

## **Chapter 5**

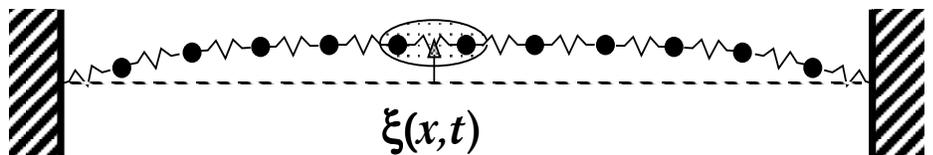
**Wave equation. Waves  
in strings. Sound**

## Introduction

The method we developed in the previous chapter makes it possible to study the vibrations of any real system. Let us discuss for example a guitar's string. We can consider the string as a chain of masses. The "masses" of the chain correspond to the atoms in the string. Because there are zillions of atoms in a string, there is a correspondingly huge number of normal modes. However, the modes we excite when we play the guitar are a few modes for which the wavelength is of the order of the size of the guitar, say 1 m. This must be contrasted with the separation between atoms, which is of the order of 1 Å, or  $10^{-10}$  m. If you look at the normal mode solutions with long wavelengths in systems with very large  $N$  you notice that nearby masses tend to have similar displacements. In the case of the string, with the enormous discrepancy between atomic separations and "music-like" wavelengths, **it is apparent that a huge number of atoms near a certain point  $x$  in the string are undergoing virtually identical displacements.** We can thus make a "continuum" approximation. Rather than labeling the atoms one by one, we assume that there is a function  $\xi(x,t)$  that gives the displacement of the atoms near position  $x$  at time  $t$ . Since the atoms are so closely spaced compared with the wavelength of the "music-like" normal modes, we can assume that the function  $\xi(x,t)$  is continuous and differentiable. This leads to the famous wave equation.

### The continuum approximation

Let us consider the many-mass chain depicted in Fig. 1



**Figure 1** The displacement pattern for the transverse oscillations of a chain of masses for one of its normal modes.

The displacement of the masses at position  $x$  is denoted by  $\xi(x,t)$ . Notice that in general there are masses only at certain values of  $x$  (given by  $x = na$ ). However, when we are consid-

ering a real guitar string, where the separation of atoms is so small compared with the length of the string, the possible values of  $x$  are so closely spaced that we can assume that  $x$  is a continuous variable. We can thus differentiate the function  $\xi(x,t)$  not only with respect to  $t$  but also with respect to  $x$ .

If the displacement of the particle is given by  $\xi(x,t)$ , its velocity is the time derivative of this quantity,  $\frac{\partial \xi}{\partial t}$ . Notice that we use the *partial* derivative notation because  $\xi$  is a function of  $x$  and  $t$  and we mean differentiation with respect to  $t$  with  $x$  fixed. Similarly, the acceleration of the particle is given by  $\frac{\partial^2 \xi}{\partial t^2}$ . We can thus rewrite Eq. (2) in Chapter 4 in terms of  $\frac{\partial \xi}{\partial t}$ . We obtain

$$m \frac{\partial^2 \xi(x,t)}{\partial t^2} = T \left[ \frac{\xi(x+a,t) - \xi(x,t)}{a} + \frac{\xi(x-a,t) - \xi(x,t)}{a} \right] \quad (1)$$

Since  $a$  is so small, the two terms on the right hand side are virtually identical to the first derivatives of  $\xi(x,t)$ . The first term is the derivative at  $x$ , and the second term is minus the derivative at  $x-a$ :

$$m \frac{\partial^2 \xi(x,t)}{\partial t^2} = T \left[ \frac{\partial \xi}{\partial x}(x,t) - \frac{\partial \xi}{\partial x}(x-a,t) \right]. \quad (2)$$

If we divide by  $a$  on the two sides, the right hand side is by definition (for very small  $a$  as in our case) the *second* derivative of  $\xi(x,t)$  evaluated at  $x-a$ :

$$\rho \frac{\partial^2 \xi(x,t)}{\partial t^2} = T \frac{\partial^2 \xi(x-a,t)}{\partial x^2}, \quad (3)$$

where  $\rho = m/a$  is the linear mass density. For a string with total mass  $M$  and length  $L$ ,  $\rho = M/L$ . In the limit  $a \rightarrow 0$ , this becomes

$$\frac{\partial^2 \xi(x,t)}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 \xi(x,t)}{\partial x^2} \quad (4)$$

which can be written as

$$\frac{\partial^2 \xi(x,t)}{\partial t^2} = c^2 \frac{\partial^2 \xi(x,t)}{\partial x^2} \quad (5)$$

This is the famous wave equation. Notice that we have derived it solely on the basis of Newton's laws combined with some type of restoring force. However, while Newton's laws are universally valid, the wave equation is only valid in the "long wavelength limit", which includes only those normal modes for which the wavelength  $\lambda$  is much longer than the distance between particles. So we cannot obtain all normal modes of a system from the wave equation, because some of those normal modes do not satisfy the conditions upon which the wave equation was derived. Still, the wave equation is extraordinarily useful and important because in many practical applications the normal modes of interest are precisely the "long wavelength" modes. The quantity  $c$  has units of speed and is called the **wave speed** for reasons that will become apparent later. For the transverse oscillations of the linear chain,  $c = \sqrt{\frac{T}{\rho}}$ . Many other processes satisfy a wave equation identical to Eq. (5). The expression for  $c$  changes according to the physical process under consideration, but it is always given by (the square root of) a quantity that measures the strength of the restoring forces divided by a quantity that measures the inertia of the system.

### **Standing-wave solutions to the wave equation**

Since the wave equation was derived as the continuum limit of the equations of motion for a chain of atoms, it is reasonable to expect that the solution to the wave equation will be given by the continuum limit of the "exact" solution, which is given by Eq. (13), Chapter 4. (For the transverse oscillations, we write  $y_n$  instead of  $x_n$  in Eq. (4), Chapter 4). In the continuum limit  $na \rightarrow x$ , so that we can try a solution of the form

$$\xi(x,t) = A \sin kx \cos(\omega t + \alpha) \quad (6)$$

If we substitute this into Eq. (5) we find that this expression

is indeed a solution to the wave equation provided that  $\omega^2 = c^2k^2$  or  $\omega = ck$ . Using  $\omega = 2\pi\nu$  and  $\lambda = 2\pi/k$ , this can also be written as

$$c = \lambda\nu, \quad (7)$$

which is easier to remember because the units of the quantities involved are very obvious.

The result  $\omega = ck$  was already derived as a long-wavelength limit of the exact solution in Chapter 4. For the continuum approximation to be valid  $\lambda \gg a$ . This is equivalent to  $ka \ll 1$ . In this case,  $\omega = \sqrt{\frac{4T}{am}} \sin \frac{ka}{2} \approx \sqrt{\frac{4T}{am}} \frac{ka}{2} = \sqrt{\frac{T}{\rho}} k$ . So  $c = \sqrt{\frac{T}{\rho}}$ , which is exactly the expression for the speed of the wave derived from the wave equation for the case of transverse oscillations in a string.

### **Change of notation in the discrete-to-continuum limit**

While the transition from a discrete system of many masses to a continuum of masses is conceptually straightforward, some subtle changes in notation make it hard to understand the physical meaning of certain expressions.

When we discuss a system of  $N$  masses connected by springs, the displacement of mass  $n$  is given by a function  $x_n(t)$  (or  $y_n(t)$  if we are discussing the transverse oscillations of the system). Here the variable  $x$  gives the instantaneous position of the particle, so that its velocity is given by  $dx_n/dt$  and its acceleration by  $d^2x_n/dt^2$ . On the other hand, the meaning of  $x$  when we make the continuum approximation is *completely different*. Here  $x$  is not the instantaneous position of the particle, which is given by  $\xi(x,t)$ , but a label that indicates the position of the particle along the string. Hence  $x$  is the continuum equivalent of  $n$ . In words,  $\xi(x,t)$  is the displacement, at time  $t$ , of a particle whose equilibrium position is  $x$ . When we work in the continuum limit, the expression  $dx/dt$  is *not* the velocity of any particle. In fact,  $x$  is not a function of time at all, it is just a label, the same way  $n$  is a label in the discrete case and we never use the derivative

$dn/dt$ . The correct expression for the velocity of the particle in the continuum limit is  $v = \frac{\partial \xi}{\partial t}$ ; the acceleration is given by  $a = \frac{\partial^2 \xi}{\partial t^2}$ .

### Standing waves in a guitar string

The standing waves in a guitar string with both ends fixed are given by Eq. (6). The possible values of  $k$  are given in Eq. (18), Chapter 4, with  $(N+1)a = L$ , where  $L$  is the length of the string. Hence the expression for the possible wavenumbers become

$$k_1 = \frac{\pi}{L}, \quad k_2 = \frac{2\pi}{L}, \quad k_3 = \frac{3\pi}{L}, \quad \dots \quad (8)$$

This expression could have been derived directly from the continuum solution Eq. (6). If both ends are fixed  $\xi(0,t) = 0 = \xi(L,t)$ . This requires  $\sin kL = 0$  or  $kL = \pi i$ .

Using  $\lambda = 2\pi/k$ , the possible values of the wavelength  $\lambda$  are given by

$$\lambda_i = \frac{2L}{i} \quad (9)$$

The first two modes are indicated in Fig. 3. Using  $v = \omega/2\pi$ ,  $\omega = ck$ , and  $c = \sqrt{\frac{T}{\rho}}$ , the frequencies of the modes in the string can be written as

$$v_i = \frac{i}{2L} \sqrt{\frac{T}{\rho}} \quad (10)$$

The lowest possible frequency, given by  $v_1 = 1/2L(T/\rho)^{1/2}$  is called the **fundamental frequency**. The other possible frequencies, called **harmonics**, turn out to be multiples of the fundamental.

### Completeness of the solutions

In the case of a discrete chain with  $N$  masses, we have  $N$  different normal modes given by the  $N$  different possible values of  $k$ . Each normal mode is a standing wave of the form

$x_n(t) = A \sin kna \cos (\omega t + \alpha)$ . There are two adjustable constants in this expression.  $A$  and  $\alpha$ . The most general solution can be written as a sum of all possible normal modes as in Eq. 16, Chapter 4. Such a solution contains  $2N$  adjustable constants, exactly the number needed to accommodate the  $2N$  initial conditions (initial position and initial velocity).

A continuum system has an infinite number of degrees of freedom. This corresponds to an infinite number of normal modes. In fact, the possible number of  $k$  values is no longer limited to  $N$ . In Eq. (8) for the guitar string, for example, any  $k$  value of the form  $k_i = \pi i/L$  gives rise to a different normal mode. This is very different from the discrete case, where the solutions start to repeat themselves after the first  $N$  values of  $k$ . In short, there are only  $N$  possible normal modes (and  $N$  values of  $k$ ) in the discrete chain because this chain has  $N$  degrees of freedom, but there is a infinite number of modes (and hence, infinite number of  $k$  values) in the continuum chain because this chain has an infinite number of degrees of freedom. The most general solution in the continuum case can be written as

$$\begin{aligned} \xi(x,t) = & A_1 \sin k_1 x \cos (\omega_1 t + \alpha_1) + A_2 \sin k_2 x \cos (\omega_2 t + \alpha_2) + \\ & + A_3 \sin k_3 x \cos (\omega_3 t + \alpha_3) + \dots \end{aligned} \quad (11)$$

where this is now sum with an infinite number of terms.

The infinite number of normal modes in the continuum chain correspond to an infinite number of initial conditions (position and velocity) for the infinite number of infinitesimal particles in this chain. Of course, we cannot list the initial conditions for an infinite number of infinitely closed particles. Instead, these conditions are given by continuous functions themselves:  $\xi(x,0) = a(x)$  is the initial position *function*, and  $\partial \xi / \partial t(x,0) = b(x)$  is the initial velocity function. The constants  $A$  and  $\alpha$  in Eq. 11 are determined from the requirement that  $\xi(x,0)$  be equal to the initial position

function  $a(x)$  and  $b(x)$  equal to the initial velocity function  $\partial\xi/\partial t(x,0)$ . The mathematical theory of *Fourier analysis* shows that these conditions are sufficient to determine the coefficients uniquely. However, we will not deal with this topic in this course.

The above discussion shows that there is a complete analogy between the discrete and continuum theory of a linear chain. The problem of  $N$  masses is completely specified and has a unique solution once the  $2N$  initial conditions are determined. Similarly, the continuum problem is completely specified and has a unique solution once the initial conditions functions  $a(x)$  and  $b(x)$  are known. The mathematical theory of Fourier analysis tells us that a solution can be found for arbitrary functions  $a(x)$  and  $b(x)$ . Physically, however, we must be careful. If any of the functions  $a(x)$  and  $b(x)$  change abruptly over very short distances, the theory of Fourier analysis predicts that the coefficients in the expansion of Eq. (11) will be different from zero for very large  $k$  values. In other words, if our initial conditions functions contain “sharp edges”, we will excite normal modes of very short wavelength. However, the wave equation was shown to be valid in continuum media for wavelengths that are long compared with the separation between particles. Therefore, we may be able to obtain an exact solution to the wave equation, but the wave equation itself may not apply to our problem if the normal mode we excite has a very short wavelength.

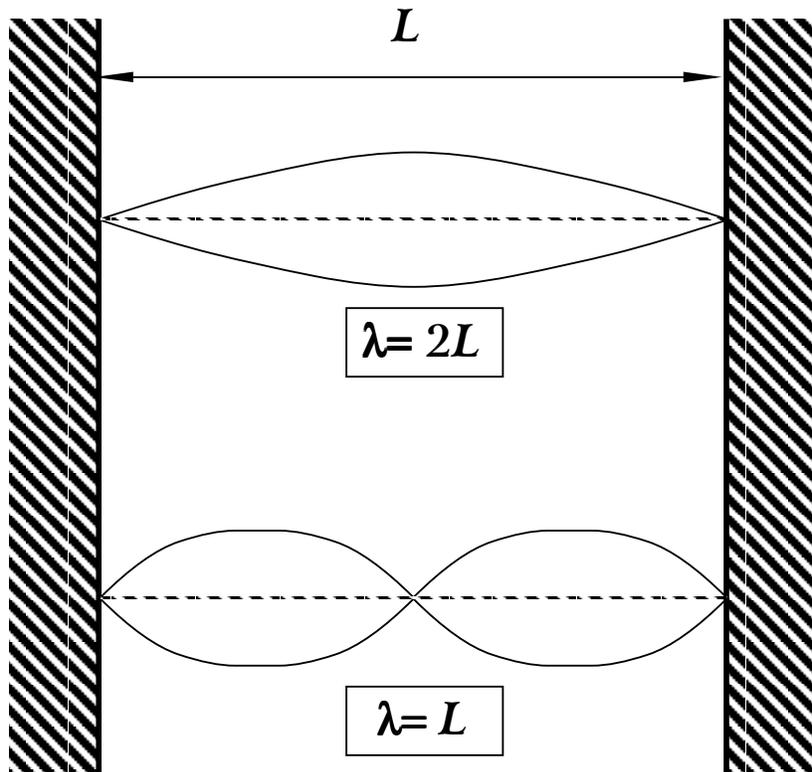


Figure 3 Standing transverse waves on a string with both ends fixed.

### Longitudinal vibrations in a string

Let us consider the continuum limit of Eq. (1), Chapter 4, for the longitudinal vibrations of a chain of masses. Using the usual transformations, this equation can be written as

$$m \frac{\partial^2 \xi}{\partial t^2} = Ka \left[ \frac{\xi(x+a) - \xi(x)}{a} + \frac{\xi(x-a) - \xi(x)}{a} \right] \quad (12)$$

which, in analogy with the section on transverse waves becomes

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{Ka^2}{m} \frac{\partial^2 \xi}{\partial x^2} \quad (13)$$

The quantity  $m/a$  is the mass per unit length of the chain, which we call  $\rho$ . On the other hand, the total spring constant of a chain with  $N$  springs of individual constant  $K$  is  $K/N$  (see homework problem). This means that the *inverse* spring constant per unit length is a constant characteristic of the

material of which the spring is made. The inverse spring constant per unit length is  $K^{-1}/a = 1/Ka$ . This can also be written as  $1/K_L L$ , where  $L$  is the length of the chain and  $K_L$  the total spring constant of the chain. Substituting these expressions, the wave equation for a linear chain becomes

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{K_L L}{\rho} \frac{\partial^2 \xi}{\partial x^2} \quad (14)$$

so that the wave speed is  $c = \sqrt{\frac{K_L L}{\rho}}$ .

### Other boundary conditions

So far, the only boundary conditions we have considered for the transverse or longitudinal standing waves in a chain are the ones that require fixed ends, *i.e.*, our chains are attached to fixed walls. Of course, the chains could also be free at these ends. The corresponding boundary conditions would therefore be different. To understand the transition from fixed end to free end, we could imagine that one of the "walls" is in fact a mass  $M$ . When the mass is very large, we are in the limit of fixed ends. When the mass  $M$  is light, we approach the limit where one of the ends is free. Let us derive the normal modes for this problem. The system is depicted in Fig. 4

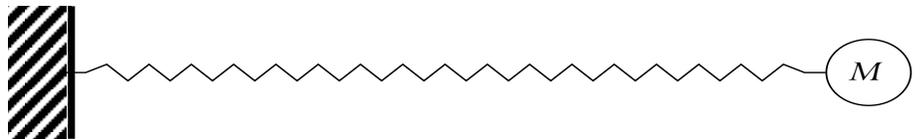


Figure 4 A spring with a mass  $M$  attached to an end.

In order to derive the boundary condition at the point where the mass  $M$  is attached, it is convenient to go back to the discrete chain and then make the continuum approximation. Let us consider the chain depicted in Fig. 5

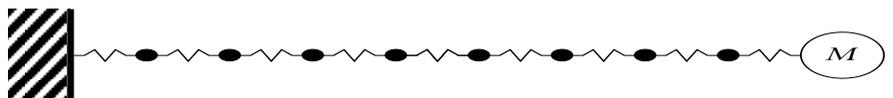


Figure 5 A discrete chain of masses and springs with a mass  $M$  attached to an end.

The acceleration of the mass  $M$  must be equal to  $\left. \frac{\partial^2 \xi}{\partial t^2} \right|_{x=L}$ , since the mass is attached to the chain. The last spring on the chain exerts a force on the mass, so that its equation of motion is

$$M \left. \frac{\partial^2 \xi}{\partial t^2} \right|_{x=L} = -K [\xi(L) - \xi(L-a)] = -Ka \frac{[\xi(L) - \xi(L-a)]}{a} \quad (15)$$

If we now make the usual continuum approximation for  $a$  small, and use the relation  $K_L L = Ka$ , we can write the above boundary condition in terms of macroscopic quantities only. We thus obtain:

$$M \left. \frac{\partial^2 \xi}{\partial t^2} \right|_{x=L} = -K_L L \left. \frac{\partial \xi}{\partial x} \right|_{x=L} \quad (16)$$

This is the new boundary condition at  $L$ . At the other end,  $x = 0$ , we still have the “old” boundary condition

$$\xi(0, t) = 0 \quad (17)$$

We now propose as a solution

$$\xi(x, t) = A \sin kx \cos \omega t \quad (18)$$

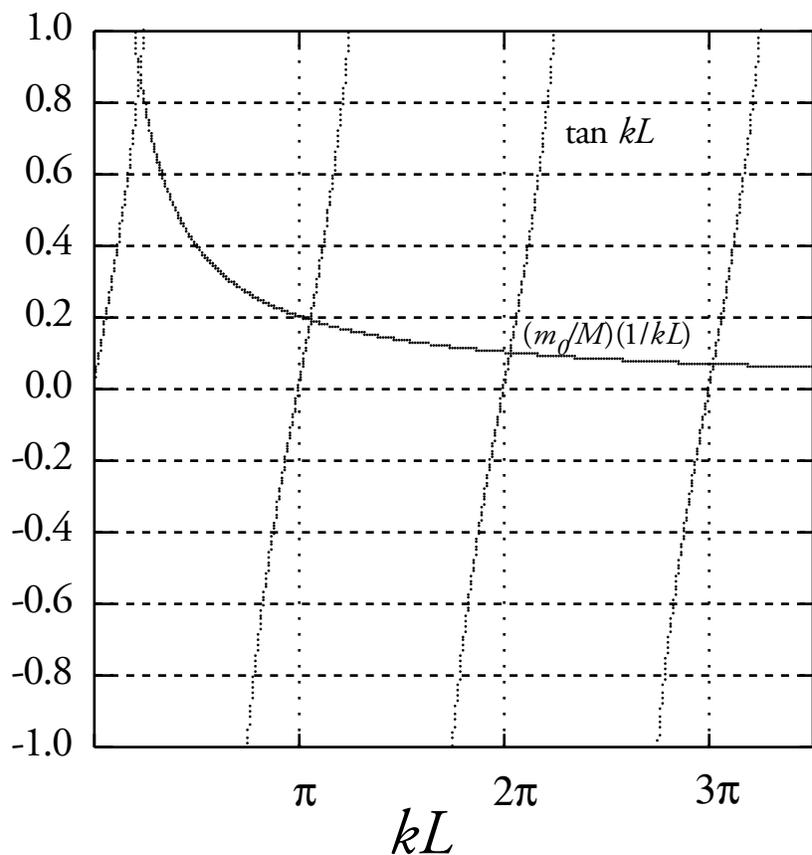
This is known to be a solution to the wave equation which in addition satisfies automatically the boundary condition at  $x = 0$ . When we substitute this solution into Eq. (16), we obtain, after dropping common factors,

$$M\omega^2 \sin kL = K_L L \cos kL \quad (19)$$

which can be rewritten, using  $\omega = ck$  and  $c^2 = K_L L/\rho$ , as

$$\tan kL = \left( \frac{m_0}{M} \right) \frac{1}{kL}, \quad (20)$$

where we have used  $m_0 = \rho L$ , the total mass of the chain. This is a **transcendental** equation for the wavenumber  $k$ . Its solution must be computed numerically or graphically. The solution is the intersection between the curves for  $\tan kL$  and the curve for  $(m_0/M) 1/kL$ , as shown in Fig. 6.



**Figure 6** Graphical method for the determination of the possible wavenumbers for the normal modes of a continuous spring chain of total mass  $m_0$  with a mass  $M$  attached to one end. The figure represents the case  $m_0/M = 0.2$

Let us now discuss the solutions to this equation. There is a set of solutions which occur for wavenumbers close to the condition  $kL = \pi i$ , with  $i = 1, 2, 3, \dots$ . This is the condition (Eq. 8) derived for a string with the two ends fixed. In fact, for very large values of  $M$ , that is, for  $m_0/M \rightarrow 0$ , the condition  $kL = \pi i$  becomes almost exact. This makes sense, because a very large mass  $M$  is equivalent to a fixed wall. When the mass  $M$  is not infinite, the  $k$  values are slightly larger than the predictions for a string with both ends fixed. This means that the wavelengths of the modes is slightly shorter.

Figure 6 shows a solution for small values of  $k$  that does not belong to the series of solutions close to the condition  $kL =$

$\pi i$ . Since the system has one extra degree of freedom relative to the chain fixed at both ends (this degree of freedom corresponds to the motion of mass  $M$ ), an additional solution is to be expected. Let us study this solution. In the limit  $m_0/M \rightarrow 0$ ,  $kL$  is very small, so that  $\tan kL$  approaches  $kL$ . We can thus rewrite Eq. (20) as

$$(kL)^2 = \frac{m_0}{M} \quad (21)$$

Using the relations  $k = \omega/c$ ,  $c^2 = K_L L/\rho$ , and  $\rho L = m_0$  this becomes

$$\omega^2 = \frac{K_L}{M} \quad (22)$$

This is exactly the expression for the frequency of a simple harmonic oscillator of mass  $M$  and spring constant  $K_L$ . Hence the meaning of the first solution in Fig. 6 is clear: in the limit of negligible spring mass, it corresponds to the solution we found in Chapter 2 for a mass attached to spring. The exact solution in Figure 6 corresponds to the case where the mass of the spring is not neglected, as we implicitly did in Chapter 2. In one of the homework problems you will derive a simple expression for the frequency of this mode for the case where  $m_0$  is small but not negligible with respect to  $M$ .

Let us now examine the opposite limit where  $M$  is much smaller than  $m_0$ . Physically, this corresponds to a chain where one of the ends is free. In this case, the intersections of the two curves occur for  $kL = \pi/2, 3\pi/2, 5\pi/2$ , etc., so that the possible wavenumbers are given by

$$k_1 = \frac{\pi}{2L}, \quad k_2 = \frac{3\pi}{2L}, \quad k_3 = \frac{5\pi}{2L}, \dots \quad (23)$$

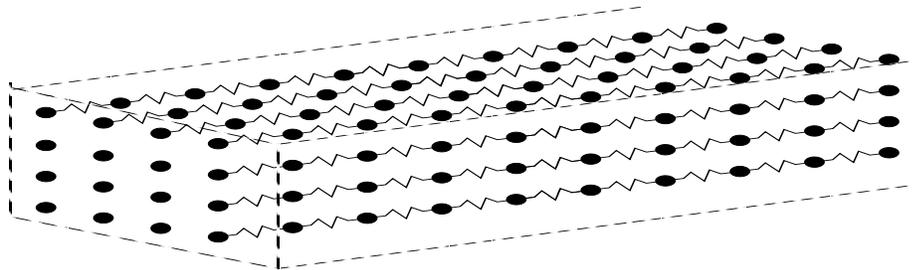
The corresponding wavelengths  $\lambda = 2\pi/k$  can thus be written as

$$\lambda_i = \frac{4L}{2i-1}, \quad i = 1, 2, 3, \text{ etc.} \quad (24)$$

Notice that the “free end” solutions have *maximum* displacement at  $x = L$ . (You can see this by using the wavenumbers from Eq. (23) in Eq. (18)). Points at which the amplitude is maximum are called **antinodes**. The existence of an antinode at  $x = L$  is consistent with Eq. (16) for  $M$  approaching zero, which requires that the first derivative  $\frac{\partial \xi}{\partial x}$  be zero. In other words, there can be no force at the free end of a chain.

### Longitudinal vibrations in solids

A solid object, such as a crystal, can be viewed as a collection of parallel linear chains of atoms



**Figure 7** A simple model of a solid object as a collection of linear chains of masses and springs.

The solid will probably feature transverse springs that help keep the chains together. Moreover, in real solids the atoms interact not only with the nearest neighbors but also with more distant ones. This implies more springs than the ones we show in the figure. We have omitted these springs because the drawing would become too messy. Even worse, the arrangement of atoms in real solids do not always follow a neat square lattice as indicated in the figure.

Within our simplified model, the effect of having parallel chains of atoms will increase the effective spring constant. For example, the effective spring constant of two equal springs acting in parallel is twice the spring constant of a single spring. If the cross-sectional area of the bar in Fig. 7 is  $A$  and its length is  $L$ , its effective spring constant can be written as

$$K_L = \frac{YA}{L} \quad (25)$$

This is because the number of chains is proportional to the cross-sectional area and the spring constant is inversely proportional to the length, as discussed above. The constant  $Y$  is a property of the bar's material, not of its shape. It is called the **Young modulus** or **elastic modulus**. Eq. (25) is very convenient because it separates the properties of the material for the shape of the object. For example, it makes no sense to talk about the spring constant of iron. On the other hand, the Young modulus of the material iron is well defined. Experiments show it is given by  $Y = 2.06 \times 10^{11}$  Nm<sup>-2</sup>. Using this value, we can calculate the spring constant  $K_L$  of any iron bar of length  $L$  and cross-section  $A$  by using Eq. 25. Inserting Eq. (25) into Eq. (14) we obtain

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{Y}{\rho_V} \frac{\partial^2 \xi}{\partial x^2} \quad (26)$$

where  $\rho_V = \rho/A$  is simply the standard volume density. Notice that the symbol  $\rho$  is always used for density, but its meaning may change from situation to situation. Sometimes, we mean *linear* density, sometimes *areal* density, and sometimes, as in the case of the solid bar, *volume* density. For simplicity, the subscripts that reveal the type of density are often dropped, and we are left to guess what type of density we are talking about. The best hint is given by the type of problem: if we are talking about a cord, we will neglect its thickness, so that the density will refer to mass per unit length. If we are talking about a bar, it makes sense to talk about the mass per unit volume, so that this will be usually the meaning of  $\rho$ . Another safe way to determine the type of density we need for our problem is to look at the units of the different factors, keeping in mind that the coefficient of the right hand side in the wave equation (see Eq. 26) has units of speed squared.

The wave speed for the problem in Eq. (26) is given by

$$c = \sqrt{\frac{Y}{\rho_V}} \quad (27)$$

Later it will become apparent that this quantity is simply the speed of sound in a solid object. Replacing values for typical materials, we find that this number is about ten times higher than the speed of sound in air, which we will determine below.

### Strain-stress

Eq. (25) suggest an experiment to determine the Young modulus  $Y$  of a given material. Consider a bar of length  $L$  and cross-sectional area  $A$ . Suppose we attach one end of the bar to a wall and exert a tensile force  $F$  on the other end. The bar will stretch, according to Hooke's law, by  $\Delta L = F/K_L$ . Using Eq. (25) this can be written as  $F/A = Y \Delta L/L$ . The quantity  $\Delta L/L$ , which gives the relative elongation of the bar, is called the **strain**  $\epsilon$ . The quantity  $F/A$ , which gives the force per unit cross sectional area, is called the **stress**  $S$ . We therefore arrive at the strain-stress relation

$$S = Y\epsilon \quad (28)$$

This is sometimes known as Hooke's law for a continuum medium. By determining the stress and the strain, one can compute the Young modulus. Notice that hard materials are those whose Young modulus is high, so that their deformation for a given stress is small.

### Shear deformations

Fig. 8 (a) shows the type of deformation produced when we measure the Young modulus. As indicated by Fig. 8 (b), it is possible to produce another type of deformation in a solid object by applying forces tangent to the surface. This type of stress is called **shear stress**.

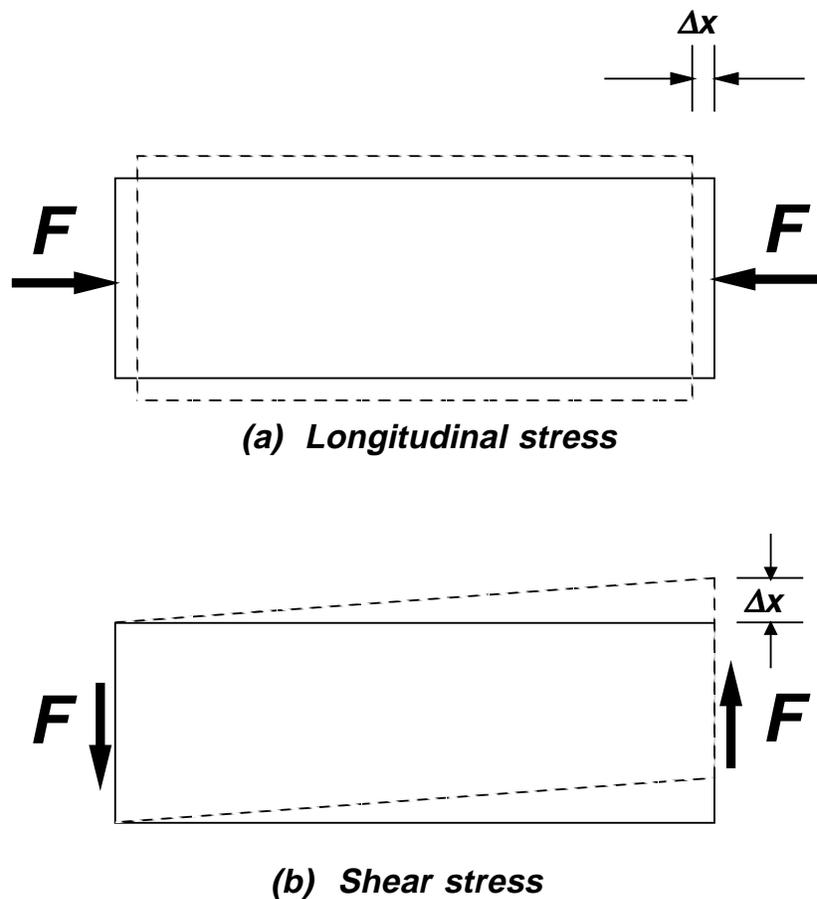


Figure 8 Longitudinal and shear stress-strain in a solid bar.

For shear stress, we define stress and strain in a similar way as in the case of longitudinal deformation,  $S = F/A$ , and  $\epsilon = \Delta x/L$ , where  $A$  is the cross sectional area,  $L$  the length of the bar, and  $\Delta x$  the *lateral* deformation. However, there is no reason why the restoring force for a longitudinal deformation has to be equal to the restoring force for shear deformation, so that in general we will have

$$S = G\epsilon \quad (29)$$

with  $G \neq Y$ . The quantity  $G$  is called the **shear modulus** of the material. When a deformation as in Fig. 8 is induced in a bar, *transverse* waves are setup. They satisfy a wave equation identical to the one we derived for longitudinal waves, but now the speed will be given by

$$c = \sqrt{\frac{G}{\rho_v}} \quad (30)$$

Thus the speed for transverse waves is different from the speed for longitudinal wave. As we will see in the next chapter, this has important implications for applied science. An important example is seismology.

When shear stress is applied to a liquid or a gas, there is no restoring force, because the liquid or the gas don't oppose being deformed as in Fig. 8 (b). In other words, for liquids and gases  $G = 0$ . Thus transverse waves are not possible in these media.

### Vibrations in a column of air: sound



**Figure 9** A column of air enclosed in a tube is similar to a spring in that it resists being compressed.

A column of air enclosed in a tube, as in Fig. 9, acts like a spring. Therefore, we can assume that the longitudinal air displacements satisfy an equation similar to Eq. (14). Our task is to calculate the spring constant  $K_L$  of the column of air. A fundamental difference between a gas and a standard spring is that the spring “constant” of the gas is actually not a constant: as the gas is compressed, it becomes harder and harder to compress it. This is what makes air such a good shock absorber. If we apply a force  $F$  to a spring of constant  $K_L$ , the spring stretches by an amount  $\Delta L = F/K_L$ , where  $\Delta L$  could in principle be a large displacement. However, if  $K_L$  is not a constant, this expression makes no sense, because  $K_L$  will be changing as the spring is compressed. This is exactly

what happens with a column of air. The right expression in this case is

$$dF = -K_L(L) dL \quad (31)$$

For air, the force is given by  $F = pA$ , where  $A$  is the area of the piston that encloses the air and  $p$  is the pressure. Thus  $dF = Adp$ . Since pressure changes as we compress the gas, it is convenient to write this expression as

$$\begin{aligned} dF &= A \left( \frac{dp}{dV} \right)_0 dV \\ &= A \left( \frac{dp}{dV} \right)_0 A dL \\ &= A^2 \left( \frac{dp}{dV} \right)_0 dL \end{aligned} \quad (32)$$

where the subscript “0” means that the derivatives are computed at equilibrium, that is, when the pressure inside is equal to the pressure outside, which usually is the atmospheric pressure. Comparing Eq. (31) with Eq. (32), we conclude that the spring “constant” of the air column is given by  $K_L = -A^2 \left( \frac{dp}{dV} \right)_0$ . Hence the speed in the wave equation is

$$\begin{aligned} c^2 &= \frac{K_L L_0}{\rho_0} = -\frac{A^2 L_0 \left( \frac{dp}{dV} \right)_0}{\rho_0} = -\frac{A L_0 \left( \frac{dp}{dV} \right)_0}{\rho_0 / A} \\ &= -\frac{V_0 \left( \frac{dp}{dV} \right)_0}{\rho_{V,0}} \end{aligned} \quad (33)$$

where all subscripts “0” refer to equilibrium. Here, as usual,  $\rho_V$  refers to the volume density (mass per unit volume). We must now compute the derivative  $dp/dV$ . For an ideal gas, we will show in later chapters that  $p = NkT/V$ , is the number of molecules in the gas,  $k$  the so-called Boltzmann constant, and  $T$  the temperature in Kelvin. If we try to compute the volume derivative of this expression, we must know how the temperature depends on the volume. The first who thought about this problem was Newton himself. He reasoned that

compressions and rarefactions in a gas occur so rapidly when a wave is excited, (thousands of times per second for standard sound waves) that the temperature remains essentially constant. With this assumption,  $dp/dV = -NkT/V^2 = -p/V$ . Using this expression in Eq. (33), we obtain  $c^2 = p_0/\rho_0$ . For air at atmospheric pressure,  $p_0 = 1.01 \times 10^5 \text{ N m}^{-2}$  and  $\rho_0 = 1.29 \text{ kg/m}^3$ . This yields  $c = 280 \text{ m/s}$ , only 15% off the known speed of sound in air,  $c = 332 \text{ m/s}$ . Not too bad. On the other hand, what could be wrong in our derivation to give a 15% error? The answer is in our assumption that the temperature remains constant.

We will show in later chapters that the energy of a gas is directly proportional to its temperature. When the pressure does *work* on the gas, its energy increases. If the temperature is to remain constant, the work done by the pressure must be compensated by an energy loss of exactly the same magnitude, usually in the form of heat. However (this is the crucial argument) it takes time for heat to dissipate. For very rapid oscillations this time is not available. Hence a reasonable assumption is not that the temperature remains constant but that the heat transfer is zero. Under such an assumption, we will show in the thermodynamics part of this course that the temperature is proportional to  $V^{\gamma-1}$ , where  $\gamma = 1.4$  for air. Hence the correct relationship between pressure and volume is  $pV^\gamma = \text{constant}$ . With this expression,  $dp/dV = -\gamma p/V$  and

$$c^2 = \frac{\gamma p_0}{\rho_{v,0}} \quad (34)$$

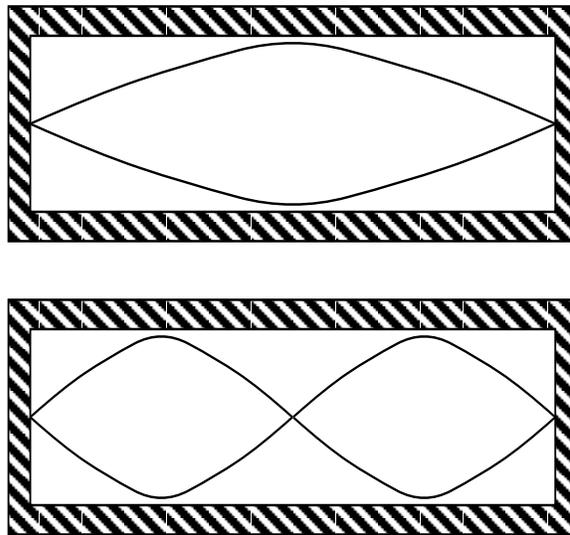
This gives  $c = 332 \text{ m/s}$ , which agrees exactly with the experimental value.

### Standing waves in tubes

The study of standing waves in tubes filled with air is important to understand how musical instruments work. Depending on whether the tubes are open or closed at the ends, the boundary conditions change and different types of standing waves are setup. From the boundary conditions, we

can determine the possible values of the wavenumber  $k$  (or, equivalently, the wavelength  $\lambda$ ). Using  $\omega = ck$ , (or  $c = \lambda\nu$ ) with  $c = 332$  /s, we readily obtain the frequencies of vibration.

The simplest case is that of closed tubes. Here the boundary conditions are quite obvious: since the tube is closed, there must be nodes at the two ends, that is, the air displacement is zero at these points. This is schematically illustrated in Fig. 10. Notice that the figure suggests that the air displacement is perpendicular to the length of the tube. This is not true! The actual air displacement is *longitudinal*. Since we cannot indicate in the figure a longitudinal wave, we have drawn a transverse one. The figure illustrates the amplitude of the oscillation, but is not a picture of the actual displacements, which occur along the axis of the tube.



**Figure 10** The first two standing waves in a closed tube filled with air.

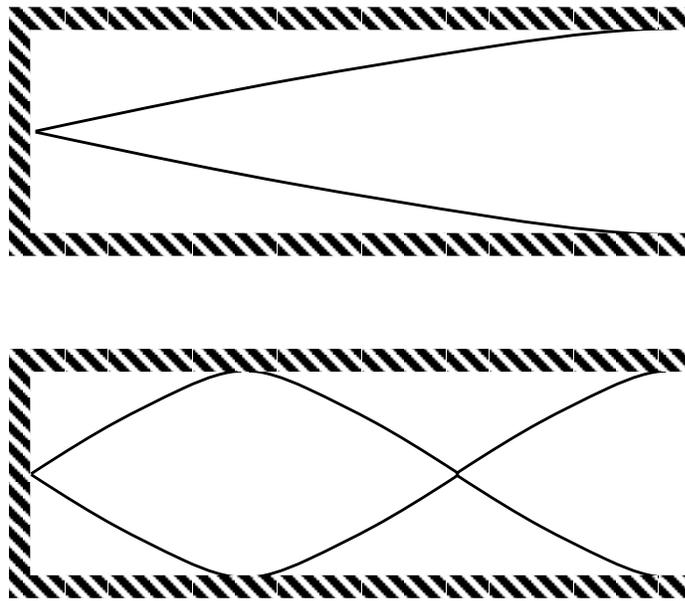
From Fig. 10 it is easy to conclude that the possible wavelengths in a closed tube are given by  $\lambda = 2L/i$ , where  $i$  is 1,2,3, etc. Hence the frequency of the wave is

$$\nu_i = i \left( \frac{c}{2L} \right) \quad (35)$$

Let us now consider the case where one of the ends is open. This is illustrated in Fig. 11. The boundary condition for the closed end remains the same: there is a node at this point. At the open end, the situation is more complicated. We can understand the boundary condition at this end by analogy with the case of a spring with one end fixed and a mass  $M$  attached to the other end. If there was no air outside the tube, this would correspond to the case  $M = 0$ . Therefore, we would obtain antinodes at the open end of the tube. Of course, there must be air outside, so that this condition is not fulfilled exactly. On the other hand, while the air inside the tube will “resist” any compression, the air outside the tube will offer very little resistance to compression. So we could “model” an open tube by a spring with large spring constant (air inside) attached to a spring of very low spring constant (air outside). The solutions to this problem are very similar to the solutions for the spring with a very small mass  $M$  at one end, because they essentially reduce to the condition that no force is applied to the system at the open end. Hence there are antinodes very close to the open end of the tube. For the solution of homework problems, we usually assume that these antinodes are *exactly* at  $x = L$ . Hence we can use Eqs. (23) and (24) to determine the wavenumbers and wavelengths for the possible standing waves. This leads to frequencies given by

$$\nu_i = (2i - 1) \frac{c}{4L} = \nu_1, 3\nu_1, 5\nu_1, \dots \quad (36)$$

where the *fundamental* frequency is given by  $\nu_1 = c/4L$ . Notice that only the odd harmonics of the fundamental are present. From Fig. 6, We notice that the exact solution features wavenumbers slightly smaller than the wavenumbers obtained from the condition that an antinode exists exactly at  $x = L$ . This means that in real tubes the antinode is slightly outside the tube.



**Figure 11** The lowest two vibrational modes in tubes open at one end.

The case of a tube open at both ends is straightforward from the discussion in the previous paragraphs and is the subject of one of the homework problems.

### **Importance of the boundary conditions**

The frequencies of oscillations of a system depend on the boundary conditions, as can be seen by comparing the possible frequencies for closed and open-end tubes. This statement is valid for all vibrating systems and is not restricted to tubes. Hence the possible frequencies of sound waves inside a room will change if we open a door. However, we typically don't notice much of a change in the sounds heard inside a room when a door is opened. We do notice changes, on the other hand, when we sing while taking a shower. Those of us who are lousy singers notice a definite improvement in the quality of the sounds we emit. The explanation for these effects is the following: The separation between a certain mode and the next mode is proportional to  $1/L$ . When the size of the vibrating system is much larger than the wavelengths of the modes, the modes are very closely spaced, so that the allowed frequencies do

not change much if we change the boundary conditions. This makes sense, because in a very large system the boundaries should eventually become unimportant. On the other hand, the frequencies will change in a noticeable way if the dimensions of the system are of the order of the wavelengths. For a sound wave which has a frequency of 300 Hz, the wavelength is of the order of 1 m. This is much smaller than the size of a typical living-room but is of the order of the distance between the wall and the shower curtain.

When we discuss sound in open places, we can imagine that we are dealing with volumes of air enclosed in very large volumes. Thus the modes are extremely close to each other. In other words, all frequencies are possible outdoors.

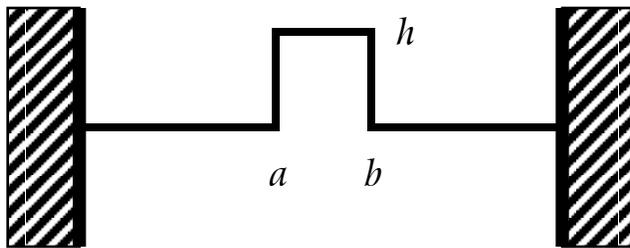
## Problems

1. Consider a chain of 100 equal masses connected by equal springs and attached to rigid walls at both ends.

a) Using the procedure developed in Chapter 4, find the frequency of the first and last normal mode of the system. In both cases, compute the ratio  $x_{59}/x_{60}$  of the displacements of mass 59 and mass 60.

b) Repeat a) by using not the exact solution but the continuum approximation developed in this chapter. Compare with a) and discuss.

2. At time  $t = 0$ , a string of length  $L = 1$  m is deformed with the shape of a square pulse between  $a = 0.4$  m and  $b = 0.6$  m. The height of the pulse is  $h = 0.05$  m.



The initial velocity of the string is zero at all points.

a) Verify (graph when needed) that the initial conditions are satisfied by Eq. (11) if one chooses  $\alpha_m = 0$  for all  $m$ ,

$$A_m = 0,$$

for  $m$  even, and

$$A_m = \frac{2h}{m\pi} [\cos mk_1 a - \cos mk_1 b]$$

for  $m$  odd. Eq. (11) is an infinite series. Of course, you cannot add all terms in an infinite series. However, you should get a good approximate answer if you take the first 20 terms or so in the expansion.

b) Graph the function  $\xi(x,t)$  at several times  $t > 0$ . Invent your own values for the speed  $c$ .

3. Show that the effective spring constant of a chain of  $N$  equal springs of spring constant  $K$  is  $K/N$ . Show that if you put the  $N$  springs in parallel, the effective spring constant becomes  $NK$ .

4. (Crawford 2.21) Find the mode configurations and frequencies for transverse oscillations of a beaded string of 5 beads with one end fixed and one free. Plot the five corresponding points on the dispersion relation  $\omega(k)$ .

5. In the problem of a mass  $M$  attached to a spring of mass  $m_0$  and constant  $K_L$ , find the frequency and plot  $\xi(x,t)$  as a function of  $x$  for the lowest frequency mode. Take  $m_0/M = 0.1$  and select the other parameters arbitrarily. Compare the graph of  $\xi(x,t)$  with the deformation of the string when it is stretched statically.

6. A way to estimate a correction to the frequency of oscillation of a mass  $M$  attached to a spring due to the mass  $m_0$  of the spring is as follows: In Eq. (20), expand the tangent to terms of order  $(kL)^3$ . In the last term of the expansion, substitute  $(kL)^2 = m_0/M$ . Finally, show that the frequency of the lowest mode of the system becomes

$$\omega^2 = \frac{K_L}{M + \frac{1}{3}m_0}.$$

Discuss the correctness of the approach used to derive this expression.

7. (Alonso 28.13) A string of length 2 m and mass  $4 \times 10^{-3}$  kg is held horizontally, with one end fixed. The other end passes over a pulley and supports a mass of 2 kg. Calculate the velocity of transverse waves in the string.

8. (Alonso 28.15) A steel ( $Y = 2.0 \times 10^{11}$  N m<sup>-2</sup>) wire having a length of 2 m and a radius of  $5 \times 10^{-4}$  m hangs from the ceiling. a) If a body having a mass of 100 kg is hung from the free end, calculate the elongation of the wire. b) Also determine the displacement and the downward pull at a point in the middle of the wire. c) Determine the velocity of longitudinal and transverse waves along the wire when the mass is attached.

9. (Alonso 34.16) How is the fundamental frequency of a string changed if a) its tension is doubled, b) its mass per unit length is doubled, c) its radius is doubled, d) its length is doubled? Repeat

the problem if the quantities listed are halved.

10. Derive an expression for the frequencies of oscillations in a tube with both ends open.

11. (Alonso 34.17) A tube whose length is 0.6 m is *a*) open at both ends, and *b*) closed at one end and open at the other. Find its fundamental frequency and the first overtone. Plot the amplitude distribution along the tube corresponding to the fundamental frequency and the first overtone for each case.

12. Compare two pipes of identical length, one of them closed at both ends and the other one open at one end. Find the length of the pipes so that the fundamental frequencies differ by 1 Hz. Would you be able to tell the difference between the sound waves produced by the two tubes? What if the length  $L$  is reduced by a factor of 100?