

# **Chapter 7**

**Water waves,  
dispersion,  
group velocity**

## Introduction

A fundamental property of the solutions to the continuum wave equation is the linear relationship  $\omega = ck$  between frequency and wavenumber, or, equivalently, the famous  $\lambda v = c$  relationship. The most important implication of this relationship is the following: if a certain perturbation is imposed on a medium, the perturbation travels without distortion. You have already found one such case in Problem 2, Chapter 5, where you started with a rectangular shape and watched this shape travel back and forth in a cord with speed  $c$  while remaining undistorted. Media where waves propagate without distortion are called **non-dispersive**.

Systems which do not satisfy the wave equation are called **dispersive**. Since the wave equation was derived as a continuum limit approximation, one would expect all media to be somewhat dispersive. Moreover, even in cases where the continuum approximation is perfectly justified, the system may not satisfy the standard wave equation. Very important examples of this are water waves and electromagnetic waves in matter. We will discuss water waves in this chapter and electromagnetic waves in the next chapter as an example of dispersive waves.

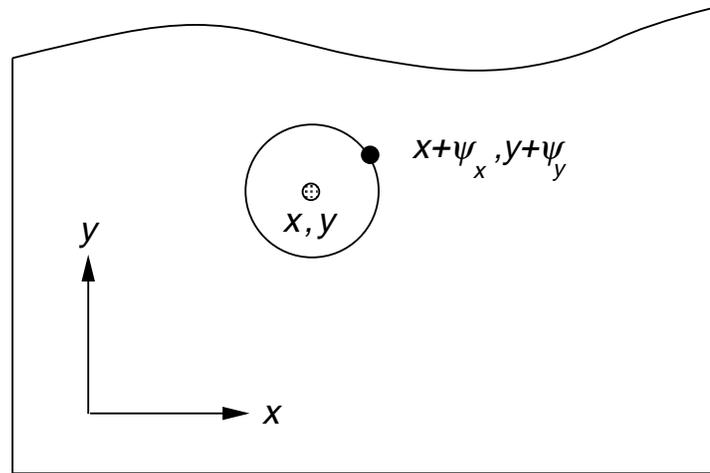
When the medium is dispersive, a wave packet will distort as it travels. This has many important consequences. The amount of information that can be transmitted with optic fibers, for example, depends on the ability of the material to maintain the separation between different wave packets. If the packets broaden due to the dispersive nature of the medium, then the initial separation must be large if one wants to avoid overlaps. The speed of propagation of the packet also becomes an issue. Packet distortion means that different points travel with different speeds. Hence it is not obvious how to define and what is the meaning of the wave velocity concept. We will show in this chapter that this problem can be addressed by introducing the concept of **group velocity**.

## Water waves

The first thing that comes to mind when one talks about waves is the beautiful pattern of waves on the surface of a quiet lake or the imposing shape of the ocean waves. Surprisingly, water waves do not satisfy the standard wave equation we derived in previous chapters, except in some special limit cases. Hence water waves are dispersive. We will study these waves as the first example of dispersive waves.

### Description of water waves

Our system is illustrated in Fig. 1



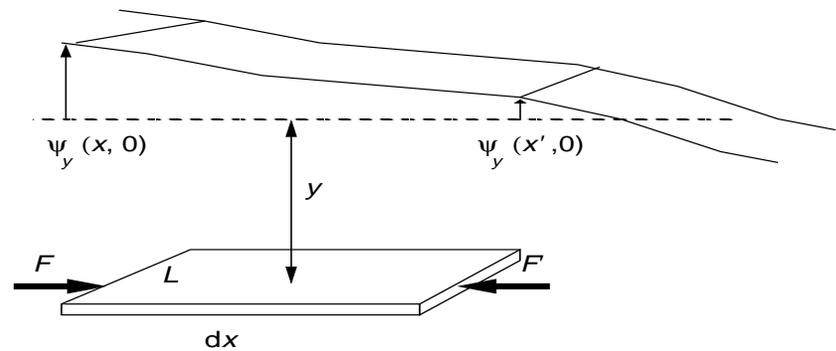
**Figure 1** The mathematical description of the position of a drop of water.

When no waves are present, a certain drop of water has an equilibrium position  $x, y$ . Under the effect of the wave, the drop will be displaced to the position  $x + \psi_x(x, y), y + \psi_y(x, y)$ . Normally, the drop will oscillate about the equilibrium position. We see immediately a fundamental difference with the wave problems we discussed in previous chapters: to describe water waves, we need *two* position functions  $\psi_x(x, y)$  and  $\psi_y(x, y)$ . The wave equation as we now it, however, is an equation for a single function. Moreover, the displacements of the drop in the  $x$ - and  $y$ -directions are coupled by a number of requirements. For example, we now that water is essentially incompressible. This means that the

amount of water entering a certain volume must equal the amount of water leaving this volume. This imposes a connection between the functions  $\psi_x(x,y)$  and  $\psi_y(x,y)$ , as we will see below. An additional connection is imposed if we consider water without whirlpools or vortices. Instead of a wave equation with a single function, the mathematical description of water waves requires several coupled differential equations. We will start with some simplified examples in the sections below.

### Equation of motion for an element of water

The starting point of all dynamical problems is Newton's second law. Let us apply this law to the  $x$ -direction motion of an element of water of volume  $L \times dx \times dy$ . (The acceleration of gravity is in the negative  $y$ -direction).



**Figure 2** The horizontal motion of an element of water. The dashed horizontal line represents the equilibrium surface of the water.

The forces  $F'$  and  $F$  acting on our element are caused by the pressure exerted by the surrounding water. In equilibrium, these forces cancel out, but in the case depicted in the figure the height of the water is different at points  $x$  and  $x'$ , so that  $F$  and  $F'$  are not exactly equal and opposite. The force  $F$  is given by  $F = pL \, dy$ , where  $L \times dy$  is the cross-sectional area of the element of volume and  $p$  the pressure. The pressure at depth  $y$  is given by  $p(y) = p_0 - \rho gy$ , where  $p_0$  is the atmospheric pressure and  $\rho$  the density of water. The difference between  $F'$  and  $F$  arises from the fact that the depth for  $F'$  is  $-y + \psi_y(x', 0)$ , while the depth for  $F$  is given by  $-y + \psi_y(x, 0)$ .

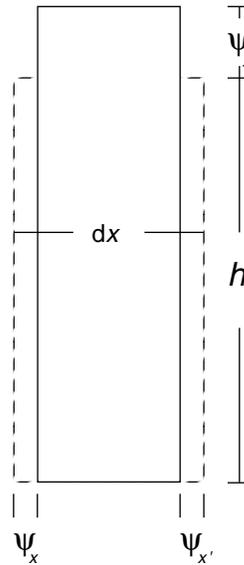
Hence the net horizontal force is

$$F - F' = -\rho g L dx \left[ \psi_y(x', 0) - \psi_y(x, 0) \right] \quad (1)$$

This must equal the mass  $\rho L \times dx \times dy$  times the horizontal acceleration. For  $x' = x + dx$  with  $dx$  small, we can approximate  $\psi_y(x', 0) - \psi_y(x, 0) \approx \left. \frac{\partial \psi_y}{\partial x} \right|_{y=0} dx$ , so that we finally obtain

$$\frac{\partial^2 \psi_x}{\partial t^2} = -g \frac{\partial \psi_y}{\partial x} \quad (2)$$

This equation is clearly different from the standard wave equation. It can be converted into a wave equation if we make the additional assumption that the horizontal motion is the same for all depths. This is the so-called “shallow water” approximation. With this assumption, let us consider a narrow rectangular section of the water, where the top is at the open surface and the bottom has depth  $h$ . (See Fig. 3)



**Figure 3** Conservation of mass in the shallow water limit

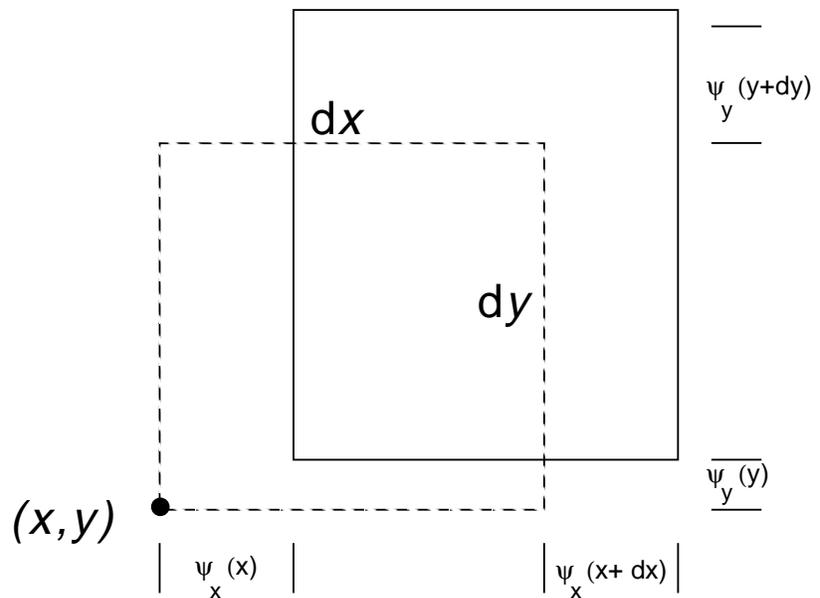
If the mass of the volume is to remain constant, we must have  $\psi_y dx = -[\psi_x(x + dx) - \psi_x(x)]h$ , or  $\psi_y = -h \frac{\partial \psi_x}{\partial x}$ . Substituting into Eq. (2), we finally obtain

$$\frac{\partial^2 \psi_x}{\partial t^2} = gh \frac{\partial^2 \psi_x}{\partial x^2} \quad (3)$$

which has the form of the wave equation with  $c = \sqrt{gh}$ .

### Dispersive water waves

The shallow water assumption cannot be true in general, for it is unreasonable to assume that the horizontal motion of the water near the bottom of a very deep lake will be the same as the motion near the surface. The derivation of the general solution for water waves requires an exact treatment of the conditions that mass be conserved and that there be no vortices. This is done following exactly the same ideas applied to the analysis of Fig. 3, except that one must consider volumes of infinitesimal dimensions in all directions. Let us consider one such volume on Fig. 4



**Figure 4** Conservation of mass in an incompressible fluid.

A certain amount of water occupies initially the volume  $dx \times dy$ . When a wave is present, the volume is distorted to  $[dx + \psi_x(x+dx) - \psi_x(x)] \times [dy + \psi_y(y+dy) - \psi_y(y)]$ . However, since the amount of water inside the two volumes is the same and the water is incompressible, the volumes themselves must be equal. This requires  $[\psi_x(x+dx) - \psi_x(x)] dy + [\psi_y(y+dy) - \psi_y(y)] dx = 0$ .

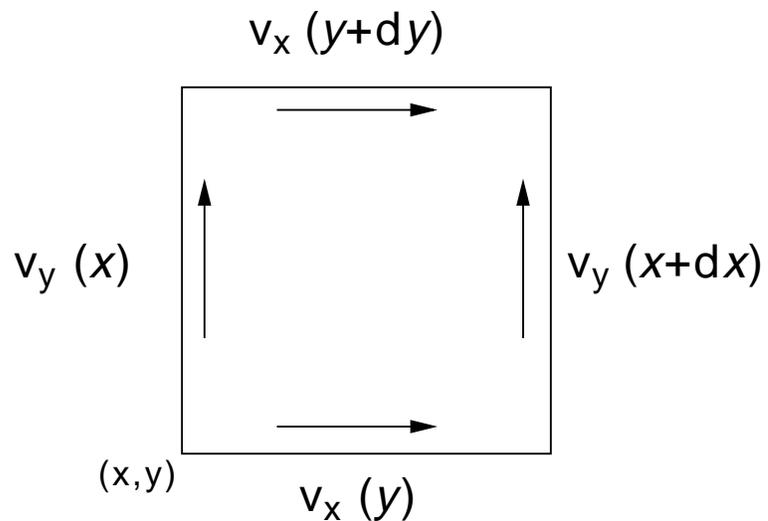
$\psi_y(y)] dx = 0$ , where we have assumed that the  $\psi$ 's are small, so that quadratic terms in  $\psi$  can be neglected. Dividing by  $dx dy$  we finally obtain

$$\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} = 0 \quad (4)$$

An additional condition linking the displacements in the  $x$ - and  $y$ -directions is obtained by requiring no vortices or whirlpools. The condition of no vortices can be written as

$$\oint \mathbf{v} \cdot d\mathbf{l} = 0 \quad , \quad (5)$$

where  $\mathbf{v}$  is the velocity vector and the path integral is performed along any closed loop. Suppose we apply this the square loop in Fig. 6



**Figure 6** Infinitesimal loop for the application of the no vortices condition.

If we integrate counterclockwise starting from the  $(x, y)$  point, we obtain  $[v_x(y) - v_x(y+dy)] dx + [v_y(x+dx) - v_y(x)] dy = 0$ . Using the definitions  $v_x = \partial \psi_x / \partial t$  and  $v_y = \partial \psi_y / \partial t$  this condition reduces to

$$\frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y} = 0 \quad (6)$$

Hence waves in water do not satisfy a simple wave equations but rather the set of coupled equations given by Eqs. 2, 4, and 6. Clearly, we cannot expect the solutions to these coupled equations to look like the solutions to the standard wave equation. On the other hand, we know that waves propagate in water, so that the solutions to the coupled equations must be similar to the solutions to the wave equation. Moreover, the only thing obviously wrong with the shallow water approximation is the neglect of any depth dependence of the displacements. This suggest that we try a solution of the form

$$\psi_y(x,y) = A \cos(\omega t - kx) f(y) \quad (7)$$

with the function  $f(y)$  to be determined. This looks like a familiar traveling wave, except for the function  $f(y)$  that should give the attenuation of the amplitude for points below the surface. Inserting this solution in Eq. (4), we find

$$\psi_x = \frac{A}{k} \sin(\omega t - kx) f'(y) \quad (8)$$

When the expressions for the  $\psi$ 's are used in the no-whirlpools condition Eq. (6), we obtain an explicit equation for the unknown function  $f(y)$ :

$$f''(y) = k^2 f(y) \quad (9)$$

This equation has the well-known solution

$$f(y) = B e^{ky} + C e^{-ky} \quad (10)$$

The boundary condition at the bottom of the volume,  $y = -h$ , is that the vertical displacement of the water be zero. Thus  $f(-h) = 0$ . This implies  $C = B e^{2kh}$ , so that we can write the final solution as

$$\begin{aligned} \psi_y &= D \cos(\omega t - kx) (e^{ky} - e^{-2kh} e^{-ky}), \\ \psi_x &= D \sin(\omega t - kx) (e^{ky} + e^{-2kh} e^{-ky}), \end{aligned} \quad (11)$$

where  $D$  is an arbitrary constant. We have not yet show that

these expressions are a solution to Eq. (2). You will do this in a homework problem. You will find that they are indeed solutions to the equation of motion provided that the relationship between  $\omega$  and  $k$  is no longer linear but given by

$$\omega^2 = gk \tanh kh \quad (12)$$

This means that water waves are dispersive. Let us consider the limits of Eq. (12). When  $kh \ll 1$ ,  $\tanh kh = kh$ , so that we obtain the shallow water limit  $\omega^2 = gh k^2$ . In this limit the waves are non-dispersive. Of course, this is the expression we derived above, but we can now understand the meaning of “shallow:” we are in the shallow water limit whenever  $kh \ll 1$ , or  $\lambda \gg h$ . The opposite limit is the “deep water” limit for which  $\lambda \ll h$  or  $kh \gg 1$ . In this case  $\tanh kh = 1$  and we obtain

$$\omega^2 = gk \quad (13)$$

Hence deep water waves, defined as those for which the wavelength is much less than the depth of the water, are dispersive. Their frequency is proportional not to the wavenumber but to its square root. A pulse of shallow water waves does not distort, but a pulse of deep water waves will become distorted as it propagates.

From Eq. (11) one can easily see (see homework problem) that a given water drop travels in an elliptical path, forward if on a crest and backward when on a trough. When friction is taken into account, the fact that there is more friction when the water is trying to go back during the troughs leads to a net displacement of the water and to the “breaking” of the waves.

## Propagation of dispersive waves

A MUCH BETTER WAY OF DOING THIS SECTION IS TO GO BACK TO THE SQUARE WAVE PROBLEM 5.2 AND TO CONVERT THE COS SIN PRODUCT INTO TRAVELING WAVES. SHOW THAT FOR  $w = ck$  ALL TERMS ARE FUNCTIONS OF  $x-ct$ , BUT NOT IF THE SYSTEM IS DISPERSIVE. IN THE PROCESS DEFINE PHASE VELOCITY. SAY THAT WE HAD USED JUST

VELOCITY BEFORE BECAUSE THERE WAS NO DISTINCTION BETWEEN GROUP AND PHASE VELOCITY. MAY BE ONE COULD USE JACKSON'S ANALYSIS PAGE 301 TO INTRODUCE GROUP VELOCITY AS A DERIVATIVE OF  $\omega$  VS  $k$

To better understand the difference between dispersive and non-dispersive waves, it is convenient to write the traveling wave as a sum of sinusoidal traveling waves. For the case of a non-dispersive wave  $a(x-ct)$ , we can write

$$a(x - ct) = \sum_k A_k \sin[kx - \omega(k)t + \alpha_k] \quad (14)$$

where the sum runs over an infinite number of terms. It is known from Fourier analysis that any wave can be written in this form. This result was also discussed in previous chapters. Notice that we have written  $\omega = \omega(k)$  to emphasize that there is a unique frequency  $\omega$  for every valid wavenumber  $k$ . The  $\omega$  vs.  $k$  relationship is the dispersion relation. For waves that satisfy the standard wave equation, we know that  $\omega = ck$ . Using this result, Eq.(14) can be written as

$$a(x - ct) = \sum_k A_k \sin k[x - ct + \alpha_k] \quad (15)$$

This means that every sin term in this equation is a traveling wave moving to the right with speed  $c$ . (As usual, by “right” we mean here the direction of the positive  $x$ -axis). If all “parts” of the wave are moving with the same speed, it is not surprising that the total wave moves with that same speed without suffering any distortion. When the relationship between  $\omega$  and  $k$  is not linear, however, the term  $c$  is no longer constant and different parts will travel with different speeds. This is the case of dispersive waves.

In this chapter, we have introduced water waves, which are an important example for which  $\omega \neq ck$ . (Except in the shallow water limit). An earlier example is the discrete chain of masses and springs. When we solved the problem of  $N$

masses  $m$  connected by springs  $K$  of equilibrium length  $a$ , we found that their dispersion relation was given by

$$\omega = \sqrt{\frac{4K}{m}} \sin \frac{ka}{2} \quad (16)$$

In these cases, we cannot use the trick we used in Eq. (15). Waves with different  $k$ 's will travel with different speeds. If we produce a distortion in the chain or in deep water, the distortion will not keep its shape as it propagates. For example, if we initially displace a single mass and keep all others in their equilibrium positions, a non-dispersive solution would be one in which the first mass stops and its neighbor starts vibrating, then the neighbor stops and the next neighbor starts its motion, etc. Instead, we see a spread of the perturbation: after a while, all masses are oscillating at any given time. The reason why the waves in a chain of masses are dispersive is that they do not satisfy exactly the standard wave equation. In fact, the standard wave equation was derived as an approximation valid only when the wavelength  $\lambda$  of the mode is much longer than the separation  $a$  between neighboring masses. In this case (recall that  $k = 2\pi/\lambda$ )  $ka \ll 1$ . Using  $\sin x \sim x$  the dispersion relation in Eq. (16) becomes  $\omega = (Ka^2/m)^{1/2} k$ . In this limit,  $\omega$  is proportional to  $k$ , so that the wave becomes non-dispersive. The proportionality constant is  $(Ka^2/m)^{1/2}$ . This is precisely the expression we found for the speed  $c$  when we solved the equations of motion in the limit where the wave equation is valid.

## The group velocity

Suppose that a certain localized perturbation  $a(x)$  is produced in a dispersive medium at time  $t = 0$ . After some time, the perturbation will have travelled a certain distance and its shape will have changed because the medium is dispersive. Although one cannot define “the” velocity of the perturbation, because different “parts” travel with different speed, one can ask what is the velocity of the “center” of

the pulse. To answer this question, we will form a very simple pulse, composed of two sinusoidal waves of frequencies  $\omega$  and  $\omega'$ . We will assume that the amplitudes are the same, so that we can write

$$\begin{aligned}\xi &= \\ &A \sin(kx - \omega t) + A \sin(k'x - \omega' t) \\ &= 2A \cos \frac{1}{2}[(k' - k)x - (\omega' - \omega)t] \sin \frac{1}{2}[(k' + k)x - (\omega' + \omega)t]\end{aligned}\quad (17)$$

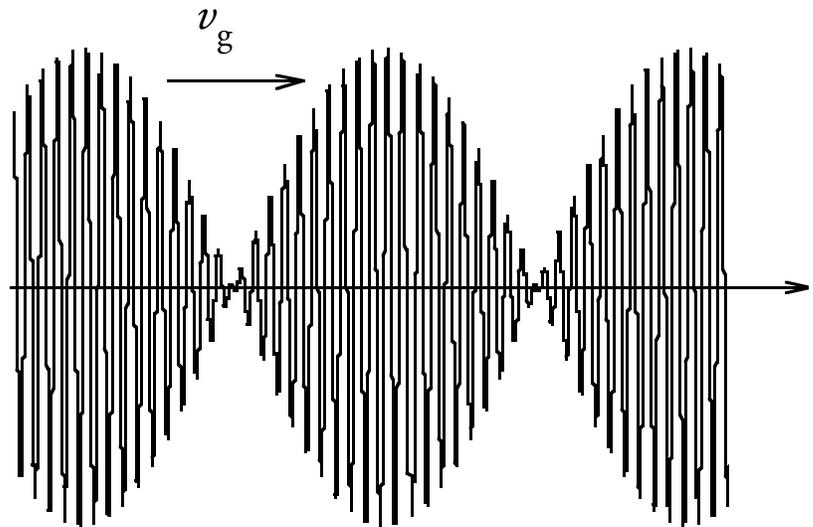


Figure 7 Group velocity.

This wave is illustrated in Fig. 7. It is not a single pulse (many sines are needed to build such a pulse) but rather a train of pulses. Our result, however, will be very general. Suppose that  $\omega$  and  $\omega'$  are very close, so that  $k$  and  $k'$  are also very close. We can thus approximate  $\frac{1}{2}(\omega + \omega')$  by  $\omega$  and  $\frac{1}{2}(k + k')$  by  $k$ . We thus obtain

$$\xi = 2A \cos \frac{1}{2}[(k' - k)x - (\omega' - \omega)t] \sin(kx - \omega t) \quad (18)$$

Hence we have a wave whose amplitude is being modulated.

The modulation travels with a velocity called **group velocity**, given by

$$v_g = \frac{\omega' - \omega}{k' - k} = \frac{d\omega}{dk} \quad (19)$$

where the last equality is valid for the case of very close frequencies. One can always define a **phase velocity**  $v(k)$  as  $\omega = v(k)k$ . Using this definition, the group velocity becomes

$$v_g = v + k \frac{dv}{dk} \quad (20)$$

The importance of the group velocity is that it can be shown that for dispersive waves the energy travels at the speed  $v_g$ .

## Problems

1. Consider the exact solutions for water waves given in Eq. (11). Derive approximate expressions valid in the deep water and shallow water limits. Show that in the deep water limit the waves are exponentially attenuated.

2. By inserting the solutions Eq. (11) in Eq. (2), derive the dispersion relation for water waves in Eq. (12).

3. Graph the motion of a drop as a function of time using the solutions Eq. (11). Show that in the deep water limit the path is a circle.

4. a) Verify that water can sustain standing waves of the form

$$\psi_y = D \cos \omega t \sin kx (e^{ky} - e^{-2kh} e^{-ky}),$$

$$\psi_x = D \cos \omega t \cos kx (e^{ky} + e^{-2kh} e^{-ky}).$$

b) What would be the boundary conditions and possible values of  $k$  for standing water waves in a rectangular aquarium?

5. (Crawford 7.30) Suppose that at the surface of the ocean there are traveling waves with 10-ft amplitude and wavelength 30 ft. If you were a fish (or a Scuba diver), how far beneath the surface should you swim if you wished the amplitude of your motion to be  $1/2$  foot?

6. *Tsunamis* are solitary ocean waves that propagate without much change of their shape. They are produced by undersea earthquakes.

a) Explain why *tsunamis* must be closer to the shallow water limit. Given the known average ocean depth of 5,000 m, what can you say about the wavelengths involved in *tsunamis*?

b) Estimate the propagation velocity of *tsunamis*.

6. Show that for non-dispersive waves the phase velocity equals the group velocity. Show that for deep water waves the group velocity is half the phase velocity.

7. Use your spreadsheet to construct a pulse as in Fig. 7 Graph this pulse for several times and verify the expression for the group velocity by measuring on the graphs the displacements of the wave packet for different times.

8. Repeat Problem 2, Chapter 5, for deep water waves. Determine from the graphs the speed of the pulse the best you can, and compare with the prediction from the group velocity expression.