

# **Chapter 2**

## **Oscillations in one-particle systems**

## Introduction

The problem of a mass attached to a spring was discussed in previous physics courses. As you remember, the mass oscillates around the point where the spring is not stretched. The reason we are reviewing this problem here is that we want to study *waves*, which are the oscillations of a system of many masses connected by many springs. As you will see, the generalization to many masses is surprisingly straightforward once you understand the problem of a single oscillating mass.

## Hooke's law

Consider the mass attached to the spring in Fig. 1

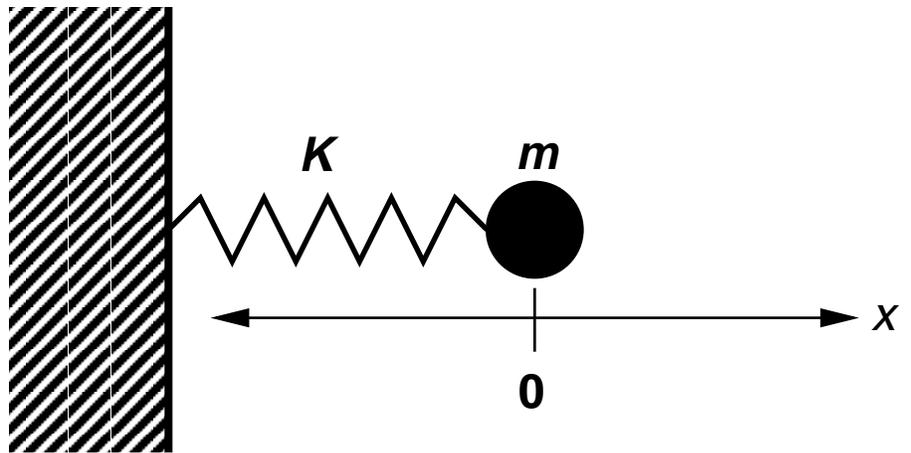


Figure 1 A mass connected to a spring is the typical example of Hooke's law.

It is known that the force exerted by the spring on the mass is proportional to the displacement of the mass, namely  $F = -Kx$ . The constant  $K$ , known as the *spring constant*, measures how strong that force is. The minus sign in the expression for the force is of fundamental importance. It means that the force opposes the displacement, ultimately leading to oscillations. Without the minus sign in the force, an object displaced by any amount would never return to the starting point.

The above force is an example of *Hooke's law*. You may wonder what is the origin of Hooke's law. In the previous chapter, we stated that there are only a few fundamental forces in Nature. We did not mention Hooke's law. This

implies that this law must be a special case of one of the fundamental interactions. Let us consider what happens when we stretch a spring, or, for that matter, any object. Clearly, the separation between atoms, and, consequently, their mutual forces, will change. The forces between atoms are *electrical* in nature. The gravitational interaction between the atoms of any human-size object is negligible, and the strong nuclear force is not involved, because the distance between protons and neutrons inside the nuclei does not change. It is only the separation between atoms which increases, and this must affect the Coulomb interaction between electrons and nuclei. But the Coulomb interaction is inversely proportional to the square of the separation between charges. This makes it hard to understand Hooke's law, which states that the force is *proportional* to the separation. The key to understanding Hooke's law is to realize that a given atom interacts with all its neighbors. When a solid object is formed, the atoms will look for the positions that *minimize* their energy. In other words, the atoms will sit at a position where their potential energy has a minimum. Near a minimum, the potential energy can always be written as  $E_p = E_0 + cx^2 + dx^3 + fx^4 + \dots$ . There can be no linear terms of the form  $bx$ , because the force must be zero at the equilibrium position. (Remember that  $F = -\frac{dE_p}{dx}$ ). For  $x$  small, the cubic and higher terms will be negligible, so that we end up with a potential energy that has a parabolic form. However, you remember that this is exactly the case for a spring that obeys Hooke's law: its potential energy is given by  $E_p = \frac{1}{2}Kx^2$ . Therefore, it is not surprising that most objects obey Hooke's law. However, there is a limitation: The displacement from the equilibrium position must be small, otherwise, the cubic and higher terms in the expansion of the potential energy become important. This can be observed experimentally: if you apply a very strong stretching force to an object, eventually the elongation ceases to be proportional to the force. This is sometimes called the transition from the elastic to the plastic regime.

In the problems we will consider, the spring constant  $K$  is a quantity that must be obtained experimentally. On the other hand, you might now conjecture that by carefully constructing the potential energy function for our material we might be able to *derive* the spring constant from the electrical forces between atoms. Unfortunately, this is not an easy task, because the interactions typically involve hundreds of electrons and nuclei and because the problem must be treated with *quantum mechanics*. It is only in the 1970's and 1980's that powerful methods have been developed which allow the calculation of spring constants from *first principles*. This is an active area of research, particularly at Arizona State University.

## The harmonic oscillator

When Hooke's law is inserted into Newton's second law, we obtain, as discussed in the previous chapter, the following differential equation of motion:

$$\frac{d^2x}{dt^2} = -\frac{K}{m}x \quad (1)$$

The solution to this equation is a function whose second derivative is proportional to the function itself. We know two types of elementary functions that have this property: exponentials and the trigonometric functions sine and cosine. As you will show in one of the homework problems, exponentials cannot be a solution to Eq. (1). We can also decide this issue on the basis of an experiment: if we displace the mass, it will oscillate. The exponential function is not oscillatory, but the trigonometric functions are. We thus explore solutions of the form

$$x(t) = A \cos \omega t \quad (2)$$

Notice that we have introduced two constants, the *amplitude*  $A$  and the *angular frequency*  $\omega$ . The amplitude  $A$  is needed for two reasons: first, the function  $x(t)$  has units of distance, while the trigonometric function cosine has no units. Thus  $A$  has units of distance and insures the dimensional consistency between the two sides of Eq. (2). On the other hand, the function cosine runs between 1 and -1. Our

mass, however, will oscillate between any two values. As you can easily show, these values will be precisely  $A$  and  $-A$ . The angular frequency  $\omega$  is introduced to compensate the units of  $t$  in the argument of the cosine. This argument is in radians. If we want time inside the argument, we must have a factor with units of  $1/\text{time}$  so that the units of the argument cancel out. This factor is  $\omega$ .

If you plug the proposed solution into Eq.(1), the following relationship obtains:

$$\omega^2 = \frac{K}{m} \quad (3)$$

This means that Eq.(2) is indeed a solution to the spring equation provided that we select a value of  $\omega$  that satisfies Eq. (3).

### The meaning of $\omega$

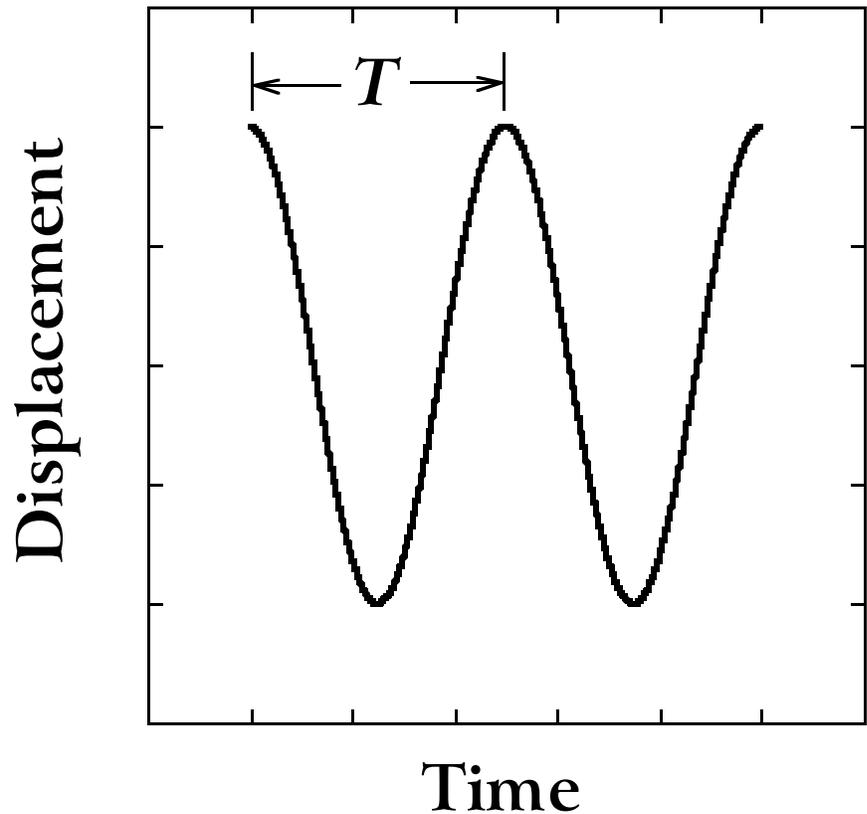


Figure 2 Oscillations of a mass attached to a spring

Fig. 2 displays the proposed solution, Eq. (2). As expected for an oscillation, the function repeats itself after a certain time. We call this time the *period*  $T$ . Its value can be obtained from the condition

$$\cos[\omega(t+T)] = \cos \omega t \quad (4)$$

which follows from the definition of period. This condition has an infinite number of solutions, given by  $\omega T = 2\pi n$ , where  $n$  is any integer. But period is defined as the shortest time for which the motion repeats itself, so that it corresponds to  $n = 1$ . Thus the period is given by

$$T = \frac{2\pi}{\omega} \quad (5)$$

When  $\omega$  is large, the period is short, so that the oscillations are very rapid. When  $\omega$  is small, the period is long, so that the oscillations are slow. Now consider Eq. (3). In view of the meaning of  $\omega$ , this equation means that the oscillations will be faster the stronger the spring force, (large  $K$ ) but slower the larger the mass. This is a very important and general result, which you'll find again and again when we discuss waves.

**The angular frequency  $\omega$  is always given by something proportional to the strength of the “restoring” force divided by a quantity that measures the inertia of the object.**

The reason why  $\omega$  is called angular frequency is that one can define a quantity, known as the *frequency*  $\nu$ , that measures the number of oscillations per unit time. Thus the frequency is the inverse of the period:

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \quad (6)$$

The angular frequency  $\omega$  is therefore proportional to the frequency  $\nu$ . From Eq. (6) it is easy to see that the reason for the word “angular” is that one oscillation per second corresponds to  $\omega = 2\pi \text{ } 1/\text{s}$ , whereby  $2\pi$  is the number of

radians in a full circle. Of course, there are no physical angles in the spring problem. Traditionally, however, the arguments of the trigonometric functions are angles. This explains why you'll find the word "angle" quite frequently in the next few sections, although - we insist - there are no geometrical angles in this problem.

### **Completeness of the solution**

It appears that we have found a solution to the equation of motion for a mass attached to a spring. We now must answer a fundamental question: is this solution complete? By "complete" we mean the following: is Eq. (2) the solution to the problem for *any* initial position and *any* initial velocity of the mass? The form of Eq. (2) suggests that we might be in trouble: the equation has two constants,  $A$  and  $\omega$ . But  $\omega$  is "used up" in Eq. (3), so that we are left with only one "adjustable" constant with which we are supposed to account for two numbers: the initial position and the initial velocity.

Let's inspect the solution at  $t = 0$ . Eq. (2) gives  $x(0) = A \cos 0 = A$ . This means that  $A$  is nothing but the initial position of our mass. What is the velocity at  $t = 0$ ? By differentiating Eq. (2), we obtain  $v(t) = -\omega A \sin \omega t$ . For  $t = 0$ , we obtain  $v(0) = -\omega A \sin 0 = 0$ . Thus the initial velocity is zero! This means that we have not yet found a complete solution to the oscillating mass problem. Our solution only applies to cases where the initial velocity is zero. For example, we might stretch the spring to a certain position, wait for a moment and release it. But if the initial velocity is different from zero, Eq. (2) does not apply. What we have found is very general: any solution to the equation of motion of a particle must have two "adjustable" constants to account for the arbitrary initial position and arbitrary initial velocity.

### **The complete solution to the spring problem**

If you go back to our discussion leading to the proposal of Eq. (2) as the solution of the problem, you may wonder why we chose the cosine function instead of the sine function. The fact is that both functions are solutions to our problem.

If you use  $x(t) = B \sin \omega t$ , you'll find that this solution satisfies Eq.(1) with the same condition specified in Eq. (3). However, in this case we obtain  $x(0) = 0$  and  $v(0) = \omega B$ . So the “cosine” solution gives a non-zero initial position and zero initial velocity, whereas the “sine” solution gives zero initial position and a non-zero initial velocity. How about combining the two solutions? It turns out that the spring equation is a *linear* differential equation with the following property: if two arbitrarily different functions are solutions to the equation, then their sum is also a solution. Hence we propose as our complete solution

$$x(t) = A \cos \omega t + B \sin \omega t \quad (7)$$

### Linear homogenous differential equations

The spring equation belongs to a special kind of differential equations that can be written as

$$A_0 x + A_1 \frac{dx}{dt} + A_2 \frac{d^2 x}{dt^2} + \dots + A_n \frac{d^n x}{dt^n} + \dots = 0$$

These equations have the following features:

- there are no terms independent of  $x$ , the function sought. Because of this, the equations are called *homogeneous*. For example  $x - t^3 = 0$  is *not* an homogeneous equation, but  $x + 2 \frac{dx}{dt} = 0$  is homogeneous.
- There are terms no higher than the *first* power of  $x$  or its derivatives. Because of this, the equation is said to be *linear*. For example,  $x^2 + \frac{dx}{dt} = 0$  or  $x + \left(\frac{dx}{dt}\right)^2 = 0$  are not linear, but the spring equation (1) is.

The fundamental property of linear differential equations is the *superposition principle* if we find two solutions,  $f_1(t)$  and  $f_2(t)$ , any linear combination of these functions,  $B_1 f_1(t) + B_2 f_2(t)$ , (where the  $B$ 's are constants) is also a solution of the equation. This is actually not a principle but an easy theorem. You will verify the superposition theorem in one of the end-of-chapter problems.

Eq. (7) has the two required “adjustable” constants to account for the initial position and initial velocity. They are given by

$$x(0) = A \quad ; v(0) = \omega B \quad (8)$$

You will find in the literature several alternative ways of writing Eq.(7). A popular one is to combine the two trigonometric functions into one by using the identities

$$\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha \cos\beta + \sin\beta \cos\alpha \\ \cos(\alpha + \beta) &= \cos\alpha \cos\beta - \sin\alpha \sin\beta \end{aligned} \quad (9)$$

Using this, we can rewrite Eq.(7) as

$$x(t) = A_{amp} \cos(\omega t + \alpha) \quad (10)$$

Notice that this expression also has two “adjustable” constants,  $A_{amp}$  and  $\alpha$ . Using the second identity in Eq. (9), we find

$$\begin{aligned} A_{amp} \cos \alpha &= A = x_0 \\ -A_{amp} \sin \alpha &= B = \frac{v_0}{\omega} \end{aligned} \quad (11)$$

so that we can solve for the constants  $A_{amp}$  and  $\alpha$ . We find

$$A_{amp} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} \quad ; \quad \alpha = \arctan\left(\frac{-v_0}{x_0 \omega}\right) \quad (12)$$

It is a matter of convenience how we write our solution. Eqs. (7) and (10) are mathematically identical. Eq. (10) has the advantage that it contains a single trigonometric function, which sometimes makes it easier to manipulate. In addition, the coefficient  $A_{amp}$  has a simple physical interpretation: it is the **amplitude** of the oscillation, *i.e.*, the mass oscillates between  $-A_{amp}$  and  $A_{amp}$ . On the other hand, the “angle”  $\alpha$ , called the “phase angle,” does not represent a geometrical angle. Its meaning is the following: at a time  $\alpha/\omega$  before  $t=0$ , the mass would be at the maximum elongation of the spring  $A_{amp}$ , with zero instantaneous velocity. The

main disadvantage of Eq.(10) is that the determination of the constants  $A_{amp}$  and  $\alpha$  from the initial conditions often leads to mistakes. The problem lies in the “tan” function, because there are *two* possible angles which have the same tangent. You must find the correct one. You do this by going back to Eqs. (11) and making sure that the *signs* you obtain for the initial position and velocity match the signs of the known initial conditions.

If you use Eq. (7), the expression is somewhat cumbersome, but the constants  $A$  and  $B$  in are trivially determined from the initial conditions. You should familiarize yourself with the two alternative ways of expressing the solutions. Trigonometric functions play a fundamental role in the description of wave phenomena. You should be able to manipulate them without hesitation.

### Same equation, same solution

In Eq. (1), the function we seek is the position as a function of time of a mass attached to a spring. From the point of view of a mathematician, however,  $x(t)$  could represent anything. Once we find a solution to the mass-spring problem, this solution will be formally identical to the solution of any other problem satisfying the same differential equation. This problem need not have anything to do with masses and springs: it could be the value of the Dow index, the public’s opinion about Bill Clinton, or the number of cockroaches in your garage. Anything whose second time derivative is proportional to minus itself, will oscillate exactly as indicated by Eq. (10). In physics, there are several phenomena that lead to oscillations (that’s why we spend so much time studying this phenomenon). A special case is that of a pendulum, which we’ll discuss next.

#### **An almost harmonic problem: the pendulum**

Let us consider Fig. 3, where we show a mass  $m$  hanging from a rod of length  $l$ . At a certain time  $t$ , we can decompose the motion into a tangential and a normal component. We arbitrarily select the bottom position as the origin from

which we measure angles, and we define as positive the counter-clockwise direction. Of course, we could select the zero and the positive direction in any way we want; no result is affected by this choice. We define the arc displacement function as  $s(t) = l \theta(t)$ . Because of our sign convention for  $\theta(t)$ ,  $s(t)$  is positive when the mass is to the right of the vertical and negative when the mass is to the left of the vertical. If we now write Newton's second law for the tangential motion, we obtain

$$-mg \sin \theta = m a_T \quad (13)$$

Notice the minus sign in front of the force term. The reason for the minus sign is that the gravitational force acts to reduce the arc when the mass is at the position of the figure.

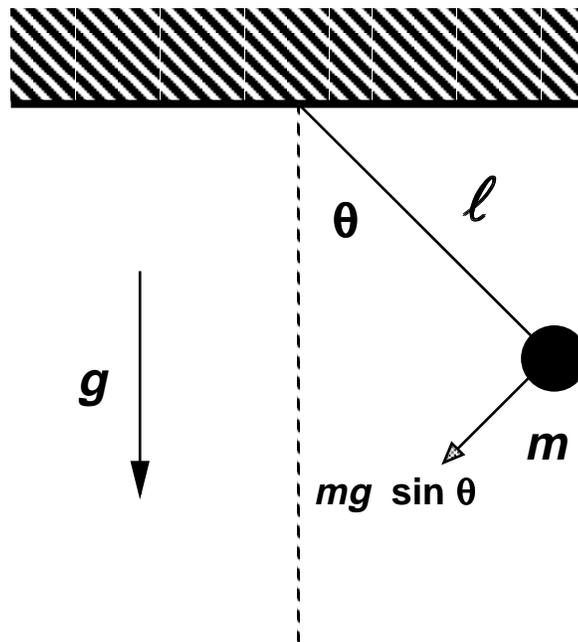


Figure 3 Pendulum

By definition,  $a_T = \frac{d^2 s}{dt^2}$ . Using the expression that gives the arc as a function of the angle, and eliminating the mass (which appears on the two sides of the equation), we obtain

$$\frac{d^2 \theta}{dt^2} = -\left(\frac{g}{l}\right) \sin \theta \quad (14)$$

This equation is *not* equivalent to the spring equation,

because the second derivative of the angle is not proportional to minus the angle but to minus the *sine* of the angle. In fact, this is a very complicated equation, if you consider that the sine of a quantity is given by an infinite series. However, we can make an approximation for the case where the angular displacement of the pendulum is small. (Not for the drawing in Fig. 3!) With that restriction, we can approximate  $\sin \theta \sim \theta$ , in which case we obtain

$$\frac{d^2 \theta}{dt^2} = -\left(\frac{g}{l}\right) \theta \quad (15)$$

This equation is formally identical to Eq. (1) so that we can immediately write the solution:

$$\theta(t) = A_{amp} \cos(\omega t + \alpha) \quad (16)$$

with  $\omega^2 = \frac{g}{l}$ . The constants  $A_{amp}$  and  $\alpha$  are determined from the initial conditions, which in this case are the initial angle and the initial angular velocity. Notice that the quantity  $\theta$  in Eq. (16) corresponds to a real physical angle, namely the angle between the pendulum and the vertical at any given time. This means that the amplitude  $A_{amp}$  has units of angle. On the other hand, the “angle”  $\alpha$  on the left-hand side of the equation does not correspond to a physical angle: as in the case of the mass and the spring, it is a parameter that is determined from the initial conditions.

## Damped harmonic oscillators

The most important difference between “real” harmonic oscillators and the “theoretical” harmonic oscillators we discussed in the previous section is the fact the oscillations die out in real ones. A real harmonic oscillator, when left alone, does not oscillate forever, as implied by our solutions in Eqs. (7) or (10). The obvious reason for this “damping” of the oscillations is the existence of friction. Friction tends to stop the oscillation because friction forces always *oppose* the motion. Thus the work they do is *negative*. This work is equal to the change in energy of the mass attached to the spring. This means that the energy of the mass

decreases and eventually becomes zero. Very often, the damping is relatively slow: it takes many oscillations for the mass to come to a final stop. In these cases, a given oscillation is very similar to the next, so that if you are interested in these short times, you might neglect the effect of the damping. Hence damping can often be ignored in many problems involving the *free* oscillations of a mass-spring system. However, damping becomes critically important when you force the oscillations from outside the system, by applying an *external* force. In these cases, the system reaches an equilibrium between energy intake and energy loss only if there is some friction. Friction is therefore essential to discuss forced harmonic oscillators, the subject of the next section. Mathematically, however, it is easier to start our discussion with the free-oscillation case.

### **The origin of the friction force**

Old-fashioned textbooks used to enumerate a long list of different forces. The friction force was a prominent member of this list. However, we know that there are only four fundamental interactions in Nature, so that friction forces must be a manifestation of one of those. The interactions between human-size objects are electrical in nature: friction must ultimately owe its origin to electrical interactions between atoms. Unlike the case of Hooke's law, however, it is not possible to derive an expression for the friction force based on general arguments. The reason is that many factors will affect friction: the shape of the surface of the objects, the atomic arrangement at the surface (which is often different than in the bulk of the material), the temperature, the strength of the atomic binding in a given material vis-a-vis the strength of the interactions with other materials, etc, etc. On the other hand, let's consider the form of the solution to the equation of motion when we include friction (you might wonder how can we possibly find a solution if we don't know yet the form of the equation. The answer is that we are in the Physics Department, not in the Math Department. We can do experiments!). If you observe a real harmonic oscillator, the oscillations will look as in Fig. 4.

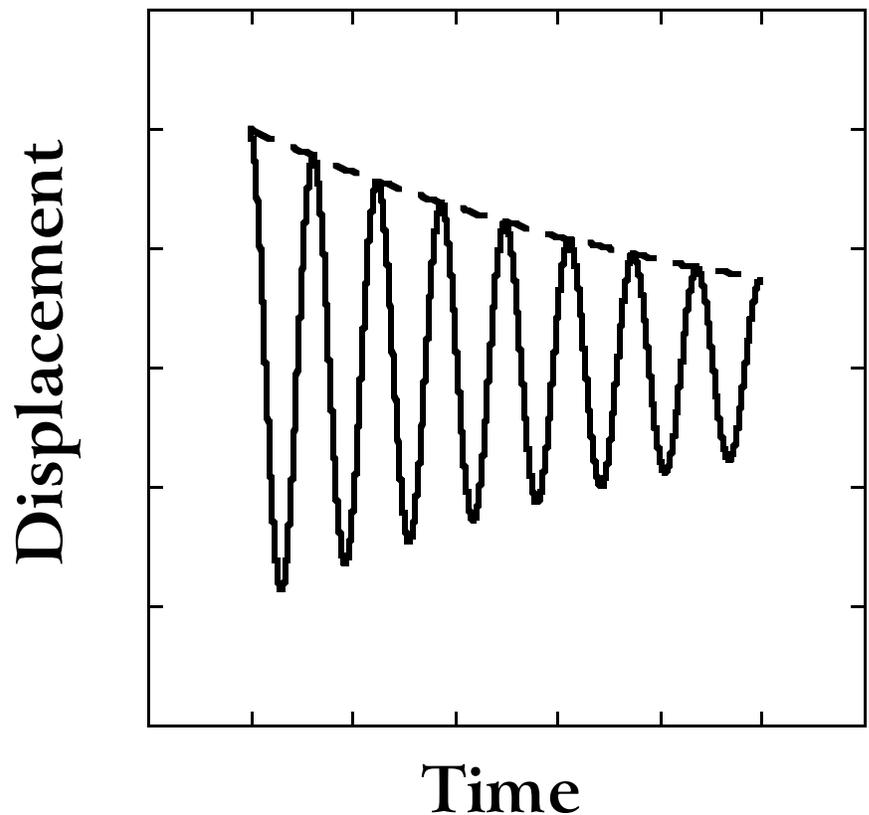


Figure 4 The motion of a real, damped harmonic oscillator

Note that the amplitude of the oscillations diminishes as a function of time. In fact, if you pass a function through the maxima of the oscillations (this is the dashed line in the figure) you will find that this function is usually proportional to a simple exponential function of the form  $e^{-\gamma t}$ , where  $\gamma$  is a constant. We might therefore adopt the point of view Newton adopted when he discovered the gravitational law. We might ask ourselves: what would be the simplest expression for the friction force that leads to the solutions depicted in Fig. 4? The answer is really *very* simple:

$$F_{\text{friction}} = -\lambda v, \quad (17)$$

where  $\lambda$  is a constant and  $v$  is the velocity of the mass. This equation has two appealing features: it automatically gives a

force that opposes the motion and, when included in the equation of motion, preserves its linear character. However, one should always be careful concerning the applicability of this expression. In many common phenomena, such as the friction experienced by an object travelling very fast in air, the friction force is better approximated by a term proportional to the *square* of the velocity. This, besides making it even more difficult to design fast cars, leads to non-linear equations of motion.

### Solution to the damped oscillator problem

Let us now show how a solution like the one depicted in Fig. 4 is obtained when a friction force proportional to the velocity is added to the spring problem. The equation of motion for the mass is now  $a = (-Kx - \lambda v)/m$ , which is equivalent to

$$\frac{d^2 x}{dt^2} + \frac{\lambda}{m} \frac{dx}{dt} + \frac{K}{m} x = 0 \quad (18)$$

In view of Fig. 4, let's try a solution of the form

$$x(t) = A_{amp} e^{-\gamma t} \cos(\omega t + \alpha) \quad (19)$$

If we plug Eq. (19) into Eq. (18), we obtain after some algebra and calculus the following (messy) equation:

$$\begin{aligned} & (\gamma^2 - \omega^2) A_{amp} e^{-\gamma t} \cos(\omega t + \alpha) + 2\gamma\omega A_{amp} e^{-\gamma t} \sin(\omega t + \alpha) \\ & + \frac{\lambda}{m} \left[ -\gamma A_{amp} e^{-\gamma t} \cos(\omega t + \alpha) - \omega A_{amp} e^{-\gamma t} \sin(\omega t + \alpha) \right] \\ & + \frac{K}{m} A_{amp} e^{-\gamma t} \cos(\omega t + \alpha) = 0 \end{aligned} \quad (20)$$

This can only be satisfied if the terms that contain cosines and sines vanish independently, leading to the following two equations

$$\begin{aligned} & (\gamma^2 - \omega^2) - \frac{\gamma\lambda}{m} + \frac{K}{m} = 0 \\ & 2\gamma\omega - \frac{\lambda\omega}{m} = 0 \end{aligned} \quad (21)$$

We can now solve for  $\omega$  and  $\gamma$ :

$$\begin{aligned}\gamma &= \frac{\lambda}{2m} \\ \omega^2 &= \frac{K}{m} - \gamma^2\end{aligned}\quad (22)$$

The expression for  $\gamma$  makes sense: if the damping force becomes zero ( $\lambda = 0$ ), then  $\gamma = 0$  and  $e^{-\gamma t} = 1$ , so that we recover our solution without friction. The expression for  $\omega$  is somewhat surprising. It implies that damping affects not only the amplitude of the motion but also its frequency. We did not anticipate this effect when we proposed the solution. The reason becomes quite obvious upon inspection of Eq. (22). Even for small values of  $\gamma$ , we can see its effect on the amplitude simply by waiting long enough. On the other hand, if  $\gamma \ll \omega$ , the frequency will be almost indistinguishable from the value without damping. Notice that our solution requires  $\omega^2 > 0$ . However, nothing prevents the expression for  $\omega^2$  in Eq. (22) to become negative. If that happens, our solution must be wrong. This is the case of “overdamping”: the friction is so strong that the mass doesn’t oscillate but approaches the equilibrium point asymptotically. The solution is a combination of two decaying exponentials, which is not surprising if you remember that exponentials and trigonometric functions are closely related via complex numbers, which would appear in this case because  $\omega^2$  becomes negative.

## Forced harmonic oscillations

Let us now consider the most complicated case: a damped harmonic oscillator subject to an external force. Let us assume that the force is periodic and can be written as  $F(t) = F_0 \cos \omega_f t$ . Adding this term to Eq.(18) we obtain

$$\frac{d^2 x}{dt^2} + \frac{\lambda}{m} \frac{dx}{dt} + \frac{K}{m} x = \frac{F_0}{m} \cos \omega_f t \quad (23)$$

This equation has a very important difference with Eq.(18). The term of the right hand side is independent of  $x$ , so that the equation is no longer *homogeneous*. Again we will find a solution to this equation by observing experimental results.

They will convince you of a fundamental property of linear, classical systems: if you excite them at a certain frequency, the system responds at the *same* frequency. This is a result worth remembering. It will play a fundamental role in our analysis of wave motion in the coming chapters. If the system responds at the same frequency, we can write the solution to Eq.(23) as

$$x(t) = A \cos \omega_f t + B \sin \omega_f t \quad (24)$$

If you plug this equation into Eq. (23) you obtain a lengthy expression that contains  $\cos \omega_f t$  and  $\sin \omega_f t$ . The terms that contain  $\cos \omega_f t$  and the terms that contain  $\sin \omega_f t$  must satisfy the equation independently. This leads to two equations whose solution is finally

$$A = \frac{F_0}{m} \frac{\omega_0^2 - \omega_f^2}{(\omega_0^2 - \omega_f^2)^2 + 4\gamma^2 \omega_f^2}$$

$$B = \frac{F_0}{m} \frac{2\gamma\omega_f}{(\omega_0^2 - \omega_f^2)^2 + 4\gamma^2 \omega_f^2} \quad (25)$$

where we have used  $\omega_0^2 = K/m$  and  $\gamma = \lambda/2m$ . In analogy with the transformation leading to Eq. (10) from Eq. (7), we can combine the two trigonometric functions in Eq. (24) into a single expression of the form

$$x(t) = A_{amp} \cos(\omega_f t + \alpha) \quad (26)$$

with

$$A_{amp} = \sqrt{A^2 + B^2} = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_f^2)^2 + (2\omega_f\gamma)^2}}$$

$$\tan \alpha = -\frac{B}{A} = \frac{2\omega_f\gamma}{(\omega_f^2 - \omega_0^2)} \quad (27)$$

Several features of Eqs. (25) and (27) are worth discussing.

Notice that while the system oscillates at the *external* frequency  $\omega_f$ , it somehow “remembers” that it “likes” to oscillate at the frequency  $\omega_0$  of the free oscillations. So when the external frequency  $\omega_f$  coincides with the natural frequency  $\omega_0$ , the amplitude  $A_{amp}$  becomes very large. It actually becomes infinity for vanishing damping and  $\omega_0 = \omega_f$ .

The other important feature about Eqs. (25) and (27) is that the constants  $A$  and  $B$  (or  $A_{amp}$  and  $\alpha$ ) have definite expressions and are not dependent on the initial conditions. (To avoid confusion, we should have used a different notation for the constants that appear in free and forced oscillations, since their physical meaning is very different.) This means that our solution, Eq. (24) or Eq. (27), cannot be the complete solution to our problem, since, as we discussed before, there should be two “free” constants to account for the initial position and velocity. Fortunately, the *linearity* of the equation of motion provides an easy way out of this problem. Eq. (23) is a linear *inhomogeneous* equation. These equations satisfy a special version of the superposition principle.

**Superposition principle for linear inhomogeneous equations**

Suppose that  $x_1(t)$  is the solution to a linear inhomogeneous equation where the external driving force is given by  $F_1(t)$ , and  $x_2(t)$  is the solution for the same equation when the external driving force is given by  $F_2(t)$ . Then, if both driving forces act simultaneously, so that the driving force becomes  $F_1(t) + F_2(t)$ , the sum  $x_1(t) + x_2(t)$  is the solution to the new problem.

This theorem can be applied to the special case where one of the external driving forces is zero. Of course, we have discussed this case: free oscillations with damping, whose solution is given by Eq. (19). Hence the complete solution to the problem of a harmonic oscillator subject to an external force  $F = F_0 \cos \omega_f t$  is given by

$$x(t) = A_{amp} \cos(\omega_f t + \alpha) + B e^{-\gamma t} \cos(\omega t + \beta) \quad (28)$$

where  $A_{amp}$  and  $\alpha$  are given by Eq. (27),  $\omega$  and  $\gamma$  by Eq.(22), and  $B$  and  $\beta$  are the “adjustable” constants that allow us to accommodate any initial position and any initial velocity.

A very remarkable feature of Eq. (28) is what happens a long time after the oscillation is started. Due to the exponential in the second term, this term vanishes for very long times, so that we are left with the first term, which has no “independent” constants. This means that two identical harmonic oscillators subject to exactly the same external force but starting with different initial conditions, will eventually perform the same motion. In other words, the system “forgets” its initial conditions. This is clearly due to the damping, for if  $\gamma = 0$  then the second term never vanishes. The result is easier to understand if you consider what happens if a damped oscillator is left alone: it eventually stops, no matter what the initial conditions are. This, combined with the superposition principle, explains the curious behavior of Eq. (28).

The first term of Eq.(28) is usually referred to as the “steady state” solution, whereas the second term is called the “transient.” The meaning of this terminology is obvious from our discussion above.

### Resonance

When the exciting frequency  $\omega_f$  approaches the *natural* frequency  $\omega_0$ , the system experiences a *resonance*. At resonance, the amplitude of the oscillation, given by Eq. (27) is dramatically enhanced. This means that the system is able to absorb a large amount of energy from the external driving force. The work done by this force is given by  $dW = [F_0 \cos(\omega_f t)] dx$ , where  $dx$  is the displacement of the oscillator. If we divide both sides by  $dt$  we obtain the instantaneous *power* given by

$$P(t) = \frac{dW}{dt} = [F_0 \cos \omega_f t] v(t) \quad (29)$$

The velocity  $v(t)$  can be obtained by differentiating Eq. (24). We obtain

$$\begin{aligned} P(t) &= F_0 \cos \omega_f t \left[ \omega_f B \cos \omega_f t - \omega_f A \sin \omega_f t \right] \\ &= F_0 \omega_f B \cos^2 \omega_f t - F_0 \omega_f A \cos \omega_f t \sin \omega_f t \end{aligned} \quad (30)$$

The function  $\cos^2 \omega t$  is positive at all times, whereas the product  $\cos \omega t \sin \omega t$  is positive and negative an equal amount of time. During a period, the  $\cos^2$  term gives an average value of  $1/2$ , whereas the  $\sin \cos$  term averages to zero. Hence the average power absorbed during a cycle is

$$P = \frac{1}{2} F_0 \omega_f B = \frac{F_0^2}{m} \frac{\gamma \omega_f^2}{(\omega_0^2 - \omega_f^2)^2 + 4\gamma^2 \omega_f^2} \quad (31)$$

This function has a maximum for  $\omega_f = \omega_0$ , which means that the maximum power absorption occurs for excitation at the natural frequency of the system.

## Problems

1. Show that  $x(t) = A e^{-at}$ , where  $a$  is a real number, cannot be a solution to the harmonic oscillator equation.

2. In problem 2, previous chapter, you studied numerically the oscillations of a harmonic oscillator. Find the analytical solution to the same problem, and plot the analytical solution on top of the numerical solution.

3. A mass  $m = 2$  kg is attached to a spring of constant  $K = 8$  N/m. The initial position is  $x(0) = 3$  m and  $v(0) = 2$  m/s.

a) By writing the solution as Eq. (7), determine all constants in that expression.

b) By writing the solution as in Eq. (10), determine all constants in that expression.

c) Graph the two solutions in parts a) and b) to show that they are identical.

4. Repeat Problem 3 with  $x(0) = -3$  m.

5. Repeat Problem 3 with  $v(0) = -2$  m/s

6. Repeat Prob. 3 with  $x(0) = -3$  m and  $v(0) = -2$  m/s.

7. (Alonso 10.13) When a person of mass 60 kg gets into a car, the center of gravity of the car lowers 0.3 cm.

a) What is the elastic constant of the springs of the car?

b) Given that the mass of the car is 500 kg, what is its period of vibration when it is empty and when the person is inside?

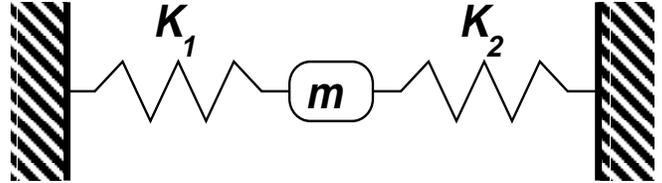
8. (Alonso 10.16) Find, for simple harmonic motion, the values of  $\langle x \rangle_{ave}$  and  $\langle x^2 \rangle_{ave}$ , where the averages refer to time.

9. (Alonso 10.17) Find the average values of the kinetic and potential energies in simple harmonic motion relative to a) time and b) position.

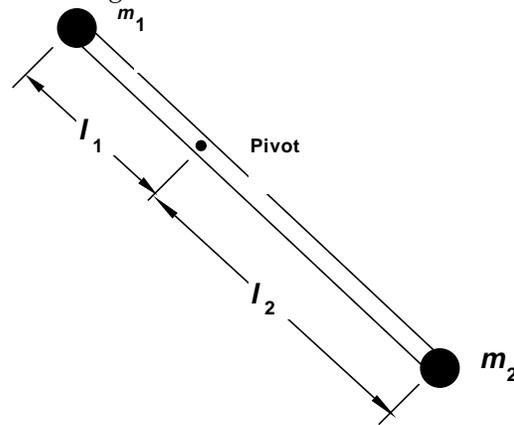
10. (Alonso 10.20) A simple pendulum whose length is 2 m is in a place where  $g = 9.80$  m/s<sup>2</sup>. The

pendulum oscillates with an amplitude of 2°. Express, as a function of time, a) its angular displacement, b) its angular velocity, c) its angular acceleration, d) its linear velocity, e) its centripetal acceleration and f) the tension on the string if the mass of the bob is 1 kg.

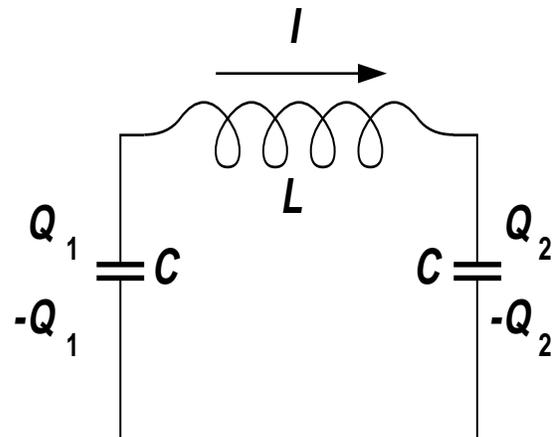
11. Find the period of oscillation for the mass in the figure.



12. Find the period of oscillation for the pendulum in the figure.



13. Show that the current in the LC circuit of the figure satisfies a harmonic oscillator equation with  $\omega^2 = \frac{2}{CL}$ .



14. Using Eq. (19), find an expression for the initial position and velocities in the case of a damped harmonic oscillator. If the oscillator of Problem 3 has a damping constant  $\lambda = 0.2$  kg/s, find a numerical expression for the position of the mass as a function of time. Graph this function.

15. Prove the theorem of superposition for linear inhomogeneous equations. Show with an example that the theorem is not valid for non-linear equations.

16. Complete the derivation leading to Eq. (25).

17. Derive Eq. (31) by computing the averages in a rigorous way and plot the average power as a function of the frequency  $\omega_f$ .

18. For the oscillator in the previous problem, find the average power given to the oscillator by the friction force. Show that this power equals minus the power given by Eq. (31). Discuss.

19. Compute and plot the instantaneous total energy of the oscillator driven by an external force. Does your result contradict your conclusions in Problem 18?

20. (Alonso 10.31) Find the limiting values of the amplitude and the phase of a forced damped oscillator when *a*)  $\omega_f$  is much smaller than  $\omega_0$  and *b*) when  $\omega_f$  is much larger than  $\omega_0$ . Determine the dominant factors in each case.