

## **Chapter 6**

# **Traveling Waves**

## Introduction

The standing waves we found as solutions to the wave equation can be used to explain any type of vibration in continuous media, since they represent the most general solution to the problem. Any arbitrary set of initial conditions can be taken care of by appropriately selecting the adjustable parameters in the expression for the standing waves. On the other hand, these standing waves are particularly inappropriate to illustrate certain kind of very common wave phenomena. Suppose for example that you drop a stone on the surface of a lake. You see a circular wave *traveling* away from the point where the stone hits the water. This appears to be a very simple (and beautiful) type of motion, yet it is by no means clear how this simplicity shows up when we write the disturbance as some type of combination of standing waves in the lake. You had a similar experience when you solved Problem 2 in Chapter 5. After a very complicated series expansion, your final solution is so simple that it can be accurately described in words: the initial perturbation splits into two equally shaped pulses that travel without distortion toward the ends of the cord. The simplicity of the **traveling waves** is such that even football fans - not necessarily the most (mathematically) gifted members of our society - have no trouble generating waves that travel back and forth around the stadium. In this chapter, we will show that the solutions to the wave equation can indeed be written as traveling waves. Traveling waves are not new, additional solutions to the wave equation. Rather, the standing wave solutions *can be written* as traveling waves. In many practical cases, the traveling wave form turns out to be much simpler than the standing wave form.

## Traveling wave solution to the wave equation

Consider a cord under tension where at time  $t = 0$  a disturbance of the form  $a(x)$  is produced, as indicated in Figure 1.

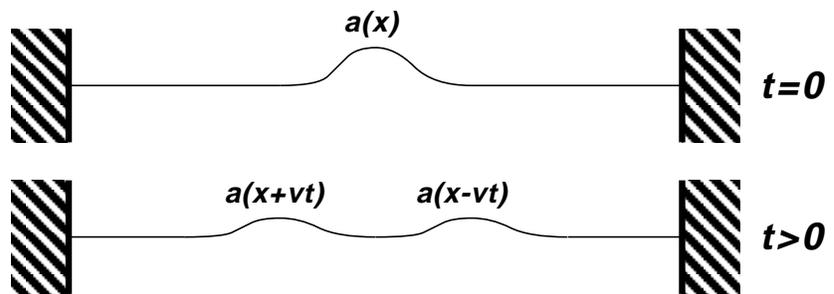


Figure 1 Evolution of a perturbation in a cord under tension.

You have already solved this problem (Problem 2, Chapter 5) within the context of standing waves, starting from the solution

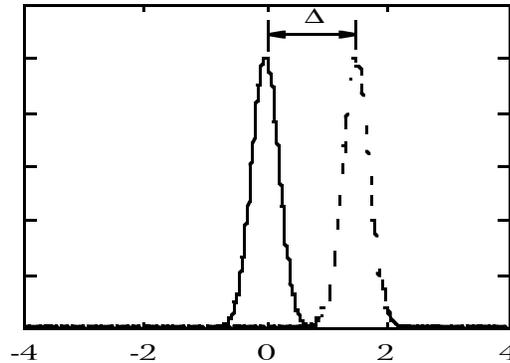
$$\xi(x,t) = A_1 \sin k_1 x \cos(\omega_1 t + \alpha_1) + A_2 \sin k_2 x \cos(\omega_2 t + \alpha_2) + A_3 \sin k_3 x \cos(\omega_3 t + \alpha_3) + \dots \quad (1)$$

The coefficients  $A_1, A_2, A_3, \dots$  and  $\alpha_1, \alpha_2, \alpha_3$  were determined from the condition that  $\xi(x,0) = a(x)$  and  $\partial\xi/\partial t(x,0) = 0$ . (The latter applies if the initial velocity of all points in the cord is zero, as assumed in this particular case). The exact procedure to determine the coefficients is not very complicated but involves Fourier analysis, which is beyond the scope of this course. In the famous Chapter 5 problem, you were given the coefficients corresponding to a rectangular initial shape, which were calculated by your instructor exclusively for you.

The standing wave solution to our problem looks quite complicated. Eq. (1) is an infinite series with an infinite number of terms. On the other hand, we found that the final solution was extremely simple: it looks like two disturbances, similar in shape to the initial one, traveling away from the origin of the perturbation. This suggests that we study the properties of “traveling” functions.

### The mathematics of a traveling function

Consider any function  $a(x)$ . For example, this function could



**Figure 2** A function can be shifted to the right by  $\Delta$  if the argument is changed from  $x$  to  $x-\Delta$ .

be the bell-shaped pulse with a maximum at  $x = 0$  displayed in Fig. 2. We would like to find the function  $f(x)$  which is equal to  $a(x)$  but shifted to the right by an amount  $\Delta$ . This function is shown as a dashed line in Fig. 2. What we want is a function such that its value at  $x+\Delta$  is equal to the value of  $a(x)$  at  $x$ . This can be written as  $f(x+\Delta) = a(x)$ . Hence the value of  $f$  at any point is the value of  $a$  at a point “ $\Delta$  behind.” An obviously equivalent form of writing this is  $f(x) = a(x-\Delta)$ . So the function  $a(x-\Delta)$  is shifted to the right of  $a(x)$  by the required amount  $\Delta$ . It is trivial to show that  $a(x+\Delta)$  is shifted to the *left* of  $a(x)$  by an amount  $\Delta$ . (Of course, *right* and *left* depend on the orientation of the  $x$ -axis. To be more precise, we should say that  $a(x-\Delta)$  is shifted in the direction of the positive  $x$ -axis by an amount  $\Delta$ , and  $a(x+\Delta)$  is shifted in the direction of the negative  $x$ -axis by an amount  $\Delta$ ).

The separation between the two functions is  $\Delta$ . If this separation increases uniformly with time, we can write  $\Delta = vt$ . The quantity  $v$  has units of velocity. It is actually the velocity with which one of the functions moves relative to the other. Hence the function  $a(x-vt)$  “moves” to the right of  $a(x)$  with speed  $v$ . The function  $a(x+vt)$  moves to the left of  $a(x)$  with speed  $v$ .

**Any traveling function is a solution to the wave equation!**

Let us now consider an arbitrary traveling function  $a(x-vt)$ . In view of Fig. 1, we would like to investigate if such a function is a solution to the wave equation

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \xi}{\partial t^2} \quad (2)$$

The usual way to verify that a function is a solution of certain differential equation is to plug the function into the equation. We would like to do this here. However, we have not specified the function  $a(x)$ . It appears that without further details we will not be able to compute the second derivatives in the wave equation. The best we can do is to write down how we would compute the derivatives of  $a(x-ct)$  if we *knew* the function  $a$ . Of course, we would use the chain rule. For an arbitrary function  $f(u)$ , with  $u = u(x,t)$ , we know that the partial derivatives are  $\frac{\partial f}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x}$  and  $\frac{\partial f}{\partial t} = \frac{df}{du} \frac{\partial u}{\partial t}$ . In our particular case  $u(x,t) = x-vt$ . Hence  $\partial u/\partial x = 1$  and  $\partial u/\partial t = -v$ . Substituting into Eq. (2) we find

$$\frac{d^2 a}{du^2} = \frac{v^2}{c^2} \frac{d^2 a}{du^2} \quad (3)$$

For  $v = c$  this is an identity *regardless* of the specific form of the function  $a(x)$ . Hence we don't need to know the form of the function  $a(x)$  to show that it satisfies the wave equation because *any* function of  $x-vt$  will be a solution with the sole condition that  $v = c$ . Thus possible solutions to the wave equation are

$$(x-ct)^3, \sin(x-ct), e^{-(x-ct)^2} \quad (4)$$

and any other function of  $x-ct$  you can think of. Of course, functions of  $x+ct$  are also solutions to the wave equation that travel to the left with speed  $c$ . Both left- and right-traveling functions are solutions to the wave equation because the velocity appears *squared*, so that its sign doesn't matter.

By studying traveling wave solutions, the meaning of the

coefficient  $c$  in the wave equation becomes clear: it is the speed with which the waves travel along the  $x$ -axis, which for this reason is sometimes called the propagation axis. We can now understand the solution we found for Problem 2, Chapter 5, in much simpler terms. We started with an initial condition  $\zeta(x,0) = a(x)$ , so that at later times the solutions must be  $a(x-ct)$  or  $a(x+ct)$ , *i.e.*, the same shape traveling away from the initial location. Since there is no preferred direction in our problem, we obtain the two traveling waves with equal “weight”. In other words, the solution to Problem 2, Chapter 5 was simply  $\zeta(x,t) = \frac{1}{2} a(x-ct) + \frac{1}{2} a(x+ct)$ , which at  $t = 0$  gives the correct initial condition  $\zeta(x,0) = a(x)$ . This solution is valid for any function  $a(x)$ , not just the square pulse discussed in the Chapter 5 problem.

### Traveling waves versus standing waves

The traveling wave solutions to the wave equation are so different from the standing wave solutions we found earlier that it is hard to believe that they are equivalent. Let us first consider the standing wave solution to the problem of Fig. 1. This solution is given by Eq. (1). Any given term in the series is of the form  $\sin kx \cos(\omega t + \alpha) = (\sin kx \cos \omega t) \cos \alpha - (\sin kx \sin \omega t) \sin \alpha$ . Using the rules for the sine and the cosine of sums of angles, we can rewrite the position- and-time dependent parts as

$$\begin{aligned}\sin kx \cos \omega t &= \frac{1}{2} [\sin(kx - \omega t) + \sin(kx + \omega t)] \\ \sin kx \sin \omega t &= \frac{1}{2} [\cos(kx + \omega t) - \cos(kx - \omega t)] \quad (5)\end{aligned}$$

But  $kx - \omega t = k(x - ct)$  and  $kx + \omega t = k(x + ct)$ , which are functions of  $x \pm ct$ . Hence our standing wave solution can be written entirely in terms of traveling waves. Conversely, the traveling waves  $a(x \pm ct)$  in Fig. 1 can be written in the standing-wave form of Eq. (1). This shows that standing waves and traveling waves are merely alternative ways of writing the *same*

solution.

Eq. (5) suggest an interpretation of the standing-wave, normal-mode solution  $\sin kx \cos \omega t$  in terms of traveling waves: a standing wave is the superposition of traveling waves moving in opposite directions. When the traveling wave  $\sin (kx-\omega t)$  reaches the boundary on the right, it is *reflected* back as  $\sin (kx+\omega t)$  The mathematics of traveling waves combined with *reflections* and *transmissions* has many advantages over the mathematics of standing waves or normal modes, as we will see in coming chapters.

For each problem, we must decide whether we want to use traveling wave or standing wave solutions. The following considerations should help in deciding which form is more convenient:

- The initial conditions are easily incorporated into the traveling wave solution. For example, if the initial position is  $\xi(x,0) = a(x)$  and the initial velocity is  $\partial\xi/\partial t(x,0) = 0$ , the solution is simply  $\xi(x,t) = a(x-ct)$ . When the initial velocity is not zero, the solution is slightly more complicated (see homework problem) but still much simpler than determining the coefficients  $A$  and  $\alpha$  for the infinite terms in the standing wave solution Eq. (1).
- When the initial perturbation is concentrated in a region of space, as in Fig. 1, we know that distant parts of the medium (the cord in our example) will not move for a while. That is,  $\xi(x,t) = 0$  until the traveling wave reaches the point  $x$ . On the other hand, each individual term in Eq. (1) will not give zero at that point. Only their sum will be zero. In fact, the coefficients  $A$  and  $\alpha$  in each term are adjusted so that the motion is zero in those distant points. But if we know that the motion is zero at these points, why should we spend time adjusting the coefficients of an infinite series to give a result we know in advance? In fact, if we produce a perturbation at a point in a cord,

we know that whatever happens initially will not depend on what is going on at distant points. We could even change the boundary conditions and nothing would happen until the traveling wave reaches the boundary. Hence traveling waves are the physically best form of the solution when we start with a perturbation concentrated in a small region.

- On the other hand, when the traveling wave reaches the boundary, it may be partially or totally reflected back. If we wait long enough, we may end up with multiple successive reflections and, consequently, many waves traveling back and forth. The traveling wave solution becomes increasingly more complicated. In these cases, it may eventually become simpler to use standing wave solutions. In the example of Fig. 1, it is probably better to use the traveling wave solution for times shorter than the time it takes to the traveling wave to reach the boundary. For times much longer than that, the standing wave approach may be better.
- The above complications are avoided in open media, where there are no boundaries. In these cases, it is always better to use traveling wave solutions. Examples include the propagation of sound and light in open spaces. As a rule of thumb, if there are one or two reflections in the problem it is better to use traveling wave solutions. If there are multiple reflections, standing waves solutions may be simpler. When we discuss traveling waves in open media, we will often consider the case of waves given by  $\xi(x,t) = A \sin(kx - \omega t)$ . However, our results will be general because any traveling wave  $a(x-ct)$  can be written as a sum of sines and cosines of  $(x-ct)$ .

## Traveling waves transmit energy

Let us consider the traveling waves set up in Fig. 1. You notice that parts of the cord do not move for a while. Clearly, these parts have zero energy. When those parts are reached by the traveling waves, however, they start vibrating,

so that they acquire some energy. It is quite apparent that the traveling wave carries energy with it. On the other hand, if we consider a standing wave, each part of the system executes a simple harmonic motion, so that its energy remains constant. Hence there is a fundamental difference between a traveling wave and a standing wave: the former carries energy, the standing wave doesn't. This can be better understood if we recall that a standing wave can be written as a sum of traveling waves moving in opposite directions, so that the net transfer of energy in a given direction is zero.

Let us now try to put these ideas in mathematical form. If we go back to the discrete chain, we can write its total energy as the sum of the kinetic energies of all its masses plus the potential energy ( $1/2Kx^2$ ) associated with each spring:

$$E = \sum_{n=1}^N \frac{1}{2} m v_n^2 + \sum_{n=0}^N \frac{1}{2} K (x_{n+1} - x_n)^2 \quad (6)$$

In the continuum limit, the displacement is  $\xi(x,t)$  and the velocity is given by  $\partial\xi/\partial t$ . In terms of these quantities, Eq. 6 can be written as

$$E = \sum_{n=1}^N \frac{1}{2} m \left( \frac{\partial \xi}{\partial t}(na,t) \right)^2 + \sum_{n=0}^N \frac{1}{2} K [\xi(na+a,t) - \xi(na,t)]^2 \quad (7)$$

We can now complete the transition to the continuous regime by equating  $a = dx$  and replacing  $m$  by  $\rho dx$ . Thus the sum becomes an integral and we obtain

$$E = \frac{1}{2} \rho \int \left( \frac{\partial \xi}{\partial t} \right)^2 dx + \frac{1}{2} K a \int \left( \frac{\partial \xi}{\partial x} \right)^2 dx \quad (8)$$

But in the previous chapter we showed that  $Ka/\rho = c^2$ , so that we can finally write this result as

$$E = \frac{1}{2} \rho \int \left[ \left( \frac{\partial \xi}{\partial t} \right)^2 + c^2 \left( \frac{\partial \xi}{\partial x} \right)^2 \right] dx \quad (9)$$

Note that in this expression there is almost no reference to the original problem (longitudinal vibrations in a chain). All quantities in the expression appear in the wave equation

itself, which is valid for all types of vibrations. The only exception is the density  $\rho$ , but we have noticed earlier that a density of some sort always appears in the expression for the wave speed. We can therefore speculate that Eq. (9) is a general expression valid for all wave phenomena. In each individual case, the formula must be adapted to the problem. In Eq. (9), which we derived for a linear chain of masses, the density  $\rho$  is a *linear* density (mass per unit length) and the integration is along the length of the chain. If we were to apply this equation to sound waves in a gas, we would use  $dV$  rather than  $dx$  (*i.e.*, we would integrate over the volume of the gas) and the density  $\rho$  would be a *volume* density (mass per unit volume). In all case, the limits of integration define the section of the cord (or the volume of gas) whose energy we want to compute.

Let us now compute expression (9) for the case of a standing wave of the form  $\xi(x,t) = A \sin kx \cos \omega t$ . Substituting the appropriate derivatives and using  $\omega = ck$ , we find

$$E = \frac{1}{2} \rho A^2 \omega^2 \sin^2 \omega t \int \sin^2 kx \, dx + \frac{1}{2} \rho A^2 \omega^2 \cos^2 \omega t \int \cos^2 kx \, dx \quad (10)$$

If we take the time-average of this expression, and use

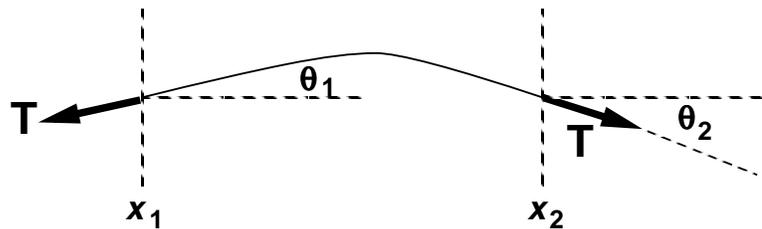
$\overline{\sin^2 \omega t} = \overline{\cos^2 \omega t} = \frac{1}{2}$  this expression reduces to

$$E = \frac{1}{4} \rho \omega^2 A^2 \int dx = \int \left( \frac{1}{4} \rho \omega^2 A^2 \right) dx \quad (11)$$

The quantity in brackets is the **energy density**  $E$ , since its integral over a region of the medium gives the energy of that region. Note the familiar aspect of the energy density. If you have a single harmonic oscillator, its energy is  $E = \frac{1}{2} K A^2$ . This can be written as  $\frac{1}{2} m \omega^2 A^2$ . This involves the same quantities as the expression for the energy density  $E$ , except that instead of the mass we use the mass density and integrate over the length (or the volume) of the vibrating medium. The expression we have derived for the energy density is very general and not limited to mass-and-springs systems. In all wave phenomena, the energy density is

proportional to the *square* of the amplitude of the vibration. As discussed above, the integration over  $dx$  means an integration over the region of the medium whose energy we want to calculate. It's an integration along a line for a cord, but becomes an integration over a three dimensional volume of waves such as sound. In this case, the density  $\rho$  is a volume density. A good mnemonic rule is that the product  $\rho \times dx$  should have units of mass: if  $dx$  is a length, then  $\rho$  is mass per unit length. If  $dx$  actually refers to a volume,  $\rho$  is mass per unit volume.

Suppose now that we have a traveling wave of the form  $\xi(x,t) = A \sin(kx - \omega t)$ . Substituting into Eq. (9), we find (homework problem) that the energy density is given by an expression similar to the standing wave case, namely  $E = \frac{1}{2}\rho\omega^2 A^2$ . There is, however an important difference between the two cases. To understand this difference, let us compute how the energy *changes* as a function of time. This quantity is given by the derivative  $dE/dt$  of Eq. (9). For definiteness, let us consider a cord under tension  $T$  and let us compute the energy of a region of the cord between  $x_1$  and  $x_2$ .



**Figure 3** A snapshot of a vibrating cord under tension.

Differentiating Eq, (9), we obtain

$$\begin{aligned} \frac{d}{dt} E(x_1, x_2) &= \\ &= \frac{1}{2} \rho \int_{x_1}^{x_2} \left[ 2 \left( \frac{\partial \xi}{\partial t} \right) \left( \frac{\partial^2 \xi}{\partial t^2} \right) + 2c^2 \left( \frac{\partial \xi}{\partial x} \right) \left( \frac{\partial^2 \xi}{\partial x \partial t} \right) \right] dx \end{aligned} \quad (12)$$

Using the wave equation, this becomes

$$\begin{aligned}
\frac{d}{dt} E(x_1, x_2) &= \\
&= \rho c^2 \int_{x_1}^{x_2} \left[ \left( \frac{\partial \xi}{\partial t} \right) \left( \frac{\partial^2 \xi}{\partial x^2} \right) + c^2 \left( \frac{\partial \xi}{\partial x} \right) \left( \frac{\partial^2 \xi}{\partial x \partial t} \right) \right] dx \\
&= \rho c^2 \int_{x_1}^{x_2} \left[ \frac{\partial}{\partial x} \left( \frac{\partial \xi}{\partial t} \frac{\partial \xi}{\partial x} \right) \right] dx \\
&= \rho c^2 \left. \frac{\partial \xi}{\partial t} \frac{\partial \xi}{\partial x} \right|_{x_2}^{x_1} \\
&= \rho c^2 \frac{\partial \xi}{\partial t} (x_2, t) \frac{\partial \xi}{\partial x} (x_2, t) - \rho c^2 \frac{\partial \xi}{\partial t} (x_1, t) \frac{\partial \xi}{\partial x} (x_1, t)
\end{aligned} \tag{13}$$

This expression has a very simple physical interpretation. In a cord, for example,  $\rho c^2 = T$ . If we call  $F(1)$  the force acting on our cord segment at  $x_1$  and  $F(2)$  the force acting on our cord segment at  $x_2$ , then for small angles (as implied by the validity of the wave equation)  $T \frac{\partial \xi}{\partial x} (x_1, t) = T \tan \theta_1 \approx T \sin \theta_1 = -F_y(1)$ . On the other hand,  $T \frac{\partial \xi}{\partial x} (x_2, t) = T \tan \theta_2 \approx T \sin \theta_2 = F_y(2)$ . The time derivative is, of course  $\partial \xi / \partial t = v_y$ , the velocity in the transverse direction (remember that in this problem we assume that the motion the direction perpendicular to the length of the cord). We can therefore write

$$\frac{d}{dt} E(x_1, x_2) = F_y(1)v_y(1) + F_y(2)v_y(2) \tag{14}$$

This obviously means that the change (per unit time) in the total energy of the segment between  $x_1$  and  $x_2$  is equal to the work (per unit time) done on the segment at  $x_1$  plus the work (per unit time) done on the segment at  $x_2$ .

Let us now compute the expression in Eq. (13) for the case of standing and traveling waves. For standing waves of the form  $\xi(x, t) = A \sin kx \cos \omega t$ , we find

$$\frac{d}{dt} E(x_1, x_2) = T\omega k [\cos kx_1 \sin \omega t - \cos kx_2 \sin \omega t] \tag{15}$$

If we average this with respect to time, we find that *both* terms on the right-hand side of this equation give zero, so that on average no work is done at either end and the total energy remains constant. This is consistent with our finding that the average energy density  $E$  is not a function of time. On the other hand, if we apply Eq. (13) to a traveling wave of the form  $\xi(x,t) = A \sin(kx - \omega t)$  we obtain

$$\frac{d}{dt} E(x_1, x_2) = -T\omega k A^2 \cos^2(kx_2 - \omega t) + T\omega k A^2 \cos^2(kx_1 - \omega t) \quad (16)$$

Averaging over time, and using  $T = \rho c^2$  and  $\omega = ck$ , we find

$$\overline{\frac{d}{dt} E(x_1, x_2)} = -\frac{1}{2} \rho c \omega^2 A^2 + \frac{1}{2} \rho c \omega^2 A^2 \quad (17)$$

which of course gives zero, as in the case of the standing wave. However, there is a fundamental difference. While in the case of the standing wave the average work at either end of the segment was zero, here we have equal and opposite non-zero quantities. This means that energy is *entering* the segment at  $x_1$  and *leaving* the segment at  $x_2$ . Notice that we obtained this result for a traveling wave of the form  $\xi(x,t) = A \sin(kx - \omega t)$ , which travels from left to right (assuming that the positive  $x$ -axis is to the right). Had we chosen a traveling wave in the opposite direction, you can easily show that the two terms in Eq. (17) reverse their sign, so that energy would *enter* the segment at  $x_2$  and leave it at  $x_1$ .

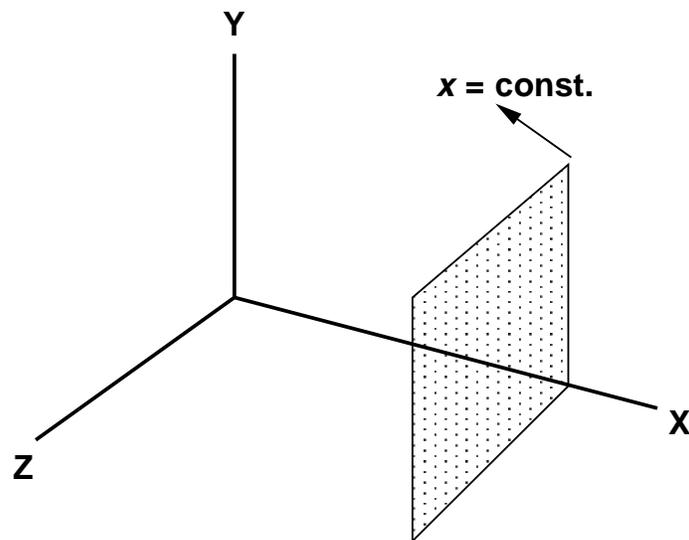
Clearly, traveling waves carry energy in the direction they travel. The reason why the average energy remains constant in our example is that we chose a special kind of traveling wave of the form  $\sin(kx \pm \omega t)$ . This wave is a continuous train of pulses traveling either to the right or to the left. Had we chosen a different type of traveling wave, such as the pulse depicted in Fig. 1, we would have the following situation: when the pulse reaches  $x_1$ , there would be positive work done on the segment, so that its energy would increase. Since the pulse does not affect the cord at  $x_2$  initially, no

work would be done at this point. Eventually, the pulse would reach  $x_2$ . Negative work would be done at this point until the energy of the segment drops again to zero as the pulse moves further to the right.

Standing waves can always be written as the sum of traveling waves moving in opposite directions, so that the energy transfer at the ends of the segments always cancel out.

### Plane waves and the definition of intensity

When we discussed waves in columns of air or in bulk solid objects, we considered this as a one-dimensional problem. Of course, we are dealing with three dimensional objects. The reason we were able to ignore the  $y$  and  $z$ -directions is that the displacement function is independent of these variables. In other words, we can write  $\xi(x,y,z,t) = A \sin(kx - \omega t)$  for a traveling wave in the  $+x$  direction.



**Figure 4** The geometry of a plane wave.

This means that all points in a plane perpendicular to the  $x$  axis, as indicated in Fig. 4, execute exactly the same type of motion. This is called a **plane wave**.

The formulas derived in the previous paragraph for the energy density and the rate at which energy changes are still valid for these waves if we understand  $\rho$  as a volume density (mass per unit volume) and use  $dV = dx dy dz$  instead of

simply  $dx$ . For example, Eq. (9) becomes

$$E = \frac{1}{2} \rho \int \left[ \left( \frac{\partial \xi}{\partial t} \right)^2 + c^2 \left( \frac{\partial \xi}{\partial x} \right)^2 \right] dx \, dy \, dz \quad (18)$$

where  $\rho$  is the volume density. Since nothing depends on  $y$  or  $z$ , the integral over these variables simply gives the area  $S$  of the wave front. For example, if we apply this to sound waves in a tube,  $S$  would be the cross-sectional area of the tube. We thus obtain

$$E = \frac{1}{2} \rho S \int \left[ \left( \frac{\partial \xi}{\partial t} \right)^2 + c^2 \left( \frac{\partial \xi}{\partial x} \right)^2 \right] dx \quad (19)$$

Hence all formulas derived in the previous section are valid if we make the change  $\rho$  (linear)  $\rightarrow \rho$  (volume)  $\times S$ . For example, if a plane wave of the form  $\xi(x,y,z,t) = A \sin(kx - \omega t)$  enters a volume, the energy per unit time that enters that volume is  $\frac{1}{2} \rho S c \omega^2 A^2$ .

In the case of a cord, the change per unit time in the energy of a segment between  $x_1$  and  $x_2$  is governed by the amount of energy per unit time that enters or leaves the segment at the two *points*  $x_1$  and  $x_2$ . In the case of a plane wave in three dimensions, the expression  $\frac{1}{2} \rho S c \omega^2 A^2$  gives the energy that enters or leaves the volume through the *planes* defined by  $x_1$  and  $x_2$ . Sometimes, these planes are very large and we are not interested in the total energy per unit time that moves through the entire plane but in the total energy per unit time *per unit area*. Suppose for example that sound from a distant source reaches your ears. (Sound from a point source is a spherical rather than a plane wave. We will discuss spherical waves in later chapters. However, far from the center the surface of a sphere is very flat and can often be approximated by a plane.) You are not interested in the amount of energy per unit time that flows from the source but in the energy that reaches your ears. If the energy per unit area is large, you will hear a strong sound. If the energy is spread over a large plane, only a tiny fraction will enter your ears and you will hear a feeble sound. Thus the quantity that is useful to characterize the intensity of the sound is

the flow of energy per unit area per unit time. This is called **intensity** ( $I$ ) and has units of  $\text{W}/\text{m}^2$ .

For a plane wave of the form  $\xi(x,y,z,t) = A \sin(kx - \omega t)$ , we found above that the energy per unit time that flows through a plane perpendicular to the direction of propagation is  $\frac{1}{2}\rho S c \omega^2 A^2$ , where  $S$  is the area of the plane. Hence the intensity is given by

$$I = \frac{1}{2} \rho v c \omega^2 A^2 \quad (20)$$

Notice that this can be written as

$$I = cE \quad (21)$$

where  $E$  is the energy density (energy per unit volume) given by  $E = \frac{1}{2}\rho v \omega^2 A^2$

### **Detector engineering by natural selection**

Humans and other similar animals carry very sophisticated wave detectors, such as ears and eyes. Throughout the course of evolution, it became apparent that it was very beneficial for the individuals of a certain species to be able to detect sound and light over a very wide range of intensities. For example, the intensity associated with a whisper is  $I = 10^{-11} \text{ W}/\text{m}^2$ , whereas the typical sound intensity in the first rows of a rock concert is  $1 \text{ W}/\text{m}^2$ . Similar requirements apply to light detection. Designing a detector over such a wide range is a monumental task. If the response of the detector to the signal were linear, we would have to deal with a system able to respond to 11 orders of magnitude in intensity. Such a “dynamical range” is virtually unattainable. For example, the state-of-the-art CCD in your camcorder may be able to detect 3 orders of magnitude of light intensity. In order to overcome this problem, nature discovered, all by itself, the logarithmic function.

If the response of the detector is logarithmic rather than linear, then eleven orders of magnitude become a factor of 11. We lose the ability to compare intensities of sound in a linear scale, but we are able to detect all sounds. Because the psychological intensity does turn out to be roughly propor-

tional to the logarithm of the intensity, it is useful to introduce a quantity that characterizes this psychological intensity. This quantity is called **intensity level  $B$** , and its unit is the **decibel (dB)**.

$$B = 10 \log \frac{I}{I_0} \quad (22)$$

where  $I_0$  is conventionally taken as  $10^{-12}$  W/m<sup>2</sup>. According to this definition, a rock concert has an intensity level of 120 dB and a whisper an intensity level of 10 dB. The ratio of 12 represents well our perception that the rock concert feels about 10 times stronger than a whisper.

## Doppler effect

The Doppler effect is the change in the detected frequency of a wave produced by either the motion of the source of the wave or the motion of the observer of the wave. In all wave phenomena, the frequency  $\nu$  is given by

$$\nu = \frac{c}{\lambda} \quad (23)$$

A change in frequency thus implies a change in the wavelength  $\lambda$  or a change in the speed  $c$  of the wave. We will consider the two cases.

### Case I: Source in motion, observer fixed

This case is illustrated in Figure 5. It is quite clear that in this case the speed of the wave does not change. This speed is a property of the medium, and does not depend on the state of motion of the source. Suppose for example that you hit an iron bar with a hammer. Sound will propagate at the speed of sound in iron. If you hit the bar while you are walking, the fact that you are moving is completely irrelevant from the point of view of the bar. What you do after or before the instant you hit the bar cannot affect the speed of propagation of the wave in the bar.



Figure 5 Doppler effect when source moves

On the other hand, the wavelength of the waves received by the observer will change. If the source doesn't move, the separation between maxima is  $\lambda$ . When the source moves toward the observer, however, this separation becomes smaller because the source moved a certain distance toward the observer from the time it emitted the  $n$ th wave to the time it emitted the  $(n+1)$ th wave. The time between successive maxima is precisely the period  $T$ , so that during that time the source travels a distance  $v_s T$ . If we use primed quantities to indicate what the observer measures, we can write for this case:

$$\begin{aligned} c' &= c \\ \lambda' &= \lambda - v_s T \end{aligned} \quad (24)$$

Although we have derived this for the source moving toward the observer, Eq. (24) also gives the right answer when the source moves away from the observer, provided we use a negative value for  $v_s$ . Combining Eq. (23) and Eq. (24), we obtain

$$v' = \frac{c'}{\lambda'} = \frac{c}{\lambda - v_s T} = v \frac{c}{c - v_s} \quad (25)$$

where  $v$  is the frequency detected when the source doesn't move. Notice that the frequency increases when the observer is approached and decreases when the source moves away from the observer, in agreement with our daily experience with ambulances. Notice also that we get an unphysical negative result whenever  $v_s$  is larger than  $c$ . This is because the source will reach the observer before its own waves if its speed is higher than the wave speed. In this case, all wave fronts accumulate in front of the source and produce an

explosion-like sound known as a **shock wave**. This occurs frequently with supersonic aircraft.

**Case II: Source fixed, observer moves**

This case is illustrated in Fig. (6).



**Figure 6** Doppler effect when the observer moves

The observer now sees no change in wavelength, because the separation between maxima does not depend of the state of motion of the observer. This basically says that the length of an object is independent of the motion of the observer, a basic fact of classical, non-relativistic mechanics. On the other hand, the waves are approaching the observer at an increased velocity, given by the sum of the two velocities. (If a car moves North-South at 40 miles/hr and a second car moves South-North at 60 miles/hour, they approach each other at a speed of 100 miles/hr). We thus have

$$\begin{aligned} c' &= c + v_o \\ \lambda' &= \lambda \end{aligned} \quad (26)$$

where  $v_o$  is the velocity of the observer. Combining this with Eq. (23), we obtain

$$v' = \frac{c'}{\lambda'} = \frac{c + v_o}{\lambda} = v \frac{c + v_o}{c} \quad (27)$$

where  $v$  is the frequency detected when the observer doesn't move. Again we note that this expression is automatically correct when the observer moves away from the source, if we take  $v_o$  as negative. Notice that when the source approaches the observer or the observer approaches the source the frequency increases, but the expressions that give the increase in frequency, Eqs. (25) and (27), are not mathematically identical. This is because the physics is different: in one case the wavelength is reduced, in the other case the

speed increases. When  $v_S$  and  $v_O$  are much smaller than the speed of the wave, however, the two expressions give very similar results (see homework problem).

When both source and observer move, we have a simultaneous change in wavelength and speed. It is then trivial to show, following the reasoning in the previous paragraphs, that the detected frequency is given by

$$v' = v \frac{c + v_O}{c - v_S} \quad (28)$$

The sign convention can be summarized as follows:  $v_O$  is positive if the observer moves toward the source, negative otherwise. On the other hand,  $v_S$  is positive if the source moves toward the observer, negative otherwise. Hence “toward” = +, “away” = -. However, you should not rely on these conventions to solve problems. The best way to avoid mistakes is to rederive the equations every time you need them. You can afford the extra time because the ideas behind the Doppler effect are very simple.

You may wonder what happens if the medium itself is moving. For example, when you study the Doppler effect for sound waves, how are the above formulas modified if wind is present? Our derivation assumed implicitly that the medium doesn't move. In other words, our derivation was made from a reference frame that is “attached” to the medium, so that the velocity of the medium is always zero. If the medium is moving,  $v_O$  and  $v_S$  refer to the velocities of the observer and the source *relative to the medium*. For example, if the source is moving to the east at 30 miles an hour and the wind blows west-east at 10 miles/hr, then the value of  $v_S$  that enters in the above formulas is 20 miles/hr.

## Problems

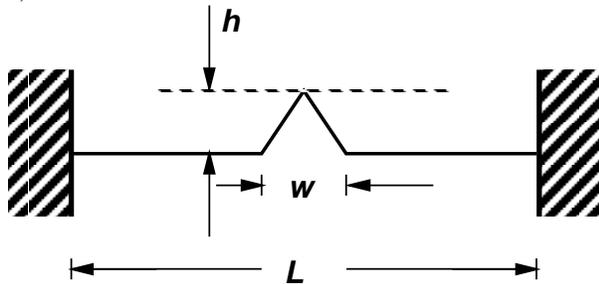
1. Consider the function  $a(x) = e^{-\frac{x^2}{2\sigma^2}}$ , known as Gaussian. Take  $\sigma = 1$ . Plot  $a(x-vt)$  at five different times, using  $v = 2\text{m/s}$ .

2. Verify (plug the proposed solution into the wave equation and compute the corresponding derivatives) that for initial conditions  $\xi(x,0) = a(x)$  and  $\frac{\partial \xi}{\partial t}(x,0) = b(x)$ , the traveling wave solution to the wave equation is  $\frac{1}{2} [a(x-ct) + a(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} b(s) ds$

3. Suppose that the initial perturbation on a cord under tension  $T$  is as given in the figure. Assume that the initial velocity is zero.

a) Give analytical expressions for the evolution of the cord as a function of time. For how long is your solution correct?

b) Plot the solution at several times.



4. (Alonso 28.2) The equation of a certain wave is  $\xi = 10 \sin 2\pi(2x-100t)$ , where  $x$  is in meters and  $t$  is in seconds. Find

a) the amplitude,

b) the frequency,

c) the velocity of propagation of the wave.

e) Draw the wave, showing the amplitude and the wavelength.

5. (Alonso 28.3) Given the wave  $\xi = 2 \sin 2\pi(0.5x - 10t)$ , where  $t$  is in seconds and  $x$  is in meters,

a) plot  $\xi$  over several wavelengths for  $t = 0$  and  $t =$

0.025 s.

b) Repeat the problem for  $\xi = 2 \sin 2\pi(0.5x + 10t)$  and compare results.

6. (Alonso 28.5) Given the equation for a wave on a string  $\xi = 0.03 \sin 2\pi(3x - 2t)$ , where  $t$  is in seconds and  $x$  and  $\xi$  are in meters,

a) at  $t=0$ , what are the values of the displacement at  $x = 0, 0.1\text{m}, 0.2\text{ m},$  and  $0.3\text{ m}$ ?

b) At  $x = 0.1\text{ m}$ , what are the values of the displacement at  $t = 0, 0.1\text{s},$  and  $0.2\text{ s}$ ?

c) What is the equation for the velocity of oscillation of the particles of the string?

d) What is the maximum velocity of oscillation?

e) What is the velocity of propagation of the wave?

7. (Alonso 28.6) Consider longitudinal waves along a rod and assume that the deformation at each point is  $\xi = \xi_0 \sin 2\pi(x/\lambda - t/T)$ .

a) Derive an expression for the force along the rod.

b) Show that the  $\xi$  and  $F$  waves have a phase difference of one-quarter wavelength.

c) Plot  $\xi$  and  $F$  against  $x$ , on the same set of axes, at a given time.

8. Because the Young modulus and the shear modulus of virtually all materials are not equal, transverse and longitudinal waves travel with different speeds.

a) Develop a method to determine the distance from the epicenter of an earthquake based on the above phenomenon. Find realistic numbers for a California earthquake, assuming you are in Phoenix. Ask somebody in geology or go to the library to find appropriate values for  $Y$  and  $G$ .

b) Suppose there is a huge oil deposit under the Colorado river bed. How would the presence of such a deposit affect your measurements in part a)?

9. Show that for traveling waves of the form  $\xi(x,t) = A \sin(kx - \omega t)$ , the energy density  $E$  is given by  $\frac{1}{2}\rho\omega^2 A^2$ , the same expression obtained in the text for standing waves.

10. (Alonso 28.20) A thin steel rod is forced to transmit longitudinal waves by means of an oscillator coupled to one end. The rod has a

diameter of  $4 \times 10^{-3}$  m. The amplitude of the oscillations is  $10^{-4}$  m and the frequency is 10 oscillations per second. Find

- the equation of the waves along the rod,
- the energy per unit volume of the wave,
- the average energy flow per unit time across any section of the rod and
- the power required to drive the oscillator.

11. Consider a column of gas enclosed in a tube positioned along the  $x$ -axis. The mass of a volume element at equilibrium is  $\rho_0 A dx$ , where  $A$  is the cross-sectional area and  $\rho_0$  the equilibrium density. In the presence of a sound wave, the volume will be distorted to  $A(dx + d\xi)$ , where  $\xi(x, t)$  gives the displacement of the air and satisfies the wave equation.

a) Show, by requiring mass conservation, that the density in the presence of a wave is given by

$$\rho = \frac{\rho_0}{1 + \frac{\partial \xi}{\partial x}}.$$

b) Assuming  $\partial \xi / \partial x$  to be small, show that  $\rho - \rho_0 = -\rho_0 (\partial \xi / \partial x)$ .

c) Assume that the pressure in the gas is a function of the density and expand the pressure as a Taylor series in the density keeping only the terms linear in the density. Show that

$$p = p_0 + \kappa \left( \frac{\rho - \rho_0}{\rho_0} \right),$$

where  $\kappa = \rho_0 (dp/d\rho)_0$  is the so-called elasticity modulus and  $p_0$  is the equilibrium pressure.

d) Show that

$$p = p_0 - \kappa \frac{\partial \xi}{\partial x}.$$

e) Show that if there is a displacement wave with amplitude  $\xi_0$ , the corresponding pressure wave has an amplitude  $c \rho_0 \omega \xi_0$ . (Hint: compute  $\kappa$  using the adiabatic condition  $p = C\rho^\gamma$  and use the expression derived in Chapter 5 for the speed of sound).

12. (Alonso 28.21) The faintest sound that can be heard has a pressure amplitude of about  $2 \times 10^{-5}$  N m<sup>-2</sup>, and the loudest that can be heard without pain

has a pressure amplitude of about 28 N m<sup>-2</sup>. Determine, in each case,

- the amplitude of the oscillations if the frequency is 500 Hz. Assume an air density of 1.29 kg m<sup>-3</sup> and a velocity of sound of 345 ms<sup>-1</sup>.
- The intensity of the sound both in Wm<sup>-2</sup> and dB.

13. Compare the Doppler effect for the cases where the source moves toward the observer with speed  $v$  or the observer moves toward the source with the same speed.

- Using realistic numerical values for sound waves, show that the change in frequency is not the same for the two cases.
- Show that you do obtain the same answer for the two cases in the limit  $v/c \ll 1$ .

14. (Alonso 28.26) A sound source has a frequency of 10<sup>3</sup> Hz and moves at 30 ms<sup>-1</sup> relative to the air. Assuming that the velocity of sound relative to air is 340 m s<sup>-1</sup>, find the effective wavelength and the frequency recorded by an observer at rest relative to the air who sees the source

- moving away and
- moving toward the observer.

15. (CW) Two students decide to use their saxophones to measure the speed of a car. The saxophones are both tuned to a frequency of 262 Hz, and each student plays this note while one rides in the car and the other remains stationary. When the car approaches the stationary student, 6.00 beats/s are heard by this student. What is the speed of the car? (Ans.: 7.68 m/s).