Chapter *3*

Oscillations in systems with a few particles: normal modes

Introduction

Our Chapter 2 discussion of the oscillations of a single mass attached to a spring was a review of material you learned in previous physics courses. In this chapter, we would like to start with a specific PHY 241 topic: the oscillations of several - perhaps many - masses connected by springs. Ultimately, this will lead to the concept of waves.

Several complications arise when we consider the motion of many interacting masses. To describe this motion, we need a position function for each mass. Hence there will be an equation of motion associated with each of these functions. However, these equations will not be independent, because the motion of one of the masses affects the motion of the others. Hence we cannot solve the equations one at a time: we must solve all the equations simultaneously! This appears to be an impossible task, particularly for cases like a chain of hundreds of beads connected by springs. Fortunately, the solution of this problem is not as complicated as it sounds at this time. Very soon, you'll solve the problem of *N* beads connected by springs, with *N* as large as you want.

The two-mass problem

Equations of motion and their solutions

Let us start with a simple problem, illustrated in Fig. 1. Two identical masses are connected by identical springs.

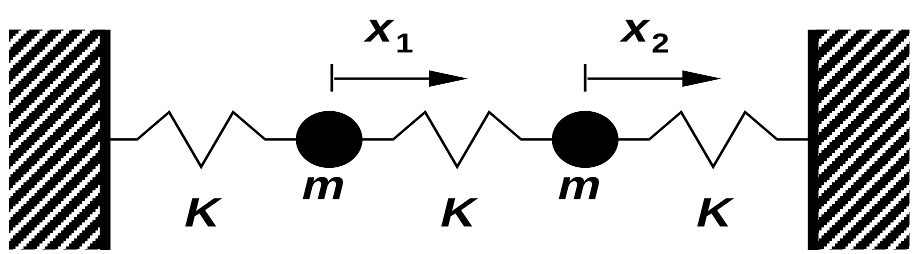


Figure 1 **Two masses connected by springs.**

Let us assume that the displacements are restricted to the horizontal direction, so that we need only two functions to characterize the motion completely: the position $x_1(t)$ for the mass on the left and the position $x_2(t)$ for the mass on the right. We also assume that the springs are not stretched when the masses are at rest. It will become clear as we proceed that it is not convenient to refer the two position functions to the same horizontal axis. Instead, as indicated in the figure, we measure $x_1(t)$ from the equilibrium position of the first mass and $x_2(t)$ from the equilibrium position of the second mass. Before we even write down the equations of motion for the system, we know that the solutions must have *four* independent constants. This is because we can specify the initial position and velocity of each particle arbitrarily .

The first step to solve our problem is to write down the equation of motion for each particle. This is, of course, Newton's second law. Consider first particle 1 (the particle on the left). There are two forces acting on this particle: a force exerted by the spring on the left, which is identical to the force exerted by a spring on an isolated mass, and a force exerted by the middle spring. The elongation of this spring depends on the *difference* x_1-x_2. This is easy to see: if the two masses are displaced by the same amount in any direction, the middle spring is not stretched and exerts no force. Hence the force exerted by this spring must be given either by $K(x_1-x_2)$ or by $K(x_2-x_1)$. The trickiest point is to determine which of these two expressions is the correct one. This is best done by inspection of the figure and considering any specific displacements. For example, if mass 2 moves to the right and mass 1 doesn't move, the force exerted by the middle spring is to the right, which is positive for our axis convention. But $K(x_1-x_2)$ would be negative in such case, so that the right expression for the force of the middle spring on mass 1 is $K(x_2-x_1)$. The equation of motion of mass 1 becomes

$$m\,\ddot{x}_1 = -K\,x_1 + K\,(x_2 - x_1) \qquad (1)$$

Note that this equation involves the motion of mass 2, as we discussed in the introduction. So we cannot solve for the motion of mass 1 alone: we must solve simultaneously the equation for mass 2. This equation can be deduced along the same lines as Eq. (1). It is given by

$$m\,\ddot{x}_2 = -K\,x_2 - K\,(x_2 - x_1) \qquad (2)$$

The first term in the right-hand side is the force exerted by the spring to the right of mass 2. The second term is the force exerted on mass 2 by the middle spring. Note that this force is equal to *minus* the force exerted by this spring on mass 1. This is consistent with Newton's third law.

The procedure to solve a set of simultaneous differential equations is similar to the approach used to solve a set of simultaneous *algebraic* equations: you multiply the equations by carefully chosen constants and add them up, so as to get an equation for a single function. In the case under consideration, the simplicity of our problem dictates a very simple linear combination. Suppose we add Eqs. (1) and (2). We obtain

$$m\,(\ddot{x}_1 + \ddot{x}_2) = -K\,(x_1 + x_2) \qquad (3)$$

If we now that we define a function $X_A(t) = x_1(t) + x_2(t)$, the second derivative of this function is $d^2X_A/dt^2 = \ddot{x}_1 + \ddot{x}_2$, so that the function $X_A(t)$ satisfies the same equation as the simple harmonic oscillator: $d^2X_A/dt^2 = -(K/m)X_A$. But we already know the solution to the simple harmonic oscillator equation, so that we can immediately write

$$X_A(t) = x_1(t) + x_2(t) = 2A_A \cos(\omega_A t + \alpha_A) \qquad (4)$$

with $\omega_A = (K/m)^{1/2}$. We have inserted here a factor of 2 for later convenience, but this obviously does not affect the generality of our solution. If we now subtract Eqs. (1) and (2), we obtain a differential equation for the difference between $x_1(t)$ and $x_2(t)$:

$$m\,(\ddot{x}_1 - \ddot{x}_2) = -3K\,(x_1 - x_2) \qquad (5)$$

This equation is also a harmonic oscillator equation for the function $X_B(t) = x_1(t) - x_2(t)$. Thus its solution is

$$X_B(t) = x_1(t) - x_2(t) = 2A_B \cos(\omega_B t + \alpha_B) \qquad (6)$$

with $\omega_B = (3K/m)^{1/2}$.. Again, we have inserted here a factor of 2 for later convenience. It is now easy to combine Eqs. (4) and (6) to obtain the final expressions for $x_1(t)$ and $x_2(t)$:

$$x_1(t) = A_A \cos(\omega_A t + \alpha_A) + A_B \cos(\omega_B t + \alpha_B) \quad (7)$$

and

$$x_2(t) = A_A \cos(\omega_A t + \alpha_A) - A_B \cos(\omega_B t + \alpha_B) \quad (8)$$

Notice that there are *four* free constants in these expressions: A_A, A_B, α_A, and α_B. These constants are determined from the four initial conditions (initial position and velocities of the two masses) corresponding to this problem. Since we can satisfy any set of four initial conditions with four constants, our solution is complete.

Normal modes

The motion of the two masses in the above problem, represented by Eqs. (7) and (8), is not simple harmonic. If you plot the functions $x_1(t)$ or $x_2(t)$, you obtain a complicated curve. This is shown in Fig. 2.

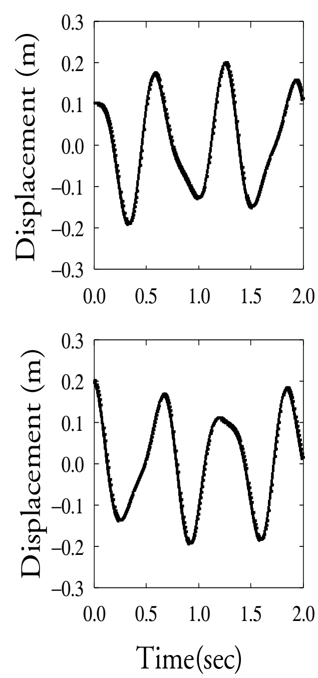


Figure 2 **Displacements of mass 1 (top figure) and mass 2 (bottom figure) for arbitrary choices of the initial conditions.**

But with an appropriate choice of the initial conditions, it is possible to make the system execute simple harmonic oscillator vibrations. For example, if $A_B = 0$, the two masses vibrate like a simple oscillator of frequency ω_A. This is called a **normal mode.** Normal mode A is shown here in Fig. 3

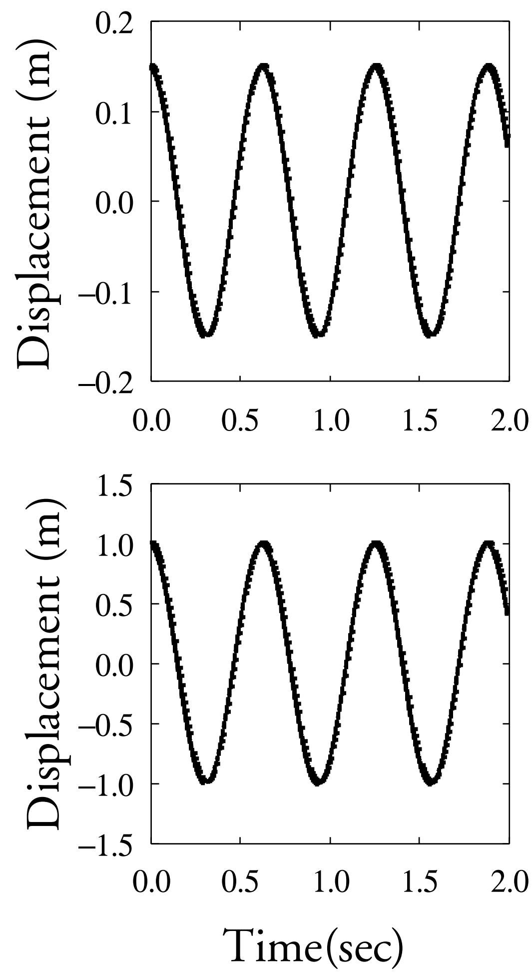


Figure 3 **Displacements of mass 1 (top panel) and mass 2 (bottom panel) for choices of the initial conditions that produce normal mode A.**

On the other hand, if $A_A = 0$, we excite a second normal mode, for which the two masses oscillate with frequency ω_B. This mode is shown in Fig. 4

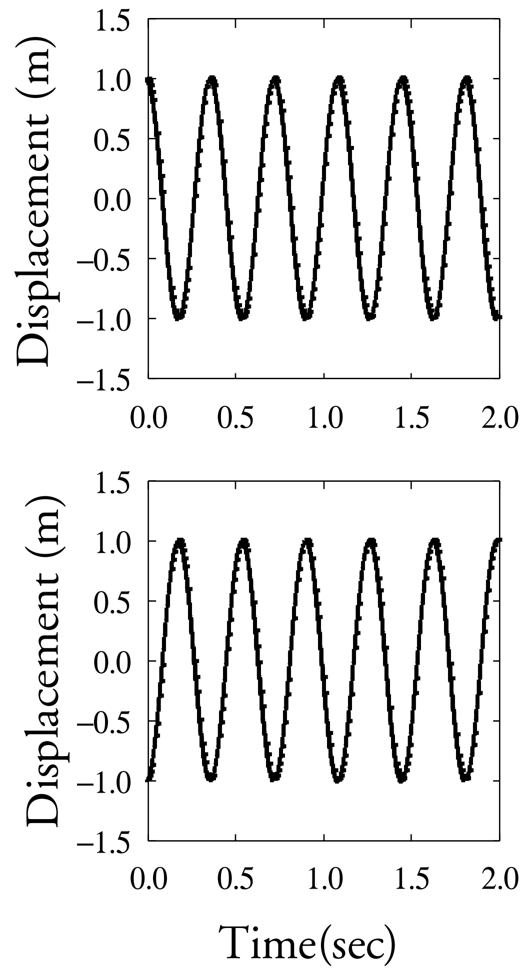


Figure 4 **Displacements of mass 1 (top panel) and mass 2 (bottom figure) for a choice of the initial conditions that leads to normal mode B.**

The initial conditions needed to excite these normal modes are particularly simple. To excite normal mode A, our initial conditions must be such that $A_B = 0$. This means that (see Eq. 6) $x_1(t) - x_2(t) = 0$ at all times. Remember that the difference $x_1(t) - x_2(t)$ satisfies a harmonic oscillator equation. The only way to keep a harmonic oscillator at rest is to place it at the equilibrium position and set its initial velocity equal to zero. Thus in our case we need $x_1(0)-x_2(0) = 0$ and $v_1(0)-v_2(0) = 0$. In other words, to excite mode A we need to select the initial displacement and velocity of the two masses in such a way that they are equal.

In order to excite mode B, we must make $A_A = 0$. This is achieved if $x_1(0) = -x_2(0)$ and $v_1(0) = -v_2(0)$.

More about normal modes

For the two normal modes of our example

$$x_1(t) = x_2(t) \quad \text{(mode A)} \tag{9}$$

and

$$x_1(t) = -x_2(t) \quad \text{(mode B)} \tag{10}$$

This means that a normal mode has a characteristic *displacement pattern.* For example, the displacement pattern for mode A is indicated in Fig. 5.

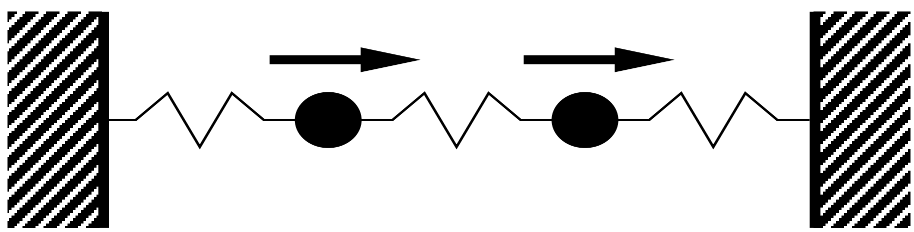


Figure 5 **Displacement pattern for the excitation of mode A**

Of course, after a certain time the direction of the arrows will be reversed, but the point is that they remain equal at all times. Similarly, the displacement pattern for mode B is given in Fig. 6.

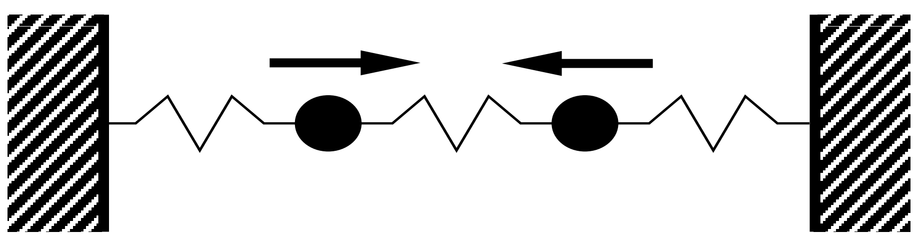


Figure 6 **Displacement pattern for the excitation of mode B**

At a later time, the two arrows will point away from each other, so that they remain equal in magnitude but opposite in direction at all times. From these displacement patterns it is easy to understand the frequencies we found for the two normal modes. For normal mode A, the middle spring is never stretched. Hence the forces acting on the masses originate only in the left and right springs, as if the masses were attached to a single spring. This explains the simple expression $\omega_A = (K/m)^{1/2}$ for the frequency of this mode. For normal mode B, the middle spring is stretched by *twice* the amount the other two springs are stretched. Hence the force acting on the masses is *three* times the force acting on them in mode A. This explains the frequency $\omega_B = (3K/m)^{1/2}$.

If the system is not in a normal mode, there is no characteristic displacement pattern. This is because the ratio $x_1(t)/x_2(t)$ is not constant, as can be seen from Eqs. 7 and 8 and Fig. 2. Since the ratio is not constant, the relative magnitude and direction of the displacement arrows will change with time. In Fig. 2, for example, $x_1(t) < x_2(t)$ initially, but at later times $x_1(t) > x_2(t)$. This is because the ratio $x_1(t) / x_2(t)$ depends on time unless the system is in a normal mode.

There are normal modes for *any* system of masses and springs. In a more complicated system (different masses, springs with different spring constants, etc.) the displacements of the masses for a normal mode X are given by

$$x_1(t) = A_X(1)\cos(\omega_X t + \alpha_X)$$
$$x_2(t) = A_X(2)\cos(\omega_X t + \alpha_X), \qquad (11)$$

where the ratio $x_1(t)/x_2(t) = A_X(1)/A_X(2)$ is a constant, but not necessarily +1 or -1as in our simple example.

A way to find the ratio $A_X(1)/A_X(2)$ (in other words, a way to find the displacement pattern for the normal mode) is to realize that in a normal mode X all masses oscillate with the same frequency ω_X. From our solution to the simple harmonic oscillator problem, we now that the square of this frequency is equal to minus the force per unit mass per unit displacement. By equating such an expression for all masses, we find an equation for the amplitudes. Let us apply this method to our simple example. From Eqs. (1) and (2), minus the force per unit mass per unit displacement is given by

$$\omega_X^2 = -\frac{-Kx_1 + K(x_2 - x_1)}{mx_1} = -\frac{-Kx_2 - K(x_2 - x_1)}{mx_2} \quad ,(12)$$

where the equal signs are only valid if the system is in a normal mode, for it is only in this case that the oscillations occur at a single frequency. But if the system is in a normal mode, we can use Eq. (11) for the displacements, so that Eq.(12) becomes

$$\omega_X^2 = -\frac{-K\,A_X(1) + K\left[A_X(1) - A_X(2)\right]}{mA_X(1)} = -\frac{-K\,A_X(2) - K\left[A_X(1) - A_X(2)\right]}{mA_X(2)} \quad (13)$$

Notice that the time-dependent cosine functions cancel out in this expression because they are identical for $x_1(t)$ and $x_2(t)$, since the system is assumed to be in normal mode X. Eq. (13) can be simplified to

$$A_X(2)\left[A_X(1) - A_X(2)\right] = A_X(1)\left[A_X(2) - A_X(1)\right] \quad (14)$$

This equation has two solutions: $A_X(1) = A_X(2)$ and $A_X(1) = -A_X(2)$. These two solutions correspond to normal modes A and B. This can be seen by comparing with Eqs. (7) and (8) and by replacing the solutions into Eq. (13), which yields the frequencies $\omega_A^2 = \frac{K}{m}$ and $\omega_B^2 = \frac{3K}{m}$, respectively.

Systematic search for normal modes

In the previous sections, we discussed two methods of finding the normal modes of a system. First, we solved the two coupled equations (1) and (2) by finding an appropriate combination of these two equations that leads to simple harmonic oscillator solutions for the two normal modes. In the previous paragraph, we found the normal modes by using the condition that the two masses oscillate with the same frequency. These methods become increasingly more complicated if the masses are not all equal, if the spring constants happen to be different from spring to spring, and, most importantly, if the number of masses connected by springs is larger than 2. Fortunately, there is a systematic way of solving the equations of motion for coupled springs that overcomes these difficulties. Suppose that we have two simultaneous equations of the form

$$\frac{d^2x_1}{dt^2} = -a_{11}x_1 - a_{12}x_2 \quad ,$$
$$\frac{d^2x_2}{dt^2} = -a_{21}x_1 - a_{22}x_2 \qquad (15)$$

where the coefficients a_{11}, a_{12}, a_{21}, and a_{22} are constants. (Note that Eqs. (1) and (2) are of this form, with $a_{11} = a_{22} = 2K/m$ and $a_{12} = a_{21} = -K/m$).Now we *assume* that there exist solutions in the form of normal modes, given by Eq. (11). This implies

$$\frac{d^2x_1}{dt^2} = -\omega_X^2 x_1, \qquad \frac{d^2x_2}{dt^2} = -\omega_X^2 x_2 \qquad (16)$$

Substituting into Eqs. (15), and rearranging, we obtain

$$(a_{11} - \omega^2)x_1 + a_{12}x_2 = 0,$$
$$a_{21}x_1 + (a_{22} - \omega^2)x_2 = 0. \qquad (17)$$

These are two standard simultaneous algebraic equations. You remember from your algebra courses that for such a system to have a solution other than the trivial solution ($x_1 = x_2 = 0$), the *determinant* of the coefficients must vanish:

$$\begin{vmatrix} a_{11} - \omega^2 & a_{12} \\ a_{21} & a_{22} - \omega^2 \end{vmatrix} \equiv (a_{11} - \omega^2)(a_{22} - \omega^2) - a_{12}a_{21} = 0. \qquad (18)$$

This is a *quadratic* equation in ω^2, from which two solutions ω_A and ω_B are obtained. These are precisely the frequencies of the two normal modes. Once these frequencies are determined, the relationship between the coefficients $A_X(1)$ and $A_X(2)$ follows from Eqs. 17:

$$\left.\frac{x_2}{x_1}\right|_{\text{mode A}} = \frac{A_A(2)}{A_A(1)} = \frac{\omega_A^2 - a_{11}}{a_{12}}$$

$$\left.\frac{x_2}{x_1}\right|_{\text{mode B}} = \frac{A_B(2)}{A_B(1)} = \frac{\omega_B^2 - a_{11}}{a_{12}} \qquad (19)$$

where we have used the first equation in (17). We could have used the second equation to obtain exactly the same result; this is because the two equations are made proportional to each other by the requirement that the determinant of the coefficients be zero. You should verify that by applying Eq. (18) and Eq. (19) to our initial problem you obtain $\omega_A^2 = \frac{K}{m}$, $\omega_B^2 = \frac{3K}{m}$, and $A_A(1)/A_A(2) = 1$, $A_B(1)/A_B(2) = -1$. This is of course the same solution we found before. The advantage of the systematic method, however, is that the solution is not more difficult if the masses or the springs are different, whereas the other methods of solving the equations of motion become more complicated.

Finally, one can write the most general solution as a linear combination of the two normal modes:

$$x_1(t) = A_A(1)\cos(\omega_A t + \alpha_A) + A_B(1)\cos(\omega_B t + \alpha_B)$$

$$x_2(t) = A_A(2)\cos(\omega_A t + \alpha_A) + A_B(2)\cos(\omega_B t + \alpha_B) \qquad (20)$$

This has six constants, but due to Eqs. (19) only four are independent. This is exactly what we need to accommodate the initial position and velocity of the two masses.

The systematic method can be generalized to an arbitrary number of masses connected by springs. **There are as many normal modes as degrees of freedom in the system**, whereby the number of degrees of freedom is the number of functions of time we are trying to find. For example, in the problem of the two masses connected to springs we are looking for the functions $x_1(t)$ and $x_2(t)$. Thus this system has two degrees of freedom and two normal modes. If we had three masses moving along a line, we would be looking at a system with three degrees of freedom and three normal modes. The systematic method can still be applied, but we would have one extra equation of motion in Eq.(15). This leads to a 3×3 determinant instead of the 2×2 system we discussed above. As more masses are added to the system, the determinant becomes larger and larger. By the time we have 5 or more masses, it must be solved numerically. Unfortunately, the numerical algorithms used to solve determinants have a serious shortcoming: the time needed by a computer to solve the problem is proportional to the *cube* of the number of masses. If 10 seconds are needed to solve the problem of 3 masses, the computer requires 80 seconds to solve a problem that involves six masses. This makes it impossible to compute the vibrations of atoms in many types of crystals. For example, in a fast state-of-the-art workstation the time needed to solve the problem of 600 interconnected masses is of the order of 1 hour. 6000 masses would require 42 days! However, even more masses are needed to study certain phenomena, such as the motion of dislocations in crystals. Worse, in a real crystal the atoms can move in three directions, so that for each mass we have 3 degrees of freedom. Hence the number of normal modes in a three-dimensional system with N masses is $3N$ and the determinant to solve has a $3N \times 3N$ size. One of the "hot topics" in condensed matter physics today is the development of "order N" methods, for which the computer time is proportional to N rather than N^3.

Guessing the normal modes

In cases where the system under investigation has some "nice" symmetries, such as the mass-spring system in Fig. 1, it is possible to guess the displacement pattern and the frequency of the normal modes. In a normal mode, all parts of the system move with the same frequency, so that the magnitude of the restoring force per unit mass per unit displacement must be the same. If all masses are equal, as in Fig. 1, this is equivalent to requiring that the restoring force per unit length be the same for all masses. Notice that this is precisely the case for the displacement patterns of Figs. 5 and 6. The restoring force per unit displacement is K for the two masses in Fig. 5 and $3K$ for the two masses in Fig. 6 Hence the frequency of the two modes are $(K/m)^{1/2}$ and $(3K/m)^{1/2}$, respectively.

Notice that it would have been impossible to guess the displacement patterns and the frequencies of the normal modes if the masses had not been equal or if all springs had had different constants. It is the *symmetry* of the problem that allows us to guess its normal modes. The relationship between normal modes and symmetry is so important that it has been investigated systematically using *group theory*, the branch of mathematics that deals with symmetry. Using group theory techniques (which are beyond the scope of this course) one can use symmetry properties to determine the form of the displacement patterns. This is very important in connection with the above-mentioned problem of solving large determinants: using group theory, one can often reduce a problem that consists of a large determinant into several smaller problems that consist of smaller determinants. A classical example is that of a periodic system: a system that repeats itself in space, such as a perfect crystal. When group theory is applied to this problem, it can be shown that the very large determinant for the normal modes of the system can be reduced to a set of small determinants whose size is equal to the number of atoms in the unit that repeats itself. This number is usually small. For example, silicon, the most important material for modern technology, has only 2 atoms per "unit cell". Hence the vibrations of silicon can be calculated solving small 6×6 determinants. (It is 6×6 and not 2×2 because each atom in a real crystal has three degrees of freedom x, y, and z).

Damped normal modes

So far we have neglected the effect of damping in our discussion of normal modes. In a real system there is always damping, for the normal mode oscillations eventually die out, in much the same way the vibration of a single mass is damped due to friction. Friction adds a term of the form $-\lambda v$ to the equations of motion of the masses in the system. For example, Eqs. (1) and (2) become

$$m\ddot{x}_1 = -Kx_1 + K\left(x_2 - x_1\right) - \lambda_1 \dot{x}_1 \qquad (21)$$

and

$$m\ddot{x}_2 = -Kx_2 - K\left(x_2 - x_1\right) - \lambda_2 \dot{x}_2 \qquad (22)$$

If we assume the same friction coefficient for the two masses, *i.e.*, $\lambda_1 = \lambda_2 = \lambda$, we obtain, by adding and subtracting these equations,

$$m\left(\ddot{x}_1 + \ddot{x}_2\right) = -K\left(x_1 + x_2\right) - \lambda\left(\dot{x}_1 + \dot{x}_2\right) \qquad (23)$$

and

$$m\left(\ddot{x}_1 - \ddot{x}_2\right) = -3K\left(x_1 - x_2\right) - \lambda\left(\dot{x}_1 - \dot{x}_2\right) \qquad (24)$$

These are damped harmonic oscillator equations for the variables $x_1(t)+ x_2(t)$ and $x_1(t)- x_2(t)$. From Chapter 2, we already know the solution to this type of equation. We obtain

$$\begin{aligned} x_1(t) + x_2(t) &= 2A_{\mathrm{A}} e^{-\gamma t} \cos\left(\omega_{\mathrm{A}} t + \alpha_{\mathrm{A}}\right) \\ x_1(t) - x_2(t) &= 2A_{\mathrm{B}} e^{-\gamma t} \cos\left(\omega_{\mathrm{B}} t + \alpha_{\mathrm{B}}\right)' \end{aligned} \qquad (25)$$

with $\gamma = \lambda/2m$, $\omega_{\mathrm{A}}^2 = \frac{K}{m} - \gamma^2$, and $\omega_{\mathrm{B}}^2 = \frac{3K}{m} - \gamma^2$. The motion of the individual masses is finally described by

$$\begin{aligned} x_1(t) &= A_{\mathrm{A}} e^{-\gamma t} \cos\left(\omega_{\mathrm{A}} t + \alpha_{\mathrm{A}}\right) + A_{\mathrm{B}} e^{-\gamma t} \cos\left(\omega_{\mathrm{B}} t + \alpha_{\mathrm{B}}\right) \\ x_2(t) &= A_{\mathrm{A}} e^{-\gamma t} \cos\left(\omega_{\mathrm{A}} t + \alpha_{\mathrm{A}}\right) - A_{\mathrm{B}} e^{-\gamma t} \cos\left(\omega_{\mathrm{B}} t + \alpha_{\mathrm{B}}\right) \end{aligned} \qquad (26)$$

Hence the solution to the damped system of connected masses is particularly simple: each normal mode behaves like an independent damped harmonic oscillator. This simplicity, however, is somewhat fortuitous. If $\lambda_1 \neq \lambda_2$, a not very unreasonable assumption, we can no longer separate the equations of motion into two independent equations for the functions $x_1(t)+x_2(t)$ and $x_1(t)$-$x_2(t)$. Physically, this means that damping can "mix" normal modes: if you select your initial conditions so that only one mode is excited, other normal modes may be indirectly excited due to the effect of damping. In addition to this complication, our mathematical treatment of friction misses an important feature of damping in systems with several moving parts. We model friction as a force acting on the vibrating masses. In real systems, however, friction forces are distributed over the entire system. In the case of masses and springs, an important component of the friction force acts on the springs themselves, which become warmer and dissipate energy as they contract and expand repeatedly. In the case of a single mass, this effect can be included as an effective friction force assumed to act on the mass. For systems with more than one mass, however, the distinction between friction from the springs and friction from the masses is important. Consider for example the two normal modes A and B depicted in Figs. 5 and 6. If the middle spring makes an important contribution to damping, its contribution will not show up in mode A, because the middle spring is never stretched in this mode. By contrast, the spring is compressed and expanded in mode B. Therefore, we would expect a larger damping for mode B. This conclusion is very general:

> **In a system with several normal modes, the normal mode with the lowest frequency is usually the one with the smallest damping. This is because there are less moving parts for this mode.**

For example, if you shake an aquarium tank, you will excite many normal modes of the water in that aquarium. After a while, however, you will see very clearly the lowest mode, for which the water goes up at one end and down at the other end of the aquarium.

Forced oscillations of normal modes

Suppose that in our system of two masses we apply a force

$F_0 \cos \omega_f t$ to the first mass. The equations of motion become

$$m\ddot{x}_1 = -Kx_1 + K(x_2 - x_1) - \lambda\dot{x}_1 + F_0 \cos \omega_f t \qquad (27)$$

and

$$m\ddot{x}_2 = -Kx_2 - K(x_2 - x_1) - \lambda\dot{x}_2 \qquad (28)$$

By adding and subtracting these equations we obtain two forced oscillator equations for the variables $x_1(t)+x_2(t)$ and $x_1(t)$-$x_2(t)$:

$$m(\ddot{x}_1 + \ddot{x}_2) = -K(x_1 + x_2) - \lambda(\dot{x}_1 + \dot{x}_2) + F_0 \cos \omega_f t$$
$$m(\ddot{x}_1 - \ddot{x}_2) = -3K(x_1 - x_2) - \lambda(\dot{x}_1 - \dot{x}_2) + F_0 \cos \omega_f t \quad (29)$$

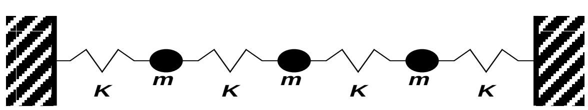
Notice that the two normal modes are driven by the same force. This is due to our particular choice for the external force. It is possible (see end of chapter problem) to apply forces on the masses so that only one of the normal modes is excited. On the other hand, even though the two normal modes are driven by the same force, their response will be different depending on the value of the frequency ω_f. If ω_f is very close to $\sqrt{\frac{K}{m}}$, mode A will be in resonance, so that its amplitude will usually be much larger. Conversely, if ω_f is close to $\sqrt{\frac{3K}{m}}$, mode B will be in resonance. If the separation between the frequencies of the modes is much larger than the broadening parameter γ, it is possible to resonantly excite only one normal mode. But if the separation is of the order of γ, one may not be able to isolate the modes by tuning the exciting frequency ω_f. Sometimes, this may lead to catastrophe. The famous Tacoma Narrows Bridge had two normal modes: an up-and-down motion and a torsional motion. Their frequency difference was of the order of γ. When the wind excited the up-and -down normal mode, the torsional motion picked up a significant amount of the external energy. Eventually, the bridge collapsed due to the torsional motion.

Problems

1. Two 1 kg masses are connected by springs as in Fig. 1. The spring constant of the three springs is $K = 100$ N/m. The mass on the left is initially displaced to the right by 0.2 m. The mass on the right is initially displaced to the right by 0.1 m. Their initial velocities are zero. Calculate the position as a function of time for the two masses *both* analytically and numerically using the step-by-step procedure discussed in previous chapters. Graph together the analytical and the numerical solutions.

2. Suppose that in the previous problem the mass on the right is $m = 2$ kg. Find the normal mode frequencies. Because the different masses destroy the symmetry of the problem, it may be easier to use the systematic approach discussed in the text.

3. Determine the mode displacement patterns and the mode frequencies for the system below. Rather than performing an actual calculation, which involves a 3×3 determinant, try to guess a displacement pattern and require that the force per unit mass per unit displacement be the same for all masses. To simplify the notation, you might want to use *a*, *b,* and *c* rather than $A_X(1)$, $A_X(2)$, and $A_X(3)$



4. Suppose that in the mass-spring system of **Fig. 1** the mass on the left is initially displaced by an amount *A.* The other mass is, initially, at the equilibrium position. The initial velocities are zero.

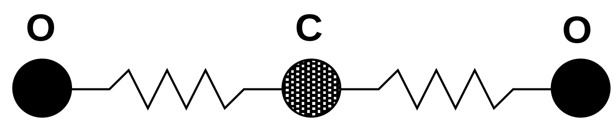
a) Find the expression that describes the position as a function of time for the two masses.

b) Show that the displacement as a function of time for the mass on the left can be written as $x(t) = A \cos \omega_1 t \cos \omega_2 t$, where $\omega_1 = (\omega_A+\omega_B)/2$ and $\omega_2 = (\omega_A-\omega_B)/2$.

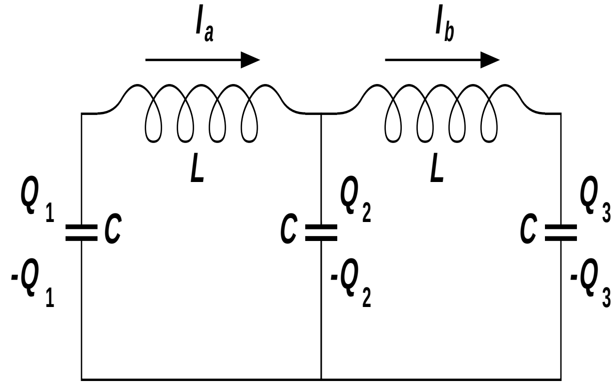
c) Guess the values of the frequencies ω_A and ω_B if the middle spring is replaced by a different spring of constant *K'*.

d) Study the motion of the mass on the left in the limit $K' \to 0$. Graph this displacement as a function of time. Invent your own parameters.

5. Discuss the normal modes of vibration of the CO_2 molecule. Go to the library, find the frequencies of the normal modes, and deduce the value of the spring constant between the carbon and oxygen atoms.



6. Find the frequencies of the two normal modes of oscillation of the coupled *LC* circuits shown in the figure. Compare your solutions with the case of two masses connected by springs.



7. Write down the steady -state solutions to Eq. (29). On the same graph, plot the amplitude of the two normal mode components of the motion of mass 1 as a function of ω_f. Invent your own parameters. Discuss the conditions under which one of the amplitudes is much larger than the other. For what values of the parameters is this possible?

8. For the two-mass system of Fig. 1, describe how to apply forces so that only one of the normal modes is excited.