}{\partial z}(\boldsymbol{x}, \tilde{z}(\boldsymbol{x}, p)) \frac{\partial \tilde{z}}{\partial p}(\boldsymbol{x}, p) = 1. \tag{3.36}$$

3.4.3. Isobaric Coordinates. For simplicity we only distinguish between $\tilde{p}$ and p, $\tilde{z}$ and z when necessary. Suppose the pressure p serves the role of independent vertical co-ordinate better than the vertical height z does. A function $f : (\boldsymbol{x}, z, t) \mapsto f(\boldsymbol{x}, z, t)$ can be transformed into a function $F : (\boldsymbol{x}, p, t) \mapsto F(\boldsymbol{x}, p, t)$ by using

$$f(\boldsymbol{x}, z, t) = F(\boldsymbol{x}, p(\boldsymbol{x}, z, t), t). \tag{3.37}$$

Examples 3.4.3.

(a) **Wind as a function of $\boldsymbol{x}$ and p.** Let $\boldsymbol{v} : (\boldsymbol{x}, z, t) \mapsto \boldsymbol{v}(\boldsymbol{x}, z, t)$ and $\boldsymbol{V} : (\boldsymbol{x}, p, t) \mapsto \boldsymbol{V}(\boldsymbol{x}, p, t)$ be field functions, respectively defined on $(\boldsymbol{x}, z)$ and $(\boldsymbol{x}, p)$ co-ordinates, describing the horizontal wind. The transformation (3.37) gives

$$\boldsymbol{v}(\boldsymbol{x}, z, t) = \boldsymbol{V}(\boldsymbol{x}, p(\boldsymbol{x}, z, t), t). \tag{3.38}$$

(b) **The equation of state for dry air.** If z is chosen as vertical coordinate we may write

$$\tilde{p}(\boldsymbol{x}, z) = \tilde{\rho}(\boldsymbol{x}, z) R \tilde{T}(\boldsymbol{x}, z).$$

If z is replaced by the left hand side of (3.33) then

$$\tilde{p}(\boldsymbol{x}, \tilde{z}(\boldsymbol{x}, p)) = \tilde{\rho}(\boldsymbol{x}, \tilde{z}(\boldsymbol{x}, p)) R \tilde{T}(\boldsymbol{x}, \tilde{z}(\boldsymbol{x}, p)),$$

which implies

$$p = \rho(\boldsymbol{x}, p) R T(\boldsymbol{x}, p).$$

A barotropic atmosphere is one in which the density depends only on the pressure, $\rho : p \mapsto \rho(p)$, so that the equation of state becomes

$$p = \rho(p) R T(\boldsymbol{x}, p).$$

Exercises 3.4.3.

(a) Prove that $\nabla_p T = 0$ in a barotropic atmosphere.

(b) Prove the following by direct differentiation of (3.37):

$$\frac{\partial f}{\partial t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial t} \tag{3.39}$$

$$\nabla_z f = \nabla_p F + \frac{\partial F}{\partial p} \nabla_z p \tag{3.40}$$

$$\frac{\partial f}{\partial z} = \frac{\partial F}{\partial p}\frac{\partial p}{\partial z}. \tag{3.41}$$

(c) **The material derivative in isobaric coordinates.** Let $w = \frac{Dz}{Dt}$ and $\omega = \frac{Dp}{Dt}$. Study the following carefully and compare with equation (3.3) on p. 59 of Holton (1992):

$$\begin{aligned}
\frac{Df}{Dt} &= (\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla_z + w\frac{\partial}{\partial z})f \\
&= (\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla_p)F + \frac{\partial F}{\partial p}(\frac{\partial p}{\partial t} + \boldsymbol{V}\cdot\nabla_z p + w\frac{\partial p}{\partial z}) \\
&= (\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla_p)F + \frac{\partial F}{\partial p}\frac{Dp}{Dt} \\
&= (\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla_p + \omega\frac{\partial}{\partial p})F. \qquad (3.42)
\end{aligned}$$

(d) **The w—ω relation**. In Section 3.5 of Holton (1992) the hydrostatic approximation is used together with scaling considerations to show that $\frac{Dp}{Dt} \approx -\rho g w$. The hypsometric equation, Holton (1992), p. 20,

$$Z - Z_0 = \frac{R < T >}{g_0}\ln\left(\frac{p_0}{p}\right),$$

with $< T >$ the layer mean temperature to be defined in (4.9), is also derived using the hydrostatic approximation. Assume p_0 and $\frac{R<T>}{g_0} = H$ are constant and that $p = \rho R < T >$. Prove by differentiation with respect to time t that

$$\omega = -\rho g_0 w.$$

3.4.4. Transformation of the Continuity equation. Holton (1992) mentions that the isobaric form of the continuity equation can be obtain by a transformation from height co-ordinates to pressure co-ordinates. He then chooses to do the simpler direct derivation, instead. The learner is now challenged: put away the Workbook and try to do the transformation yourself! However, if you need help, the following exercises may put you back on track. The equations we need are

$$\nabla_z p + \left(\frac{\partial p}{\partial z}\right)_{\boldsymbol{x}} \nabla_p z = 0 \qquad (3.43)$$

$$\frac{\partial p}{\partial z} = -\rho g \qquad (3.44)$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla_z\cdot\boldsymbol{v} + \frac{\partial w}{\partial z} = 0 \qquad (3.45)$$

and

$$\boldsymbol{v}(\boldsymbol{x}, \tilde{z}(\boldsymbol{x}, p, t), t) = \boldsymbol{V}(\boldsymbol{x}, p, t). \qquad (3.46)$$

Exercises 3.4.4.

(a) Replace z in (3.38) with $\tilde{z}(\boldsymbol{x}, p)$, and use (3.34) to prove (3.46).

(b) Let g be constant as indicated in the footnote on p. 60 of Holton (1992). Complete the following argument and check every step. Write out in more detail if necessary.

$$
\begin{aligned}
-g\frac{D\rho}{Dt} &= \frac{D}{Dt}\frac{\partial p}{\partial z} \\
&= (\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla_z + w\frac{\partial}{\partial z})\frac{\partial p}{\partial z} \\
&= \frac{\partial}{\partial z}(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla_z + w\frac{\partial}{\partial z})p - \cdots\cdots \\
&= \frac{\partial}{\partial z}\frac{Dp}{Dt} + \frac{\partial \boldsymbol{v}}{\partial z}\cdot\frac{\partial p}{\partial z}\nabla_p z - \cdots \\
&= \frac{\partial \omega}{\partial z} + \frac{\partial p}{\partial z}(\frac{\partial \boldsymbol{v}}{\partial z}\cdot\nabla_p z - \frac{\partial w}{\partial z})
\end{aligned}
$$

(c) Use (3.44) and (3.45) to rewrite the result of Exercise (b) as

$$\frac{\partial \omega}{\partial z} - \rho g(\nabla_z\cdot\boldsymbol{v} + \frac{\partial \boldsymbol{v}}{\partial z}\cdot\nabla_p z) = 0. \tag{3.47}$$

(d) Differentiate (3.46) with respect to x_i to obtain

$$\partial_i\boldsymbol{v} + \frac{\partial \boldsymbol{v}}{\partial z}\partial_i z = \partial_i\boldsymbol{V}, \quad i = 1 \text{ or } 2. \tag{3.48}$$

(e) Use $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{V} = (V_1, V_2)$ to write (3.48) in component form. By choosing appropriate values for i in the two resulting equations and adding, show that

$$\nabla_z\cdot\boldsymbol{v} + \frac{\partial \boldsymbol{v}}{\partial z}\cdot\nabla_p z = \nabla_p\cdot\boldsymbol{V}. \tag{3.49}$$

(f) Substitute (3.49) together with

$$\frac{\partial \omega}{\partial z} = \frac{\partial \omega}{\partial p}\frac{\partial p}{\partial z} = -\rho g\frac{\partial \omega}{\partial p}$$

in (3.47) to obtain the transformed continuity equation

$$\frac{\partial \omega}{\partial p} + \nabla_p\cdot\boldsymbol{V} = 0. \tag{3.50}$$

3.4.5. A Generalized Vertical Coordinate. Consider the transformations

$$f(\boldsymbol{x}, z, t) = F(\boldsymbol{x}, s(\boldsymbol{x}, z, t), t) \tag{3.51}$$

$$f(\boldsymbol{x}, z(\boldsymbol{x}, s, t), t) = F(\boldsymbol{x}, s, t), \tag{3.52}$$

where s is a single-valued monotonic function of height.

Exercise 3.4.5. Prove the following and compare with p. 23 in Holton (1992):

$$\nabla_z f = \nabla_s F + \frac{\partial F}{\partial s}\nabla_z s$$

$$\nabla_z f + \frac{\partial f}{\partial z}\nabla_s z = \nabla_s F.$$

Chapter 4.

The quasi-geostrophic approximation

Gone with the wind. Zbigniew Sorbjan (1996) relates the following frightening incidents in his book: *In 1930, a deadly demonstration of the hail formation process was provided by five glider pilots in Germany. The pilots soared into a thunderstorm in the region of the Rhön Mountains, and were trapped in the cloud's powerful drafts. After losing control of their planes, all five parachuted out. Powerful updrafts immediately lifted them to regions of subfreezing temperatures. Then they fell down with the downdraft, only to be raised up and frozen again. After a series of lifts and falls they literally became human hailstones. When they finally reached the Earth, they were all frozen stiff an only one survived. Similarly in 1982, an Australian parachutist was trapped for half an hour in a cumulonimbus cloud. When he opened his parachute, he was lifted by an updraft from a height of 2 km to 4 km. The parachutist rescued himself by cutting out from his main parachute. As a result of free-falling, he was able to move through the cloud. About 500 m above the surface, he opened his reserve parachute and then landed safely on the ground.*

Tropical cumulonimbus in the western Pacific.
Photo obtained from http://www.photolib.noaa.gov.

Introduction. The purpose of this chapter is to briefly show how the conservation laws for momentum, mass and thermodynamic energy can be combined and simplified to become a vehicle for qualitative understanding of the movement and development of atmospheric systems. Under certain restrictions the horizontal momentum equation and the thermodynamic energy equation may be combined to give an equation describing the isobaric vertical velocity ω (the omega equation). Another equation, describing the geopotential tendency $\frac{\partial \phi}{\partial t}$, can be obtained similarly. The omega equation can be rewritten in such a form that the forcing of the vertical motion is expressed in terms of the divergence of a horizontal vector forcing field (the $\boldsymbol{Q}$-vector formulation). The geopotential tendency equation can be rewritten in conservation form (the quasi-geostrophic potential vorticity equation). For a qualitative description of how these equations may be used to explain the development and movement of weather systems under the quasi-geostrophic approximations, see Holton (1992).

After introduction of the concepts of the *geopotential* (Section 4.1) and the *thickness* of an atmospheric layer (Section 4.2), it is shown how atmospheric density can be eliminated from the pressure gradient force (Section 4.3), by use of the isobaric co-ordinate system (see Section 3.4 for a motivation and a discussion of the transformation of conservation equations). The approximate balance on synoptic scale between the horizontal pressure gradient force and the horizontal part of the Coriolis term leads to the definition of the geostrophic wind (Section 4.4) (see also Exercise 1.5.3.1(e)). Following naturally is the definition of the ageostrophic wind (Section 4.5) as the difference between the horizontal velocity and the geostrophic wind. Next, assuming quasi-geostrophic approximations, the vorticity equation (Exercise 4.6(a)), the thermodynamic energy equation (Exercise 4.6(b)), the omega equation (Exercise 4.6(c)) and the tendency equation (Exercise 4.6(d)) follow. In Section 4.7 the Q-vector formulation of the omega equation is derived in such a way that it can clearly and easily be seen that purely geostrophic motion (Section 4.7.3) will tend to destroy the thermal wind relationship. We close (Section 4.8) by showing how the quasi-geostrophic potential vorticity equation can be used to link vertical motion with vorticity.

For the sake of simplicity, all the derivations in this chapter are done in Cartesian co-ordinates, in accordance with the notation in Section 1.3.4. The more advanced learners may look at Burger & Riphagen (1990) for an exposition of the basic equations in more general co-ordinates.

4.1. The geopotential. In Exercise 1.2(c) we defined the geopotential ϕ by

$$\phi(z) = \int_0^z g\, dz', \tag{4.1}$$

which conforms to Holton's (1992) equation (1.8). Here we emphasize that ϕ is a function of z only and that z' is a dummy integration variable. By direct differentiation of (4.1), we obtain $\frac{d\phi}{dz} = g$ and a combination with the hydrostatic equation $\frac{\partial p}{\partial z} = -\rho g$ gives

$$\frac{d\phi}{dz} = g = -\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{RT}{p}\frac{\partial p}{\partial z}. \tag{4.2}$$

Integration of (4.2) gives

$$\phi(z_2) - \phi(z_1) = -\int_{p_1}^{p_2} \frac{1}{\rho}\,dp = -\int_{p_1}^{p_2} \frac{RT}{p}\,dp, \tag{4.3}$$

where

$$p_1 = p(z_1) \text{ and } p_2 = p(z_2). \tag{4.4}$$

Exercise 4.1. Make the derivation mathematically more neat by meticulously using the transformation rule (1.7) to show that

$$\int_{z_1}^{z_2} \frac{1}{\rho(p(z))}\frac{\partial p}{\partial z}(z)\,dz = \int_{p(z_1)}^{p(z_2)} \frac{1}{\rho(p)}\,dp \tag{4.5}$$

and that (4.3) follows.

Note that the dependency of p on $\boldsymbol{x}$ is suppressed in (4.3) – (4.5). Also note that Holton (1992) uses the expression $d\ln p = \frac{1}{p}\,dp$.

4.2. The thickness of an atmospheric layer. As in Holton (1992), p. 20, we define

$$\left.\begin{aligned} Z &= \phi(z)/g_0 \\ Z_1 &= \phi(z_1)/g_0 \\ Z_2 &= \phi(z_2)/g_0 \end{aligned}\right\}. \tag{4.6}$$

The thickness of the atmospheric layer between the pressure surfaces p_2 and p_1, is given by

$$Z_T = Z_2 - Z_1 = \frac{R}{g_0}\int_{p_2}^{p_1} \frac{T}{p}\,dp \tag{4.7}$$

$$= \frac{R}{g_0} < T > \ln\frac{p_1}{p_2} \tag{4.8}$$

where the mean temperature $< T >$ is defined as

$$< T >= \frac{\int_{p_2}^{p_1} T\,d\ln p}{\int_{p_2}^{p_1} d\ln p} \tag{4.9}$$

and $p_1 > p_2$.

Exercise 4.2. Motivate (4.9) by showing that $< T >= T_1$ if a constant temperature $T = T_1$ is assumed. Apply integration of knowledge to enhance your understanding of averaging and also revise Exercise 3.4.3(d).

4.3. The geopotential in isobaric co-ordinates. The right hand side of (4.3) leads us to define

$$\Phi(\boldsymbol{x},p) = R\int_p^{p(\boldsymbol{x},0)} \frac{T(\boldsymbol{x},p')}{p'}\,dp'. \tag{4.10}$$

Note that by (4.3) and (4.5)

$$\phi(z_2) - \phi(0) = -R\int_{p(\boldsymbol{x},0)}^{p(\boldsymbol{x},z_2)} \frac{T(\boldsymbol{x},p')}{p'}\,dp' = \Phi(\boldsymbol{x},p(z_2)).$$

so that

$$\phi(z) = \Phi(\boldsymbol{x},p(z)) \text{ for every } z. \tag{4.11}$$

A vector form of the momentum equations is given in Exercise 3.3.1(b) and in component form they are given in equations (2.19) – (2.21) of Holton (1992). Exercises 4.3 will now show that the pressure gradient force $\frac{1}{\rho}\nabla_z p$ may be replaced by the geopotential gradient, $\nabla_p\Phi$, which does not contain the atmospheric density ρ.

Exercises 4.3.

(a) Prove that

$$\begin{aligned}\frac{\partial\Phi}{\partial p} &= -\frac{RT}{p}\\ \nabla_p\Phi + \frac{\partial\Phi}{\partial p}\nabla_z p &= 0\\ \nabla_p\Phi - \frac{1}{\rho}\nabla_z p &= 0.\end{aligned}$$

(b) Show that a combination of (3.43), (3.44) and (4.11) again gives $\nabla_p\Phi = \frac{1}{\rho}\nabla_z p$. Hint: use (4.11) to show that $\Phi(\boldsymbol{x},p) = \phi(z(\boldsymbol{x},p))$ and $\nabla_p\Phi = g\nabla_p z$.

4.4. The geostrophic wind. Referring to synoptic scaling of the horizontal momentum equation, Holton (1992) uses the approximate balance between the Coriolis term and the pressure gradient force

$$-fv \approx -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{4.12}$$

$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{4.13}$$

to define the *variable-f geostrophic wind* $\boldsymbol{V}_g$ by

$$\boldsymbol{V}_g \equiv \boldsymbol{k}\times\frac{1}{\rho f}\nabla_z p \tag{4.14}$$

(see Section 2.4.1 in Holton (1992)).

A *constant-f geostrophic wind* is defined in Holton (1992) by use of a constant reference latitude value f_0 for the Coriolis parameter f.

Exercise 4.4. Assume that all the acceleration terms except the horizontal Coriolis terms are negligibly small. Consider the approximate horizontal momentum equation, $\frac{D\boldsymbol{V}}{Dt} + f\boldsymbol{k}\times\boldsymbol{V} = -\frac{1}{\rho}\nabla_z p$. Use the triple product rule given in Section 1.6.3 to show that (4.14) holds. The aim of this exercise is to obtain (4.14) by vector manipulation *without* using the component form (4.12) – (4.13).

4.5. The ageostrophic wind. Define the ageostrophic wind $\boldsymbol{V}_a = (u_a, v_a)$ by

$$\boldsymbol{V}_a = \boldsymbol{V} - \boldsymbol{V}_g$$

so that $\boldsymbol{V} = \boldsymbol{V}_g + \boldsymbol{V}_a$.

Exercise 4.5. Integration of knowledge. Revise the concepts of components of a vector, Exercise 1.5.1(b) and Exercise 1.5.3.1(e).

4.6. The quasi-geostrophic prediction equations. Using synoptic scaling considerations the total derivative following the geostrophic wind is defined by

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{V}_g\cdot\nabla = \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}.$$

The quasi-geostrophic momentum equation, (6.11) in Holton (1992), can be written component wise as

$$\frac{D_g u_g}{Dt} - f_0 v_a = \beta y v_g$$
$$\frac{D_g v_g}{Dt} + f_0 u_a = -\beta y u_g,$$

with f_0 and $\beta = (\frac{df}{dy})_{\phi_0}$ constants originating from the retention of only the first two terms in a Taylor series expansion of the Coriolis parameter f about a reference latitude ϕ_0.

Exercises 4.6.

(a) Combine the above two approximate horizontal momentum equations to give the quasi-geostrophic vorticity equation

$$\frac{\partial\zeta_g}{\partial t} = -\boldsymbol{V}_g\cdot\nabla(\zeta_g + f) + f_0\frac{\partial\omega}{\partial p}.$$

Hint: use $\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$, assume $\nabla\cdot\boldsymbol{V}_g = 0$ and write the answer in terms of ω by use of (3.50).

(b) Under adiabatic conditions ($\frac{Ds}{Dt} = 0$), the thermodynamic energy equation $c_p\frac{D\ln\theta}{Dt} = \frac{Ds}{Dt}$, (see (2.46) in Holton (1992)) may be approximated for quasi-geostrophic conditions by

$$\frac{D_g T}{Dt} - S_p\omega = 0,$$

with $S_p = -\frac{T}{\theta}\frac{\partial\theta}{\partial p}$. Prove this statement.

(c) Assume $\zeta_g = \frac{1}{f_0}\nabla^2\Phi$ by Exercise 1.5.3.1(e) and $T = -\frac{p}{R}\frac{\partial\Phi}{\partial p}$, by Exercise 4.3(a) so that $\frac{\partial\zeta_g}{\partial t} = \frac{1}{f_0}\nabla^2\frac{\partial\Phi}{\partial t}$ and $\frac{\partial T}{\partial t} = -\frac{p}{R}\frac{\partial}{\partial p}\frac{\partial\Phi}{\partial t}$. Let $\sigma = \frac{R}{p}S_p$ be a constant. Derive the omega equation

$$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega = \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\boldsymbol{V}_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right] + \frac{1}{\sigma}\nabla^2\left[\boldsymbol{V}_g\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right].$$

Hint: eliminate the tendency $\frac{\partial\phi}{\partial t}$ from the quasi-geostrophic vorticity equation in (a) and the thermodynamic energy equation in (b).

(d) Assume $\zeta_g = \frac{1}{f_0}\nabla^2\Phi$, $T = -\frac{p}{R}\frac{\partial\Phi}{\partial p}$ and $\sigma = \frac{R}{p}S_p$ as in (c). Put $\chi = \frac{\partial\Phi}{\partial t}$. Derive the geopotential tendency equation

$$\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = -f_0\boldsymbol{V}_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + \frac{\partial}{\partial p}\left[\frac{f_0^2}{\sigma}\boldsymbol{V}_g\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right].$$

Hint: eliminate ω from the quasi-geostrophic vorticity equation in (a) and the thermodynamic energy equation in (b).

4.7. The Q-vector. Using Cartesian co-ordinates, we follow the first part of the derivation in Holton's (1992) Section 6.4.2 to derive two equations for $\boldsymbol{Q}$ ((4.31) and (4.33)) from which it can clearly and easily be seen that purely geostrophic motion (Section (4.7.3)) will tend to destroy the thermal wind relationship.

Our derivation may be summarized as follows: the quasi-geostrophic equations on the *f* plane (Holton (1992), Section 6.4.2, p. 170-175)

$$\frac{D_g u_g}{Dt} - f_0 v_a = 0 \tag{4.15}$$

$$\frac{D_g v_g}{Dt} + f_0 u_a = 0 \tag{4.16}$$

$$\frac{D_g T}{Dt} - S_p\omega = 0 \tag{4.17}$$

may be combined with the thermal wind relationship (to be proven in Exercise 4.7)

$$\frac{\partial\boldsymbol{V}_g}{\partial p} = -\frac{R}{f_0 p}\boldsymbol{k}\times\nabla_p T = -\frac{R}{f_0 p}\left(-\frac{\partial T}{\partial y}\boldsymbol{i} + \frac{\partial T}{\partial x}\boldsymbol{j}\right), \tag{4.18}$$

the fact that the *constant-f* geostrophic wind is divergence free

$$\nabla\cdot\boldsymbol{V}_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0, \tag{4.19}$$

and the continuity equation

$$\nabla\cdot\boldsymbol{V}_a + \frac{\partial\omega}{\partial p} = \nabla\cdot(\boldsymbol{V}_g + \boldsymbol{V}_a) + \frac{\partial\omega}{\partial p} = \nabla\cdot\boldsymbol{V} + \frac{\partial\omega}{\partial p} = 0 \tag{4.20}$$

to yield the so called **Q**-vector form of the omega equation:

$$\sigma\nabla^2\omega + {f_0}^2\frac{\partial^2\omega}{\partial p^2} = -2\nabla\cdot\boldsymbol{Q}, \tag{4.21}$$

with

$$\boldsymbol{Q} = (Q_1, Q_2) = -\frac{R}{p}\left(\frac{\partial \boldsymbol{V}_g}{\partial x}\cdot\nabla T,\ \ \frac{\partial \boldsymbol{V}_g}{\partial y}\cdot\nabla T\right). \tag{4.22}$$

Exercise 4.7. Prove (4.18), (4.19) and (4.20). Does (4.19) hold for the *variable-f* geostrophic wind? Hint for (4.18): write (4.14) in isobaric co-ordinates, and use Exercises 4.3. For (4.19) refer to Exercise 1.5.3.1(e). For (4.20) revise Section 3.4.4. For the horizontal divergence of the *variable-f* geostrophic wind look at Holton's (1992) Problem 3.19.

The Proof of (4.21) follows in Sections 4.7.1 to 4.7.4.

4.7.1. A differentiated form of the momentum equation (4.15).

Differentiation ($\frac{\partial(4.15)}{\partial p}$) and manipulation gives

$$\frac{\partial}{\partial p}\left[\frac{\partial u_g}{\partial t} + \boldsymbol{V}_g\cdot\nabla u_g\right] = f_0\frac{\partial v_a}{\partial p}$$

$$\frac{D_g}{Dt}\left(\frac{\partial u_g}{\partial p}\right) + \frac{\partial \boldsymbol{V}_g}{\partial p}\cdot\nabla u_g = f_0\frac{\partial v_a}{\partial p}. \tag{4.23}$$

Using (4.18), we may expand the second term in (4.23) as follows

$$\begin{aligned}
\frac{\partial \boldsymbol{V}_g}{\partial p}\cdot\nabla u_g &= -\frac{R}{f_0 p}\left(-\frac{\partial T}{\partial y}\boldsymbol{i} + \frac{\partial T}{\partial x}\boldsymbol{j}\right)\cdot\nabla u_g \\
&= \frac{R}{f_0 p}\left(-\frac{\partial T}{\partial x}\boldsymbol{j} + \frac{\partial T}{\partial y}\boldsymbol{i}\right)\cdot\nabla u_g \\
&= \frac{R}{f_0 p}\left(-\frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y} + \frac{\partial T}{\partial y}\frac{\partial u_g}{\partial x}\right) \\
&= \frac{R}{f_0 p}\left(-\frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y} - \frac{\partial T}{\partial y}\frac{\partial v_g}{\partial y}\right), \quad \text{using (4.19)} \\
&= -\frac{R}{f_0 p}\nabla T\cdot\frac{\partial \boldsymbol{V}_g}{\partial y} \\
&= \frac{1}{f_0}Q_2
\end{aligned} \tag{4.24}$$

so that

$$f_0\frac{\partial \boldsymbol{V}_g}{\partial p}\cdot\nabla u_g = Q_2. \tag{4.25}$$

Substitution of (4.25) in f_0 times (4.23) gives

$$\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) + Q_2 = {f_0}^2\frac{\partial v_a}{\partial p}. \tag{4.26}$$

Exercise 4.7.1. A differentiated form of the momentum equation (4.16). Use similar reasoning to prove that

$$\frac{D_g}{Dt}\left(\frac{\partial v_g}{\partial p}\right) + \frac{\partial \boldsymbol{V}_g}{\partial p}\cdot\nabla v_g = -f_0\frac{\partial u_a}{\partial p} \tag{4.27}$$

$$f_0\frac{\partial \boldsymbol{V}_g}{\partial p}\cdot\nabla v_g = -Q_1 \tag{4.28}$$

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right) - Q_1 = -{f_0}^2\frac{\partial u_a}{\partial p}. \tag{4.29}$$

A vector combination of (4.29) and (4.26) is

$$\begin{aligned}(Q_1, Q_2) - {f_0}^2\frac{\partial}{\partial p}(u_a, v_a) &= (Q_1 - {f_0}^2\frac{\partial u_a}{\partial p},\ \ Q_2 - {f_0}^2\frac{\partial v_a}{\partial p})\\ &= \left(\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right),\ \ -\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right)\right)\\ &= \frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p},\ \ -f_0\frac{\partial u_g}{\partial p}\right)\\ &= \frac{D_g}{Dt}\left(-\frac{R}{p}\frac{\partial T}{\partial x},\ \ -\frac{R}{p}\frac{\partial T}{\partial y}\right) \quad \text{by (4.18)}\\ &= -\frac{D_g}{Dt}\left(\frac{R}{p}\nabla T\right) \end{aligned}\tag{4.30}$$

which becomes

$$\boldsymbol{Q} - {f_0}^2\frac{\partial \boldsymbol{V}_a}{\partial p} = -\frac{D_g}{Dt}\left(\frac{R}{p}\nabla T\right). \tag{4.31}$$

4.7.2. A differentiated form of the energy equation (4.17). Assuming S_p to be constant, another expression for the right hand side of (4.31) can be obtained from (4.17) as follows: $\frac{\partial(4.17)}{\partial x}$ implies

$$\frac{\partial}{\partial x}\left[\frac{\partial T}{\partial t} + \boldsymbol{V}_g\cdot\nabla T\right] = S_p\frac{\partial \omega}{\partial x}$$

$$\frac{D_g}{Dt}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial \boldsymbol{V}_g}{\partial x}\cdot\nabla T = S_p\frac{\partial \omega}{\partial x} \tag{4.32}$$

and $\frac{\partial(4.17)}{\partial y}$ implies

$$\frac{D_g}{Dt}\left(\frac{\partial T}{\partial y}\right) + \frac{\partial \boldsymbol{V}_g}{\partial y}\cdot\nabla T = S_p\frac{\partial \omega}{\partial y}$$

which, combined with (4.32) and the definition of $\boldsymbol{Q}$ in (4.22), gives

$$\frac{D_g}{Dt}(\nabla T) - \frac{p}{R}(Q_1, Q_2) = S_p\nabla\omega$$

or

$$\frac{D_g}{Dt}\left(\frac{R}{p}\nabla T\right) - \boldsymbol{Q} = \sigma\nabla\omega, \quad \sigma = \frac{RS_p}{p},$$

or

$$\boldsymbol{Q} + \sigma\nabla\omega = \frac{D_g}{Dt}\left(\frac{R}{p}\nabla T\right). \tag{4.33}$$

4.7.3. Pure geostrophic motion. If $\boldsymbol{V}_a = \mathbf{O}$ then $\frac{\partial\omega}{\partial p} = -\nabla\cdot\boldsymbol{V}_a = 0$, from (4.20). Thus ω is constant with height and since there is no vertical movement on the surface, we have $\omega = 0$. Then (4.31) and (4.33) may be written as

$$\boldsymbol{Q} = -\frac{D_g}{Dt}\left(\frac{R}{p}\nabla T\right)$$
$$\boldsymbol{Q} = \frac{D_g}{Dt}\left(\frac{R}{p}\nabla T\right),$$

which means that either $\boldsymbol{Q} = \mathbf{O}$, or the thermal wind equation does not hold.

4.7.4. Ageostrophic motion. If $\boldsymbol{V}_a \neq \mathbf{O}$ then (4.31) + (4.33) gives

$$2\boldsymbol{Q} + \sigma\nabla\omega - {f_0}^2\frac{\partial\boldsymbol{V}_a}{\partial p} = \mathbf{O}$$
$$2\nabla\cdot\boldsymbol{Q} + \sigma\nabla^2\omega - {f_0}^2\frac{\partial}{\partial p}(\nabla\cdot\boldsymbol{V}_a) = 0$$
$$2\nabla\cdot\boldsymbol{Q} + \sigma\nabla^2\omega + {f_0}^2\frac{\partial^2\omega}{\partial p^2} = 0,$$

where the last term follows from $\frac{\partial\omega}{\partial p} = -\nabla_p\cdot\boldsymbol{V}_a$, i.e. equation (4.20). **Thus (4.21) follows** and dividing by σ we obtain

$$\left(\nabla^2 + \frac{{f_0}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{2}{\sigma}\nabla\cdot\boldsymbol{Q}. \tag{4.34}$$

Note the similarity between the partial differential operators on the left hand side of the omega equation (Exercise 4.6(c)), the tendency equation (Exercise 4.6(d)) and the one on the left hand side of equation (4.34).

4.7.5. Remark. Adams (1990) uses (4.24) and (4.28) as definition of the Q-vector and then derives (4.22).

4.8. The quasi-geostrophic potential vorticity equation. In Holton's (1992) Section 6.3.2 the quasi-geostrophic potential vorticity q is defined by

$$q = \frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\Phi}{\partial p}\right).$$

The tendency equation can be written as a conservation equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}_g \cdot \nabla\right)\left[\frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right] = \frac{D_g q}{Dt} = 0$$

for q. This conservation equation can be written in the form

$$\frac{D_g}{Dt}(\zeta_g + f) - \frac{D_g}{Dt}\left(\frac{f_0}{S_p}\frac{\partial T}{\partial p}\right) \approx \frac{D_g}{Dt}\left[\frac{1}{f_0}\nabla^2\Phi + f + \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)\right] = 0 \qquad (4.35)$$

by use of the geostrophic vorticity $\zeta_g = \frac{1}{f_0}\nabla^2\Phi$ from Exercise 1.5.3.1(e) and the approximation $\frac{f_0}{S_p}\frac{\partial T}{\partial p} \approx \frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)$ from Holton's (1992) Section 6.3.2.

4.8.1. Vertical motion (sinking) accompanied by vorticity and geopotential height changes. From (4.35) follows

$$\frac{D_g}{Dt}(\zeta_g + f) = \frac{f_0}{S_p}\frac{D_g}{Dt}\frac{\partial T}{\partial p}$$

so that for $f_0 \leq 0$ (in the Southern Hemisphere) $\zeta_g + f$ increases (resp. decreases) with time if and only if $\frac{\partial T}{\partial p}$ decreases (resp. increases) with time.

Exercises 4.8.1.

(a) Derive the quasi-geostrophic potential vorticity equation.

(b) Ignoring density changes with time, show that $\zeta_g + f$ increases (resp. decreases) with time if and only if $\frac{\partial T}{\partial z}$ increases (resp. decreases) with time in the Southern Hemisphere.

(c) Consider the adiabatic sinking of a vertical atmospheric column depicted in Figure 4.1. Show that the geostrophic vorticity and the geopotential must decrease if f remains constant. You may consider a Southern Hemispheric situation. Hint: use the sketch to determine whether $\frac{\partial T}{\partial z}$ increases or decreases with time, use (b) and assume $\nabla^2\Phi = -c\Phi$ with c a constant.

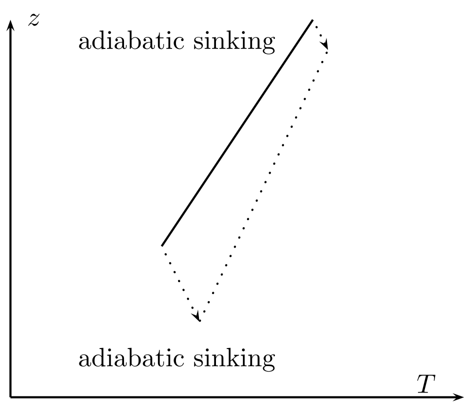

Figure 4.1

Chapter 5

Atmospheric modelling and simple numerical examples

A true story. Zbigniew Sorbjan (1996) tells the following about Richardson's forecasting plans: *The first step toward mathematical weather forecasting was done by Lewis Fry Richardson (1881–1953). Richardson believed that it might be possible to solve the complex equations of atmospheric motion by working them out in step by step computations. He described his ideas in his book* Weather Prediction by Numerical Process. *The first draft of it was prepared in May 1916, and revised in 1916–1918, during World War I. During the battle of Champagne, in April 1917, the working copy was sent to the rear lines and was lost. Fortunately, it was recovered a few months later under a heap of coal. Finally, the manuscript was printed in 1922.*

To put his idea into practice, Richardson required data from about 2000 permanent weather stations collecting both surface and upper-air measurements around the Earth. He also envisioned the globe divided up as a checkerboard into rows and columns. He thought that 32 individuals could compute a forecast at one column to keep pace with the weather. If the column spacings were 200 km, he believed that 2000 active columns would suffice to complete a forecast on the globe. Consequently, $32 \times 2000 = 64000$ *calculators would be needed to predict the weather for the entire globe. Such an amount of computations would require a "forecast factory".*

Introduction. The purpose of Chapter 5 is to give the learner a glimpse of the world of numerical modelling by using a few examples from previous chapters to introduce the new concepts. It will be fruitful to solve some problems on a computer. First single equations in one independent variable need to be solved and then systems of algebraic equations. It is not the intention of this chapter to be self contained and complete.

5.1. Mathematical models. The learner has encountered mathematical and meteorological problems, has proposed solutions for them and practised integration of knowledge. The process of mathematical modelling, which embraces mathematical equations and their numerical solution, is summarised for the atmosphere in Fig. 5.1.

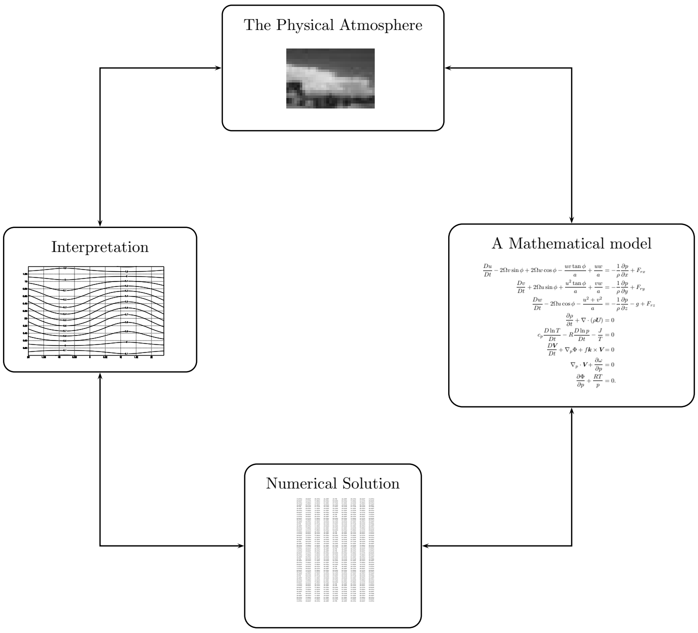

Figure 5.1

5.2. Atmospheric modelling. We summarize the dynamical meteorological equations, using as independent variables x, y, p, t and dependent variables u, v, ω, Φ, T.

$$\frac{D\boldsymbol{V}}{Dt} = -\nabla_p \Phi - f\boldsymbol{k} \times \boldsymbol{V} \tag{5.1}$$

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p} \tag{5.2}$$

$$\frac{DT}{Dt} - \frac{RT}{pc_p}\omega - \frac{J}{c_p} = \frac{T}{c_p}\left(c_p \frac{D \ln T}{Dt} - R\frac{D \ln p}{Dt} - \frac{J}{T} \right) = 0 \tag{5.3}$$

$$\nabla_p \cdot \boldsymbol{V} + \frac{\partial \omega}{\partial p} = 0. \tag{5.4}$$

Boundary values must be specified at all times throughout the forecasting period. An approximate model topography will represent the real topography and the model top will represent the upper atmospheric boundary. For a global model the lower and upper boundary values must be specified at all times throughout the forecast. For a Limited-Area Model (LAM) the lateral boundary values are needed in addition. The lateral boundary values are normally obtained from a global model.

Initial values must be specified throughout the three dimensional model domain at the start of the forecast.

5.3. Numerical models.

5.3.1. First order derivatives.

Examples 5.3.1.

(a) **The geopotential.** Consider Exercise 1.2(c). If we combine Newton's gravity law (allowing for centripetal acceleration) and the definition (4.1) of the geopotential, then a first order ordinary differential equation with lower boundary value (at $z = 0$) arises:

$$\Phi'(z) = \frac{g_0}{(1+z/a)^2}$$
$$\Phi(0) = 0.$$

(b) **Time rate of change of temperature.** In equation (5.3), $\frac{DT}{Dt} = \frac{RT}{pc_p}\omega + \frac{J}{c_p}$, the temperature T depends on time t as well as position $\boldsymbol{x}$. Consider the time rate of change first. The partial derivatives may be expanded to give

$$\frac{\partial T}{\partial t} = -\boldsymbol{U}\cdot\nabla T + \frac{RT}{pc_p}\omega + \frac{J}{c_p}. \tag{5.5}$$

Approximation of first order derivatives. In view of Taylor's theorem,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(c)$$

it seems natural to look for approximations of the first order derivatives, based on a difference form, *viz.*

$$f'(x) = \frac{f(x+h)-f(x)}{h} - \frac{h}{2}f''(c). \tag{5.6}$$

Since Φ is only a function of z in Example 5.3.1(a), the learner should have no difficulty in calculating Φ for a discrete set of points z_i, say (Exercise 5.3.1(a)). For the purpose of illustration consider now Example 5.3.1(b). We may write (5.5) in the form

$$\frac{\partial T}{\partial t} = F(J,\boldsymbol{U},\omega,T)$$

with F indicating the right hand side of (5.5). Using (5.6) to approximate the time derivatives, the *finite difference scheme*

$$\frac{T(t+\delta t)-T(t)}{\delta t} \approx F(J,\boldsymbol{U},\omega,T)$$

is obtained, implying

$$T(x_i,y_j,p_k,t_{l+1}) = T(x_i,y_j,p_k,t_l) + \delta t G(x_i,y_j,p_k,t_l), \tag{5.7}$$

where G represents the value of F at the discrete points in space (x_i, y_j, p_k) and an instant of time t_l at which all variables are known (p denotes a vertical co-ordinate). The temperature T may thus be calculated at a future instant of time t_{l+1}.

This algorithm gives an explicit 'marching' ahead in time. By truncating the Taylor series, errors are introduced. The accuracy of the finite difference approximation for $f'(x)$, given by (5.6), is indicated by the symbol $O(h)$. This symbol means that a positive number A exists such that the omitted part is less than or equal to Ah in absolute value. A higher order finite difference approximation for the first order derivatives can be obtained by subtracting the following two equations

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f^{(3)}(c_1) \tag{5.8}$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f^{(3)}(c_2) \tag{5.9}$$

to give

$$f(x+h) - f(x-h) = 2hf'(x) + O(h^3)$$

resulting in a *central difference* formulation

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2). \tag{5.10}$$

Likewise a *forward difference* and *backward difference* formulation may be obtained from (5.8) and (5.9) respectively, but with accuracy $O(h)$.

Exercises 5.3.1.

(a) Solve the problem in Example 5.3.1(a) numerically.

(b) Assume numerical values $\frac{R}{pc_p}\omega = -\frac{287}{85000\times1004} \times 0.01\ \mathrm{s}^{-1} = -3\times10^{-8}\ \mathrm{s}^{-1}$, $\boldsymbol{U} = \mathbf{O}$ and $J = 0$ in (5.5). Solve (5.5), with $T(t=0) = 300$ Kelvin, using a forward finite difference scheme and compare with the analytical solution, $T = 300e^{-0.00000003t}$ Kelvin.

(c) Assume numerical values $\frac{R}{pc_p}\omega = -3\times10^{-8}\ \mathrm{s}^{-1}$ as in (b), $\boldsymbol{U} = (1,0,0)$ and $J/c_p = 9\times10^{-6}\mathrm{Ks}^{-1}$ in (5.5). Solve (5.5), subject to $T(x,t=0) = 300+\sin x$ Kelvin, using a forward finite difference scheme for the time derivative and a central spatial derivative, as in (5.10) and compare with the analytical solution, $T(x,t) = 300 + e^{-0.00000003t}\sin(x-t)$ Kelvin.

(d) Solve (5.5) as in (c), but using a backward spatial derivative, $\frac{\partial T}{\partial x} \approx \frac{T(x,t)-T(x-\delta x,t)}{\delta x}$ and again compare with the analytical solution.

5.3.2. Higher order derivatives. A finite difference scheme for the second order derivatives can be obtained by adding equations like (5.8) and (5.9), involving up to fourth order derivatives. The result

$$f(x+h) + f(x-h) = 2f(x) + h^2f''(x) + O(h^4)$$

can be rewritten as

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2). \tag{5.11}$$

Example 5.3.2. Consider the oscillations of a parcel of air in a statically stable atmosphere. Equation (2.52) in Holton (1992) can be written as

$$f''(t) = -N^2 f(t),$$

where N represents the Brunt-Väisälä frequency and the function $t \mapsto f(t)$ the parcel displacement. If we know the initial displacement $f(0)$ and if we can approximate the displacement at a small time step δt later on in terms of the previous time steps (time t and time $t - \delta t$), then we may be able to obtain an approximate solution. Alternatively, we may use an iterative method based on the discrete form $\frac{f(t+h)-2f(t)+f(t-h)}{h^2} = -N^2 f(t)$ which implies

$$f(t) = \tfrac{1}{2}(f(t+h) + f(t-h) + h^2 N^2 f(t)).$$

Exercises 5.3.2.
(a) Prove (5.11).
(b) Use the approximation proposed in (5.11) in both the x- and y- directions to give the following finite difference form of the horizontal Laplace operator (defined in Exercise 1.5.3.1(e))

$$\nabla^2 f \approx \frac{f_{i,j+1} + f_{i,j-1} + f_{i+1,j} + f_{i-1,j} - 4f_{i,j}}{h^2}. \tag{5.12}$$

5.4. The quasi-geostrophic vorticity equation. Let $\zeta_g = \partial_x v_g - \partial_y u_g$ and assume

$$\frac{\partial \zeta_g}{\partial t} = -\boldsymbol{V}_g \cdot \nabla(\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p},$$

from Holton (1992), p.155, Section 6.2 and

$$\zeta_g = \frac{1}{f_0} \nabla^2 \Phi, \tag{5.13}$$

from Holton (1992), p.154, Section 6.2.2.
Exercise 5.4. Compare the form of the vorticity equation above with that of the thermodynamic equation (5.3) and propose a ‘marching’ ahead in time similar to (5.7).

5.5. A boundary value problem. Let $f_0 = 10^{-4}$ s^{-1} and $k = l = \frac{\pi}{2} \times 10^{-6}$ m^{-1} as in Holton (1992), p.157. Let $N = 2 \times 10^6$ m denote the east-west extent and $M = 1 \times 10^6$ m the north-south extent of the domain $R = [-N, N] \times [-M, M]$ of interest. Consider the boundary value problem

$$\left.\begin{aligned} \nabla^2 \Phi &= -(k^2 + l^2)\sin(kx)\cos(ly) \text{ in } R \\ \Phi(x, -M) &= 50000 + 10 f_0 M + 800 \sin(kx)\cos(lM) \\ \Phi(x, M) &= 50000 - 10 f_0 M + 800 \sin(kx)\cos(lM) \\ \Phi(-N, y) &= 50000 - 10 f_0 y - 800 \sin(kN)\cos(ly) \\ \Phi(N, y) &= 50000 - 10 f_0 y + 800 \sin(kN)\cos(ly). \end{aligned}\right\} \tag{5.14}$$

Note that the Partial Differential Equation (PDE) in (5.14) is of the form (5.13) with $\zeta_g f_0 = -(k^2 + l^2)\sin(kx)\cos(ly)$.

Exercises 5.5.

(a) Use the discrete form (5.12) of the Laplace operator to write the Poisson equation (5.13) in the finite difference form

$$\Phi_{i,j} = \frac{1}{4}(\Phi_{i,j+1} + \Phi_{i,j-1} + \Phi_{i+1,j} + \Phi_{i-1,j} - h^2 f_0 \zeta_{g_{i,j}}). \tag{5.15}$$

Use the Gauss-Seidel method (5.15) to solve the boundary value problem (5.14).

(b) Write down the heat equation, obtained in Exercise 3.3.1(a), for the temperature function $T : (x,t) \mapsto T(x,t)$. Find the numerical solution of the heat conduction problem, comprising of the heat equation in the spatial one dimensional interval $[0,1]$, boundary conditions $T(0,t) = 0, T(1,t) = 0$ for every t and the initial condition $T(x,0) = \sin(\pi x)$ for every $x \in (0,1)$.

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