

# **Chapter *1***

## **Review of Newtonian Mechanics**

## Introduction

Physics can be defined as the science of *predicting the motion of objects*. A science is supposed to be quantitative: the goal of physics is to predict the function  $x(t)$  that gives the position of an object as a function of time. (More precisely, in a three-dimensional space we need the three functions  $x(t)$ ,  $y(t)$ , and  $z(t)$  that determine the three coordinates of our object) The *possibility* of a physical science is obvious from our daily observation of regularities in the motion of objects. We notice that an object under well-defined conditions will always move in the same way. This knowledge is essential to our interactions with Nature. In fact, our brain is so good at predicting motion that it makes many decisions automatically. Think about the complicated process of walking. How much time do you spend planning the next step? Even animals can do an excellent job at predicting motion. Just watch the fascinating TV program *This is my dog*. The claim that physics is one of the most difficult subjects is therefore ridiculous.

## Elements of a theory of motion

The mere existence of a position function  $x(t)$  does not guarantee that the mathematical derivation of this function will be simple. It could actually be so complicated that the mathematical description of motion might become impossible. On the other hand, we notice that many types of motion in Nature *are* simple: we see circular waves in water, we find that the orbits of the planets are elliptical, we realize that a flying stone describes a parabola in air. These are simple geometrical forms; it is reasonable to expect that the mathematical theory of motion, at least for these examples, will not be too complicated.

### The mathematics of motion prediction: Kinematics

From a mathematical point of view, the problem of predicting motion can be formulated as follows: what do we need to know at time  $t_0$  to calculate the function  $x(t)$  at a later time  $t_1$ ? A possible answer can be found in the well-known Taylor expansion formula

$$\begin{aligned}
 x(t_1) = & x(t_0) + \left. \frac{dx}{dt} \right|_{t=t_0} (t_1 - t_0) + \frac{1}{2} \left. \frac{d^2x}{dt^2} \right|_{t=t_0} (t_1 - t_0)^2 + \\
 & + \frac{1}{3!} \left. \frac{d^3x}{dt^3} \right|_{t=t_0} (t_1 - t_0)^3 + \dots + \frac{1}{n!} \left. \frac{d^nx}{dt^n} \right|_{t=t_0} (t_1 - t_0)^n + \dots \quad (1)
 \end{aligned}$$

which provides exactly what we need: the value of a function at time  $t_1$  based on information for time  $t_0$  only: the function itself and all its time derivatives evaluated at time  $t_0$ . There is a catch, however: first of all, the type of functions that can be expanded in a Taylor series is very limited, because they must have well-defined derivatives to all orders. We don't know yet whether the position function  $x(t)$  for all kinds of motion will satisfy this restrictive condition. Worse, we need an infinite amount of information to predict motion with Eq. (1), because we must know the function and *all* its derivatives at time  $t_0$ . Let's analyze the physical meaning of those terms to see if we can understand the origin of the problem.

The first term in Eq. (1),  $x(t_0)$ , is the position of our object at time  $t_0$ . That we need this information makes perfect sense: if we want to predict the motion of an object, we need to know where it was at the initial time. If a tennis ball is served in Wimbledon, it will land in Wimbledon. If it is served in Roland Garros, it will land in Roland Garros. There is no way we can tell the landing point if we don't know the starting point, even if the trajectories of the two balls are identical.

The second term in Eq. (1),  $\left. \frac{dx}{dt} \right|_{t=t_0} \equiv v(t_0)$  is the **velocity** at time  $t_0$ . That this quantity is also needed to predict motion follows from our daily experience. We note that the trajectory of a flying object can be changed by changing the initial velocity (the science of ballistics depends on this fact). We also know that if an object is moving very fast it is easy to anticipate where the object is going to be an instant later. Hence from the present velocity, we know how to "extrapo-

late” into the future.

The third term in Eq. (1) contains the factor  $\left. \frac{d^2x}{dt^2} \right|_{t=t_0} \equiv \left. \frac{dv}{dt} \right|_{t=t_0} \equiv a(t_0)$  which is the **acceleration** at the initial time  $t_0$ . Do we need this information? We certainly do; however, there is a fundamental difference between position and velocity on one side and acceleration on the other side. Let us consider the case of an object near the surface of the Earth (a ball, a stone, etc.) We know that we can change the trajectory of the object by changing its initial position and velocity. However, the same is *not* true for the acceleration: no matter what acceleration the object has while on our hands, the acceleration becomes  $9.8 \text{ m/s}^2$  (vertically down) the moment the object leaves our hands. The position and velocity of the ball immediately after it is released from our hands depends on the position and velocity it had while still on our hands. Its acceleration, however, reverts “magically” to  $9.8 \text{ m/s}^2$  no matter what its value was while in our hands. Moreover, the acceleration remains at  $9.8 \text{ m/s}^2$  for the entire flight of our object. So it appears that Nature takes care of the acceleration completely.

If the acceleration of an object is determined by Nature at all times, then Eq. (1) is not needed: the solution to our problem is readily obtained by integrating the equation  $\frac{dv}{dt} = a(t)$ , from which the velocity  $v(t)$  is given by

$$v(t) = v(t_0) + \int_{t_0}^t a(t') dt' \quad (2)$$

Next we obtain the position from

$$x(t_1) = v(t_0) + \int_{t_0}^{t_1} v(t) dt \quad (3)$$

For example, in the case of a falling object, taking  $x$  as the vertical displacement (positive direction upward), we have  $a(t) = -g$ , with  $g = 9.8 \text{ m/s}^2$ . Thus Eq. (2) gives  $v(t) = v(t_0) -$

$g(t-t_0)$ , which we can plug into Eq. (3) to obtain the familiar expression  $x(t_1) = x(t_0) + v(t_0)(t_1-t_0) - 1/2g(t_1-t_0)^2$ .

Of course, so far we have only shown that there is a case in which the acceleration is completely determined by Nature. Our analysis does not demonstrate that this will be always so. On the other hand, if motion in Nature can be understood from a simple and unified theory, the free fall case we just discussed should be a special application of a more general principle. We therefore assume that Nature determines accelerations and that the solution to our problem is given by Eq. (2) and Eq. (3). Ultimately, our assumptions will have to be verified by experiments.

If acceleration is the central quantity that determines the motion of an object, we need a set of rules to obtain the acceleration. These rules are Newton's laws.

## How Nature determines acceleration: Dynamics

The key idea is the observation that accelerations are related to *interactions*. The “magic” number  $9.8 \text{ m/s}^2$  for the acceleration at the surface of the Earth depends on the interaction between the Earth and our object. If the object is placed on the Moon, its acceleration is different. So we'll assume that *the interactions between different objects in Nature is what causes their accelerations*. Consistent with this idea, we can state that the acceleration will be zero if the interactions vanish. This almost trivial statement is Newton's first law.

### Newton's First Law

*A free object (no interactions) moves with constant velocity (zero acceleration).*

Trivial as it may seem in our context, this law runs against some intuitive ideas. Suppose that you kick a chair so that it acquires a initial horizontal velocity of  $2 \text{ m/s}$ . You know that the chair will not slide forever at  $2 \text{ m/s}$ : it will eventually stop. If you want the chair to move at a constant velocity, you must push on it. Hence it appears that an interaction is required to keep something moving at constant velocity, in

direct violation of Newton's First Law. The first to challenge this "common sense" belief was Galileo Galilei, who is the true father of the First Law. Galileo accepted the obvious fact that objects set into motion and left unattended will eventually stop, but decided to analyze the reason why they stop. Suppose that you drop some oil on the floor and give your chair the same kick. Now you'll see that it travels much further. This suggests that your object stops because there is an interaction (friction) with the floor. Galileo's insight was to extrapolate this conclusion to an ideal situation where there is absolutely no friction, that is, no interactions: then the velocity would not change at all, and the acceleration would be zero. This is precisely the statement made by the First Law. The reason why the First Law appears to fail in everyday situations is that for it to be true the interactions must vanish completely. In real situations, there are always residual interactions in the form of friction, air drag, etc., so that the acceleration is not exactly zero. We will never be able to set up an experiment in which the First Law is obeyed exactly, because it is practically impossible to eliminate absolutely all forms of interactions between objects. However, we do see good approximate examples: an ice-skater keeps her initial velocity for a long time, our car tends to slip on icy roads, etc. In all cases, we find that the more we eliminate the interactions, the closer our experimental results are to the predictions of the First Law. This is why we strongly believe in the correctness of this law, even though we have never seen an "exact" example, and we don't expect to ever see one.

### **Newton's Second Law**

Having discussed the case of no interactions, we must now tackle the more interesting case of non-vanishing interactions. How do we determine the acceleration when interactions are present? Our experience with the acceleration of gravity seems to indicate that all we need is a list of accelerations for different interacting objects. For example, the acceleration of an object near the Earth is  $9.8 \text{ m/s}^2$ , the acceleration of an object on the Moon is  $1.6 \text{ m/s}^2$ , etc.

Unfortunately, things are much more complicated. On the one hand, the acceleration produced by the interaction between objects is not always the same. The acceleration of a stone at the surface of the Earth is indeed  $9.8 \text{ m/s}^2$ , but if the stone is at a height of 10,000 km the acceleration is much less. A more serious problem is that in cases where the interactions appear to be the same, different objects have different accelerations. Consider for example the case of a professional tennis player. His service is so precise, that it is reasonable to assume that for every service the interaction between the racket and the ball is essentially the same. Accordingly, the motion of the tennis ball is very much the same every time the player serves the ball. However, imagine that somebody puts some lead in the center of one of the balls. If the player were to hit this ball in exactly the same way he hits normal balls, the ball would move very differently. The player hits the ball the same way he hits other balls, the *interaction* racket-ball is the same, but the *acceleration* of the lead ball will be very different. It appears that the *acceleration of an object depends not only on its interactions with other objects but on some property of the object itself*. In order to take this into account, we introduce two new concepts: *force* and *mass*.

- **Force** is a vectorial quantity that measures the strength of the interaction between objects. The total force acting upon an object is the vector sum of all individual forces that act on this object.
- **Mass** is a scalar quantity that measures the inertia of an object, defined as its ability to “resist” acceleration when subject to a force.

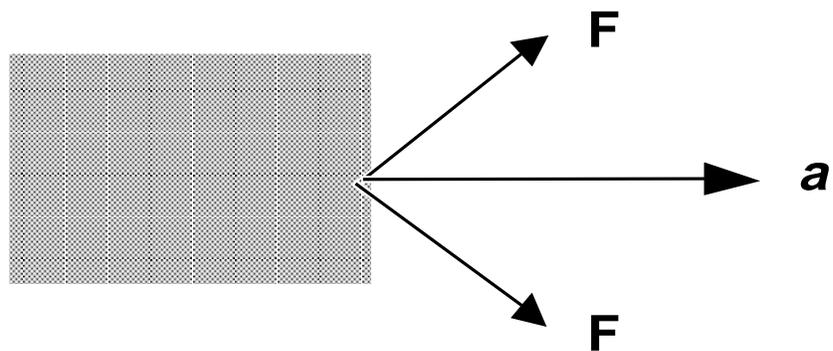
Newton’s second law states that the acceleration  $\mathbf{a}$  of an object is given by

$$\mathbf{a} = \frac{\mathbf{F}}{m}, \quad (4)$$

where  $\mathbf{F}$  is the total (or net) force acting upon the object

and  $m$  the mass of the object.

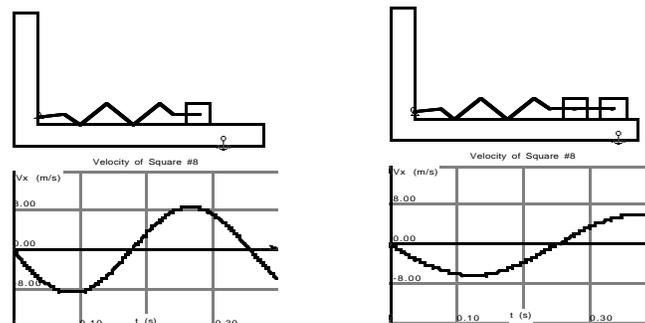
Eq. (4) is probably the most important equation in physics. It gives the acceleration of an object in terms of the force acting on it and the mass of the object. It is not at all clear, however, how we are supposed to use it, because we still don't know what the forces are and we don't have a quantitative definition of mass that we can use to measure masses. The usual definition of mass as "amount of matter" is not specific enough. We'll address these serious problems below. For the time being, let's make sure that Eq. (4) at least makes sense. The equation implies that the acceleration is proportional to the force. This is in agreement with our intuitive concept of force. Suppose that you go to the parking lot, put your car in neutral and push on it. Your car will accelerate, and you can measure this acceleration. Next you ask a friend to help you push the car and measure the acceleration again. You'll find that the acceleration has roughly doubled. This is in agreement with Eq. (4), if the force  $F$  in that equation is understood as the *net* force acting on our object, that is, as the sum of the individual forces exerted on it. Additional evidence that the acceleration is proportional to the sum of all forces can be obtained from an experiment as in Figure 1, where we drag a box by applying two forces as indicated.



**Figure 1** The acceleration produced by two forces is in the direction of the vector sum of the forces.

The direction of the resulting acceleration coincides with the direction of the vectorial sum of the forces. This

observation is sometimes called *principle of superposition*. The other important feature of Eq. (4) is that the acceleration is expected to be *inversely* proportional to the mass. This also agrees with experience. Consider for example the experiment of Figure 2. Different masses of some material, say wood, are attached to identical springs. If the spring is stretched by the same amount, it is reasonable to assume that the initial force exerted by the springs on the masses is the same. Yet if one mass is twice the size of the other, we obtain different velocities when we let go. The initial slope of the velocity versus time curve for the smaller mass is exactly two times larger than the slope corresponding to the larger mass. But this slope is the acceleration, so that we verify the Second Law: If the force doesn't change, an increase in the mass by a certain factor leads to a decrease in the acceleration by the same factor.



**Figure 2** The velocity versus time curve for a mass attached to a stretched spring (left) and for two masses equal to the first mass attached to an identical spring (right). The two springs are initially stretched by the same amount.

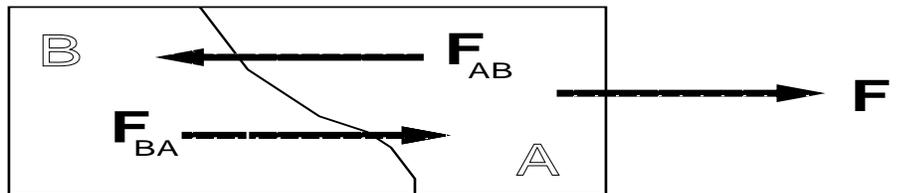
What about the gravitational case discussed earlier, where the acceleration seems to be independent of the masses of the objects? This does not necessarily contradict Eq. (4). If the gravitational force itself turns out to be proportional to the mass (remember  $F = mg$ ), the dependence on the mass cancels out in Eq. (4).

Eq. (4) is written very often as

$$\mathbf{F} = m\mathbf{a} \quad (5)$$

This is obviously equivalent to  $\mathbf{a} = \mathbf{F}/m$ . Conceptually, however, this version of Eq. (4) is somewhat misleading because it looks like an expression to calculate the force, while our idea is to use the force to calculate the acceleration.

### Newton's third law



**Figure 3** Any object can be considered as being composed of two or more parts. Applying the second law to the different parts should give consistent results.

Figure 3 shows a possible problem with Eq. (4). Suppose that we apply a certain force  $\mathbf{F}$  on a rigid, extended object of mass  $m$ . According to the second law, the acceleration of the object is  $\mathbf{a} = \mathbf{F}/m$ . On the other hand, the second law should also be true if we apply it to different parts of this object. Consider the left half of the object, which we call  $B$ . Clearly, the force  $\mathbf{F}$  is not applied on  $B$ , so that the motion of  $B$  must be caused by a force  $\mathbf{F}_{BA}$  produced by  $A$ . On the other hand, the motion of  $A$  is due to force  $\mathbf{F}$  and to a possible force  $\mathbf{F}_{AB}$  produced on  $A$  by  $B$ . We don't know if there is such a force, but, on the other hand, there is no reason to assume that  $B$  exerts no influence on  $A$ , so that we include it, "just in case". According to the second law, the accelerations of the two parts are given by

$$\mathbf{a}_A = \frac{\mathbf{F} + \mathbf{F}_{AB}}{m_A} \quad (6)$$

and

$$\mathbf{a}_B = \frac{\mathbf{F}_{BA}}{m_B} \quad (7)$$

where  $m_A$  and  $m_B$  are the masses of parts  $A$  and  $B$ , respectively. Combining these two equations we find that  $\mathbf{F} + \mathbf{F}_{AB} + \mathbf{F}_{BA} = m_A \mathbf{a}_A + m_B \mathbf{a}_B$ . But all parts of the object move together, so that  $\mathbf{a} = \mathbf{a}_A = \mathbf{a}_B$ . Using this fact, and  $m_A + m_B = m$ , we easily (see Problem 1) come to the conclusion that  $\mathbf{F}_{AB} + \mathbf{F}_{BA} = 0$ , or  $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$ . Since this result must be true for the Second law to be consistent, Newton elevated it to the status of law. This **Third Newton's Law** can be expressed in words as follows: *If an object  $A$  exerts a force  $\mathbf{F}_{BA}$  on an object  $B$ , then object  $B$  exerts a force  $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$  on object  $A$ .*

These pairs of equal and opposite forces are frequently called *action and reaction* pairs. The Third Law itself is frequently called the Law of Action and Reaction. Notice that, usually, equal and opposite forces cancel out. In this case, however, each component of these pairs is applied on a *different* object, so that when we write Newton's Second Law for a given object, only one of the components of the pair appears. This is clearly the case above in Eqs. (6) and (7).

Newton's Third law has some remarkable implications. If a truck collides with a motorcycle, Newton's Third law tells us that the force exerted by the motorcycle on the truck is equal to the force exerted by the truck on the motorcycle. This sounds strange, but it makes sense. Since the mass of the motorcycle is much less than the mass of the truck, the acceleration of the motorcycle will be much larger than the acceleration of the truck, even though the forces on both objects are equal in magnitude.

**Mass versus Weight**

With the first and second laws, we have the desired connection between interactions and accelerations. Clearly, the second law will play a fundamental role in our studies of motion. However, we still don't know how to apply it, because we don't know how to measure the mass of an object and we don't know how or where to find the force between objects.

You might think that measuring masses is not really a problem. Once we define the mass of one liter of water as 1 kg, the only thing we need to do is compare the **weight** of other objects with the weight of our liter of water. For example, if a certain amount of lead weighs twice the weight of our liter of water, we might say that the mass of lead is 2 kg. This works because we assume that the weight of an object is given by  $mg$ . However, this is an experimental statement about the gravitational law, whose accuracy we don't know for sure. If a certain amount of lead weighs slightly more than our liter of water, it could be that this is simply due to the fact that the Earth attracts lead with a slightly stronger intensity than it attracts water. In fact, these deviations from the gravitational law have received widespread attention of a late as some papers claimed to have evidence for it. It is apparent that our definition of mass must be independent of the specific form of the natural forces, so that a definition in terms of weight, while practical, is not acceptable. The masses we were talking about when we introduced Newton's second law measure the "reluctance" of an object to move under an applied force (inertia). Hence they are sometimes called *inertial* masses. They need not have anything to do with the strength of the attraction by the Earth! In fact, it is a most surprising fact that the gravitational interaction does depend on the inertial masses. This doesn't happen for other interactions. Electric forces, for example, depend on the electric charges. They are independent of the mass of the charged objects.

There is a way to measure masses - based on Newton's

Second and Third Laws - that is independent of the nature of the forces between objects. Suppose that two masses  $m_A$  and  $m_B$  experience a collision. Let's assume that we carefully eliminate all "external" forces, so that the only possible forces are those exerted by one object on the other object. According to Newton's third law, these forces are equal and opposite. Combining Newton's Second and Third laws, we obtain

$$m_A \mathbf{a}_A = \mathbf{F}_{AB} = -\mathbf{F}_{BA} = -m_B \mathbf{a}_B \quad (8)$$

If we watch these two objects for a time  $\Delta t$ , we can measure the *change* in their velocities produced by the acceleration. These changes are  $\Delta \mathbf{v}_A = \mathbf{a}_A \Delta t$  and  $\Delta \mathbf{v}_B = \mathbf{a}_B \Delta t$ . If we multiply both sides of Eq (8) by  $\Delta t$ , we obtain

$$m_A \Delta \mathbf{v}_A = -m_B \Delta \mathbf{v}_B \quad (9)$$

This can be used to measure masses: once we know one mass, say  $m_A$ , that we define arbitrarily (such as 1 kg = mass of 1 liter of water) we make it collide with an unknown mass and measure the change in velocity of the two masses. Then we can use Eq.(9) to determine the unknown mass.

**Finding the force**

We now have all ingredients in place to calculate the accelerations from the forces, except that we don't know how to find the force between objects! Imagine you are Isaac Newton trying to “sell” your three laws to an skeptical scientific audience. Somebody is bound to complain: “You are asking us to use your three laws to calculate the acceleration, but you give us no clue on how we are supposed to find the forces! The only thing we can do with your laws is measure the acceleration and calculate the force using  $F = ma$ ! As far as we can tell, your laws might even be right, but they are useless when it comes to calculating accelerations.” Of course, Newton himself had worried about this. He solved the problem (in part) with an astonishing discovery: he found the gravitational force between objects. This was his second fundamental contribution to physics. (Historically, the way Newton discovered the gravitational law and his three dynamical laws was much more complicated. It was a trial and error approximation in which he found parts of his three laws, parts of the gravitation law, invented calculus in-between, made some silly mistakes, and finally was able to formulate his ideas as a coherent whole.).

Newton's gravitational force between two objects of masses  $m_1$  and  $m_2$  separated by a distance  $r$  is given by  $F = \frac{Gm_1m_2}{r^2}$ . With this very simple expression, combined with his three dynamics laws, Newton was able to explain all known gravitational phenomena. The complicated motion of planets, the weight of the objects near the surface of the earth, the tides, and an impressive list of other natural phenomena can be explained on the basis of that simple expression for the gravitational force between objects. The fact that such an expression *exists* lends enormous credibility to Newton's approach, for it is unthinkable that such a simple mathematical description could be found on the basis of a wrong theory. This is the artistic/religious component of science: if you find a very simple and mathematically

appealing explanation for a physical phenomenon, more often than not you will have found the right answer. Conversely, if your explanation is mathematically too complicated, your approach is likely to be mistaken. (Avoid excessive boldness, though: exceptions abound).

### **Forces in Nature**

A further confirmation of the usefulness of the Newtonian concept of force was the discovery of the form of the electric interaction between a charge  $q_1$  and a charge  $q_2$  separated by a distance  $r$ , which is given by Coulomb's law  $F = K_e \frac{q_1 q_2}{r^2}$ . Again, the observation that electrical phenomena can be explained in terms of this simple force lends additional credibility to Newton's dynamics.

A significant discovery over the last three centuries is that the number of fundamental interactions in Nature is limited: in addition to the *gravitational* and *electric* interaction there is the *magnetic* interaction, the *strong* interaction (which holds together the positively charged protons in the nuclei of atoms), and the *weak* interaction (responsible for some radioactive decay phenomena). The realization that there were only a few interactions lead to an intriguing idea: could it be that there is in fact a *single* interaction, which depending on the experimental conditions manifests itself as one or the other of the known fundamental forces? This started the search for a "unification theory," which, as you can see, is motivated again by an almost religious belief in simplicity and beauty. The first great triumph in the search for unification was accomplished last century by Maxwell: his famous equations effectively unify the electric and magnetic interactions. We now use the expression "electromagnetic interaction." The second fundamental unification step was accomplished by Salam, Glashow, and Weinberg, who received the 1979 Nobel Prize for showing that the electromagnetic and weak interactions are manifestations of a fundamental "electroweak" interaction. Further "unification" progress may depend on you!

## Solutions to the force equation

Once the force relevant to the problem is known, the task of the physicist is to find the position function  $x(t)$  given the initial position  $x(0)$  and the initial velocity  $v(0)$ . This solution can be given as an *analytical* expression, such as  $x(t) = A \sin \omega t$ , or in *numerical* form, *i.e.*, as a table listing the values  $x(t_i)$  for selected times  $t_i$ . The advantage of analytical expressions is that they allow the calculation of the position at any desired time. By contrast, if that desired time is not in the list of values  $t_i$  of the table listing the numerical solution, the best we can do is interpolate between the closest values  $t_{i-1}$  and  $t_i$ . Of course, if the separation between the successive times  $t_{i-1}$  and  $t_i$  is very small, the error incurred will also be small. In other words, the numerical solution can be made as accurate as desired by calculating the position at many closely spaced times  $t_i$ . This alternative, which requires many computations, was impractical in Newton's times. He had to invent calculus and limit himself to problems with analytical solutions. The power of today's personal computers makes it feasible and sometimes convenient to use a numerical approach, as you will find out in this course.

The advantage of the numerical approach is that it can handle problems for which there are no analytical solutions, such as the motion of three interacting bodies. On the other hand, the distinction between analytical and numerical solutions is not as clear-cut as the above discussion seems to imply. Consider the above example  $x(t) = A \sin \omega t$ . When your calculator computes the value of the sine function for a certain angle, it makes a numerical approximation, since the *exact* value is given by an *infinite* series which of course cannot be summed up.

### Numerical solutions

The basic idea of the numerical approach is to divide the interval between the initial time  $t_0$  and the final time  $t_1$  into very small intervals of duration  $\Delta t$ . (The intervals need not be equal. For simplicity, however, we will assume they are). For very small intervals  $\Delta t$  we obtain:

$$v(t + \Delta t) = v(t) + \int_t^{t+\Delta t} a(t') dt \approx v(t) + a(t)\Delta t, \quad (10)$$

where we have used the definition of integral to drive the last approximate relation. Similarly,

$$x(t + \Delta t) \approx x(t) + v(t)\Delta t \quad (11)$$

Equations (10) and (11) together with Newton's second law, Eq. (4) constitute the basis for a numerical solution of the equations of motion. On a personal computer, this can be most easily achieved by using a spreadsheet program.

The biggest practical problem is the selection of an appropriate value of  $\Delta t$ , small enough to preserve accuracy and large enough to avoid unnecessary computations. Professional physicists can write computer programs that adjust the value of  $\Delta t$  to the specific conditions of the system: when the object moves slowly, the time interval grows, when the object speeds up, the interval decreases, so that there is no loss of accuracy. We will not be that fancy in this course. We will just guess a good value of  $\Delta t$  and modify it later if we are not satisfied with the result. Since the computer will do all calculations, there is little added pain in trying different values of the time interval.

## SPREADSHEET PHYSICS

A *Spreadsheet* is a popular type of program that consists in a two- (or three-) dimensional array of cells labeled with letters (columns) and numbers (rows). Each cell has a unique label: A3, B2, etc. (Most programs will also let you name cells. For example, you could name a cell “mass” and enter there the value of the mass of your object). The program lets you perform operations with the entries in different cells and store the result in the cell of your choice. We would like to set up a spreadsheet to calculate  $x(t+\Delta t) = x(t) + v(t)\Delta t$  and  $v(t+\Delta t) = v(t) + a(t)\Delta t$  in a step-by-step fashion. Suppose that the object is mass  $m = 2.0$  kg attached to a spring of constant  $K = 0.5$  N/m, so that the force is given by  $F = -Kx$ . Let us take the time interval  $\Delta t$  as 0.1 s. The initial time is  $t = 0$  s. Suppose that at this time, the object is at  $x(0) = 3.0$  m and has a velocity  $v(0) = 0.0$  m/s. Our spreadsheet could look as follows:

	A	B	C	D
1	0.1	2.0	0.5	
2	0.0	3.0	0.0	-\$C\$1*B2/\$B\$1
3	+A2+\$A\$1	+B2+C2*\$A\$1	+C2+D2*\$A\$1	-\$C\$1*B3/\$B\$1
4	A3+\$A\$1	B3+C3*\$A\$1	C3+D3*\$A\$1	-\$C\$1*B4/\$B\$1

We have entered  $\Delta t$  in cell A1,  $m$  in cell B1 and  $K$  in cell C1. In the second row we enter the initial conditions:  $t=0$  in A2,  $x(0) = 3.0$  m in B2, and  $v(0) = 0.0$  m/s in C3. In cell D2 we have entered a *formula*. (Note that when you enter a formula in a cell, what you usually see in the cell is the result of the formula. The formula itself is usually displayed in a special cell at the top of the screen.) Because spreadsheets also accept normal text in a cell, the program must have a way to distinguish between a formula and a word. This is usually accomplished with the following convention: if you start your expression with a “-“ or “+” sign, the program understands you are typing a formula. The formula we entered in D2 tells the program to multiply whatever it finds in cell C1 times B2 and divide by the quantity in cell B1 (We’ll explain the \$ signs below). The initial minus sign is equivalent to multiplying by (-1). What we have entered in D2 is clearly  $-Kx(0)/m$ , which is the acceleration at  $t = 0$ .

### SPREADSHEET PHYSICS (continuation)

Now consider the formulas in the third and fourth rows. In column A, the formulas give the successive times. In columns B and C, you'll recognize the approximate expressions for the position and velocities at a certain time in terms of the position and velocity at an earlier time. In column D we see formulas giving the acceleration at the corresponding times in column A. At this point, you may wonder what is the usefulness of the spreadsheet: do you have to type the formulas for every time you want to compute? If the answer were yes, you might as well compute the positions and velocities with your pocket calculator. Enter one of the most powerful features of spreadsheet programs. *Once you type the formulas in row 3, you can COPY them to the rows below. The program automatically upgrades the references.*

When you copy the formula containing A2 to the row below, the formula is automatically changed by the program so that A2 becomes A3. This is exactly what you want. Sometimes, however, you want the program *not* to upgrade the reference. This is what you accomplish by adding \$ signs. (Not all programs use \$ signs for this, but all of them have this feature. Look it up in the manual of your spreadsheet under "absolute" references). For example, when you copy the formula in A3 to A4, you get A3+A1. If you copy this formula down, you get A4+A1. If you copy the new formula to the row below, you get A5+A1, etc. It is clear that by copying the formulas in row 3 to the 100 rows below it, you can compute the position and velocity at 100 different times. This can be done with a couple of keystrokes or mouse clicks.

Most modern spreadsheet programs will let you graph your results. This usually consists in invoking the graph feature and selecting the columns you want as  $x$  and  $y$  axis.

### Analytical solutions

When the force is a function of time only, the solution to the equations of motion is straightforward. From Eq. (4), we easily obtain the acceleration as a function of time. This function is plugged into Eq. (2) to obtain the velocity as a function of time. Finally, the position as a function of time is obtained from Eq. (3) by using the velocity as a function of time computed in the previous step.

Unfortunately, forces in Nature are not always known as a function of time. In fact, most fundamental forces in nature depend on the velocity or the position of the particle. This creates a problem: in order to calculate the position as a function of time, we need to compute the force. But in order to be able to compute the force we must know the position as a function of time! To illustrate this difficulty,

let's consider the case of a mass attached to a spring, which we just solved numerically. The force is given by  $F = -Kx$ , so that Eqs. (2) and(3) become

$$v(t) = v(0) - \frac{K}{m} \int_0^t x(t') dt' \quad (12)$$

and

$$x(t) = x(0) + \int_0^t v(t') dt' \quad (13)$$

It is quite clear that these equations are coupled. When you have a system of couple *algebraic* equations, such as  $x + 3y = 8$ ,  $x - y = 3$ , the usual approach is to use one of the equations to express one of the unknowns in terms of the other, (for example,  $x = y + 3$ ) and substitute this expression into the remaining equation, which then becomes an equation with only one unknown. The same approach can be used here. However, the presence of the integrals poses a serious problem. To eliminate the integrals, we differentiate the equations and use the fundamental theorem of calculus. We then obtain

$$\frac{dv}{dt} = -\frac{K}{m} x(t), \quad (14)$$

and

$$\frac{dx}{dt} = v(t) \quad (15)$$

Now we can easily eliminate  $v(t)$  by differentiating Eq. (15) one more time and substituting into Eq. (14). We finally obtain

$$\frac{d^2x}{dt^2} = -\frac{K}{m} x(t) \quad (16)$$

This is a *differential* equation. Stated in words, the solution to our spring problem is a function with the following property: its second time derivative is proportional to minus the

function itself. Notice that Eq. (16) is simply  $a = F/m$ . Hence the differential equation relevant to any problem is Newton's second law.

When the force is a function of time only, the differential equation can be trivially integrated, as discussed above. When the force is a function of the position or the velocity, as in Eq. (16), we must use the mathematical theory of differential equations to solve our problem. *Note that the distinction does not arise if we are solving the problem numerically:* we can always use the expressions in Eq. (10) and (11), because at every instant we are simultaneously calculating time, position, and velocity.

When solving differential equations, mathematicians and physicists take somewhat different approaches. For a mathematician, it is important *a)* to discover a systematic approach to solving these equations, and *b)* to demonstrate that the solution found is unique. A physicist is usually happy if she finds a solution, no matter how the solution is found. She might even cheat by performing an experiment and watching the solution in real life. This might help in guessing what function might be the mathematical solution to the differential equation. A mathematician would undoubtedly find this approach disgusting. Also, physicists worry much less about the uniqueness of the solutions. Very often, they *know in advance* that there must be only one solution to the problem. For example, under well defined initial conditions a projectile will fly with a well-defined trajectory. So you know that if you find a solution to the equations of motion, this must be *the* solution. If we were to find that there are two possible solutions to the problem, this would probably mean that Newton's law are wrong, for these things don't happen in Nature. Since we have little doubt concerning Newton's equations, we don't worry very much about the uniqueness of the solutions we find.

## Problems

1. Show that in Fig. 3  $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$  by requiring that the acceleration of all parts of the object be the same and using  $m_A + m_B = m$ .

2. Set up a spreadsheet to solve numerically (don't use the exact solution!) the problem of a mass  $m$  attached to a spring of constant  $K$ . Choose any value you want for these parameters. Also, select whatever initial conditions you like. Experiment with different values of  $\Delta t$  until you find a reasonable one. Use the graphic capabilities of your spreadsheet to plot the position and the velocity as a function of time. These functions should look like sines or cosines; use this to assess how good your  $\Delta t$  is.

3. *a)* Show (analytically) that using the approximations  $x(t+\Delta t) = x(t) + v(t)\Delta t$  and  $v(t+\Delta t) = v(t) + a(t)\Delta t$  in the previous problem, the total energy is not conserved exactly from one step to the next, but with an error given by

$$\Delta E = \frac{K}{m} E (\Delta t)^2.$$

Discuss.

*b)* Show that the above expression leads to an *exponential* increase of the total energy, with an exponent given approximately by  $K\Delta t/m$ . Compare with numerical calculations done with your spreadsheet. (*Hint:* This is a difficult problem. Don't expect to solve it in five minutes.)