Chapter *11*

Energy in a system of particles

Introduction

In this brief chapter we review some elementary topics in the physics of systems of particles with the idea of applying these concepts to the study of systems with many particles, where thermodynamic concepts such as temperature can be defined.

The work-energy theorem for a system of particles.

Kinetic energy

Let us consider a system of particles, such as the one depicted in Fig. 1. The particles inside the system interact with each other via **internal forces**. According to Newton's third law, these are action-reaction pairs $\mathbf{F_{nm}} = -\mathbf{F_{mn}}$. There are also **external forces**, produced on the system particles by other particles outside the system. These forces are indicated as $\mathbf{F_n}$. Of course, these forces also appear as action-reaction pairs, but the reaction force acts on a particle outside the system, so we need not worry about it for the time being.

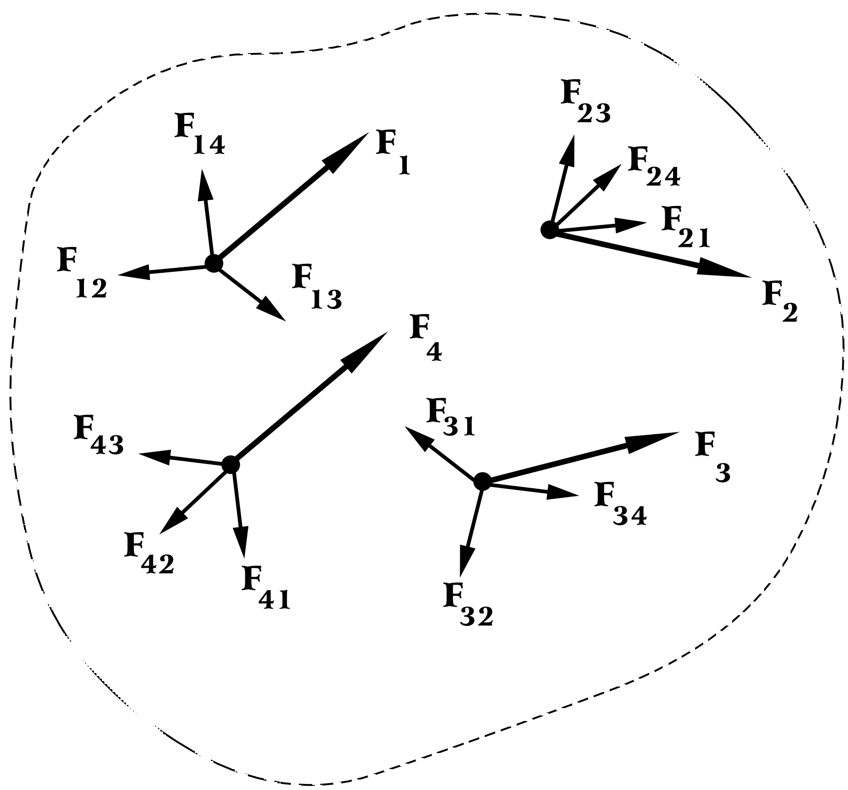

Figure 1 **A system of particles with internal and external forces.**

The kinetic energy of such a system of particles can be *defined* as

$$E_k = \sum_{i=1}^{N} \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 + \tfrac{1}{2} m_3 v_3^2 + \ \ldots \ , \qquad (1)$$

where N is the total number of particles. In the example of Fig. 1, we have $N = 4$.

Work

The work done by the external forces is, by definition

$$W_{\text{ext}} = \sum_{i=1}^{N} \int \mathbf{F_i} \cdot d\mathbf{r_i} \qquad (2)$$

where the limits of integration (not indicated for simplicity) represent the initial and final position of each particle. On the other hand, if particles 1 and 2 undergo infinitesimal displacements $d\mathbf{r}_1$ and $d\mathbf{r}_1$, the elementary work done by their mutual force is $\mathbf{F}_{12} \cdot d\mathbf{r}_1 + \mathbf{F}_{21} \cdot d\mathbf{r}_2$. Using $\mathbf{F}_{12} = - \mathbf{F}_{21}$, this becomes $\mathbf{F}_{12} \cdot (d\mathbf{r}_1 - d\mathbf{r}_2) = \mathbf{F}_{12} \cdot d\mathbf{r}_{12}$. So the total work by the internal forces can be written as

$$W_{\text{int}} = \sum_{all\ pairs} \int \mathbf{F}_{ij} \cdot d\mathbf{r}_{ij} \qquad (3)$$

Work-energy relationships

With the above definitions, the change in the kinetic energy of the system equals the total work:

$$\Delta E_k = W_{\text{ext}} + W_{\text{int}} \qquad (4)$$

This is the work-energy theorem for a system of particles. In many cases of interest, the work can be written in terms of a potential energy. For example, if the internal forces are conservative, $\mathbf{F}_{12} \cdot d\mathbf{r}_{12} = -dE_{p,12}$, etc., so that

$$W_{\text{int}} = -(E_{p,\text{int}}^{final} - E_{p,\text{int}}^{initial}) \ , \qquad (5)$$

where $E_{p,\text{int}}$ is the total **internal potential energy**. For example, for a system of charges $E_{p,\text{int}} = \frac{1}{4\pi\varepsilon_0} \sum_{all\ pairs} \frac{q_i q_j}{r_{ij}}$. We can thus define the so-called **proper energy** U of the system as

$$U = E_k + E_{p,\text{int}} = \sum_{\textit{all particles}} \tfrac{1}{2} m_i v_i^2 + \sum_{\textit{all pairs}} E_{p,ij} \qquad (6)$$

In terms of the proper energy, the work-energy theorem Eq. (4) becomes

$$\Delta U = W_{\text{ext}} \qquad (7)$$

This is a very convenient expression for thermodynamic considerations because it neatly separates "inside" (left side of the expression) from the outside world (right side of the equation). If the external forces are also conservative, we can also define a potential energy for them, in which case we can introduce the total energy E as $E = E_k + E_{p,\text{int}} + E_{p,\text{ext}}$. With this definition, the work energy-theorem becomes $\Delta E = 0$, expressing the familiar concept that the total energy is conserved. However, this form of the energy theorem is not as convenient for our purposes because it does not allow us to isolate the system from the outside world. It also requires the external forces to be conservative, something we cannot guarantee.

From the above discussion we conclude that the most convenient expression for the work-energy theorem in a system of particles is given by Eq. (7).

The internal energy

Let us consider two balls at different temperatures. Our intuitive notion of temperature (to be confirmed later in a more rigorous way) associates higher temperature with higher kinetic energy of the particles (atoms) that form the system "ball." Let us assume that the relative distance between these atoms doesn't change much, so that the internal potential energy remains constant. Can we say that the hotter ball has a higher proper energy U? No! The colder wall could be flying at a very high speed, whereas the hotter ball could remain on the floor. It is clear that if we want the energy U to represent temperature in any form we must subtract the translational kinetic energy of the system as a whole (we also need to subtract the rotational kinetic energy, but we will not worry about this technical point for now). The way to subtract the translational kinetic energy of the system as a whole is to measure the kinetic energy relative to the center of mass. We thus define the **internal energy** as

$$U_{\text{int}} = E_{k,\text{CM}} + E_{p,\text{int}} \;, \qquad (8)$$

where $E_{k,\text{CM}}$ is the kinetic energy of the system measured with respect to its center of mass. A well-known theorem in mechanics shows that if the proper energy U is measured relative to an arbitrary inertial system L, then the proper energy and the internal energy are related by $U = U_{\text{int}} + {}^1/_2 M(v_{\text{CM}})^2$, where M is the total mass of the system and v_{CM} is the speed of the center of mass measured from the inertial system L. This relationship makes it easy to transform the energy from one system into the other.

From now on, we will use the internal energy U_{int} as the fundamental quantity for our energy considerations in thermodynamic systems. In the next couple of chapters we will refer so often to U_{int} that we will end up dropping the subscript "int" and calling it simple "energy." However, you must keep in mind the meaning of the internal energy to understand thermodynamics. In terms of the internal energy, the work-energy theorem reads:

$$\Delta U_{\text{int}} = W_{\text{ext}} \qquad (9)$$

which of course means that the internal energy of a system is conserved if there is no external work done on it.

Problems

1. Consider a system of two particles with masses m_1 = 1.2 kg and m_2 = 2.2 kg. The particles have charges $q_1 = 1.5 \times 10^{-3}$ C and $q_2 = 2.9 \times 10^{-4}$ C. Their initial positions are (x_1 = 0.11 m, y_1 = 0.15 m, z_1 = 0.0 m) and (x_2 = 0.81 m, y_2 = 0.95 m, z_2 = 0.0 m). Their initial speeds are v_1 = 310 m/s and v_2 = 250 m/s. At a later time, the particles are found in positions (x_1 = 0.07 m, y_1 = 0.55 m, z_1 = 0.0 m) and (x_2 = 0.91 m, y_2 = 0.45 m, z_2 = 0.0 m), with speeds v_1 = 280 m/s and v_2 = 500 m/s.

a) What is the initial kinetic energy? What is the final kinetic energy?
b) What is the initial internal potential energy? What is the final internal potential energy?
c) What are the initial and final potential proper energies U?
d) What are initial and final internal energies U_{int}?
e) What is the work W_{int} done by the internal forces when the system goes from the initial to the final state?
f) What is the external work W_{ext} done on the system when it goes from the initial to the final state?