Introduction to continuous systems

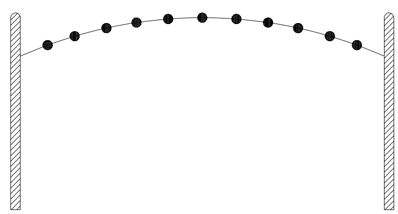
Although all matter is composed of atoms, it is often convenient to pretend the matter is continuous rather than discrete. We shall discuss the physical justification for such approximation in subsequent chapters on fluids and elastic solids. To derive dynamical equations, Lagrangians and Hamiltonians (together with the apparatus of canonical coordinates and momenta implicit therein) we mentally subdivide the continuous system into discrete segments that obey Newton's laws. Assuming the appropriate physical quantities may be represented by sufficiently well-behaved mathematical functions of coordinates and time we use Taylor's series to relate infinitesimal quantities of the same order. The resulting partial differential equations comprise our mathematical model of a continuous system.

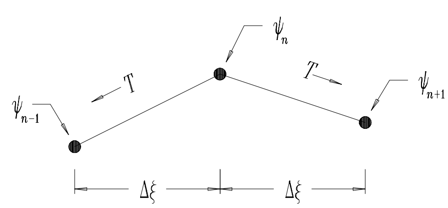
1. The stretched string

We imagine a uniform string stretched between two fixed supports. The tension in the string is T, meaning that that is the force the string exerts on each support. In equilibrium, of course, the string's curve is a straight line (neglecting gravitation). We replace this string by a set of discrete masses,

$$\Delta m = \mu\, \Delta\xi$$

where μ is the mass per unit length and $\Delta\xi$ is the spacing between the nodes where the masses are located.





Deviations from equilibrium are described by the instantaneous transverse displacement, $\psi_n(t)$, at the n'th node along the string. We can apply Newton's Second Law to the motion of, say, node n:

$$\Delta m \ddot{\psi}_n = -\frac{\tau}{\Delta\xi}\left(\psi_n - \psi_{n-1} + \psi_n - \psi_{n+1}\right) = \frac{\tau}{\Delta\xi}\left(\psi_{n-1} - 2\psi_n + \psi_{n+1}\right)$$

where we have taken the transverse component of the restoring force,

$$f_{rest} = -\tau \sin\theta \approx -\tau \frac{\Delta\psi}{\Delta\xi}\,.$$

Transition to the continuum limit

If we imagine $\psi_n(t)$ is really a continuous function,

$$\psi(x,t)\Big|_{x=n\Delta\xi}$$

evaluated at the n'th node, we may expand in Taylor's series:

$$\psi_{n-1} - 2\psi_n + \psi_{n+1} \equiv \psi(x-\Delta\xi) - 2\psi(x) + \psi(x+\Delta\xi)$$

$$\begin{aligned}
&\approx \psi(x) - \frac{\partial\psi(x)}{\partial x}\Delta\xi + \frac{1}{2}\frac{\partial^2\psi(x)}{\partial x^2}(\Delta\xi)^2 + \ldots \\
&\quad \psi(x) + \frac{\partial\psi(x)}{\partial x}\Delta\xi + \frac{1}{2}\frac{\partial^2\psi(x)}{\partial x^2}(\Delta\xi)^2 + \ldots \\
&\quad - 2\psi(x) \\
&\approx \frac{\partial^2\psi(x)}{\partial x^2}(\Delta\xi)^2 .
\end{aligned}$$

This approximation leads to the equation of motion of a uniform, continuous, stretched string:

$$\mu\Delta\xi\,\frac{\partial^2\psi}{\partial t^2} = \tau\,\Delta\xi\,\frac{\partial^2\psi}{\partial x^2} + O\Big((\Delta\xi)^2\Big)$$

or, in the limit as $\Delta\xi \to 0$,

$$\mu\,\frac{\partial^2\psi}{\partial t^2} = \tau\,\frac{\partial^2\psi}{\partial x^2} .$$

Although it is not possible for the tension τ to vary along the string, the linear mass-density μ *can* vary, in which case the p*hase velocity*

$$u^2 = \frac{\tau}{\mu}$$

becomes a function of position. In terms of the phase velocity we rewrite the string equation in the form

$$\frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} = 0$$

which is manifestly a *wave equation*. Were the string both uniform and infinite in extent, the preceding equation would have solutions of the form

$$\psi_{\pm}(x,t) = \begin{cases} f(x-ut) \\ f(x+ut) \, . \end{cases}$$

On a finite string supported at both ends (located at $x=0$ and $x=l$) the solution is subject to the boundary conditions

$$\psi(0,t) = \psi(l,t) = 0 \, .$$

Lagrangian of a stretched string

The kinetic energy of a system of discrete masses is

$$KE = \frac{1}{2}\,\Delta\xi \sum_k \mu(k\Delta\xi)\,\dot{\psi}_k^2$$

and the potential energy is

$$V = \frac{1}{2}\,\frac{\tau}{\Delta\xi} \sum_k \left(\psi_{k+1} - \psi_k\right)^2 \approx \frac{1}{2}\,\tau\,\Delta\xi \sum_k \left.\left(\frac{\partial\psi}{\partial x}\right)^2\right|_{k\Delta\xi}$$

hence the Lagrangian becomes

$$L = \Delta\xi \sum_k \left[\frac{1}{2}\,\mu(k\Delta\xi)\left(\frac{\partial\psi(k\Delta\xi,\,t)}{\partial t}\right)^2 - \frac{1}{2}\,\tau \sum_k \left(\frac{\partial\psi(k\Delta\xi,\,t)}{\partial x}\right)^2\right]$$

$$\underset{\Delta\xi \to 0}{\longrightarrow} \int_0^l dx \left[\frac{1}{2}\,\mu(x)\left(\frac{\partial\psi(x,\,t)}{\partial t}\right)^2 - \frac{1}{2}\,\tau\left(\frac{\partial\psi(x,\,t)}{\partial x}\right)^2\right].$$

The integrand is called the *Lagrangian density*, $\mathscr{L}\left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial t}, \psi, x\right)$.

Hamiltonian of a stretched string

Returning to the original system of discrete masses we see that the canonical momentum corresponding to the "generalized coordinate" ψ_k is just

$$\pi_k = \Delta\xi\;\mu(k\Delta\xi)\;\dot{\psi}_k\,,$$

hence the Hamiltonian becomes

$$H = \Delta\xi \sum_k \left[\frac{1}{2}\,\mu(k\Delta\xi)\left(\frac{\partial\psi(k\Delta\xi,\,t)}{\partial t}\right)^2 + \frac{1}{2}\,\tau \sum_k \left(\frac{\partial\psi(k\Delta\xi,\,t)}{\partial x}\right)^2\right]$$

$$\underset{\Delta\xi \to 0}{\longrightarrow} \int dx \left[\frac{1}{2}\,\mu(x)\left(\frac{\partial\psi(x,\,t)}{\partial t}\right)^2 + \frac{1}{2}\,\tau\left(\frac{\partial\psi(x,\,t)}{\partial x}\right)^2\right].$$

Correspondingly, the integrand is called the *Hamiltonian density*,
$\mathscr{H}\left(\pi(x,t),\ \psi(x,t),\ x\right)$.

Solutions of the wave equation for a string

To determine the possible solutions of the partial differential equation

$$\frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} = 0$$

we employ a standard trick: *separation of variables*. Write

$$\psi(x, t) = T(t)X(x) ,$$

substitute in the wave equation, and divide by $T(t)X(x)$ to get

$$\frac{\ddot{T}(t)}{T(t)} = u^2(x)\frac{X''(x)}{X(x)} .$$

Since the left-hand side depends on t only, and the right on x only, and since these two variables are independent, it must be true that they are both equal to the same constant.

Suppose this constant λ were positive: then because the solutions of

$$X''(x) = \lambda u^2(x)X(x)$$

can be either monotonically increasing or decreasing, it will be impossible for them to vanish at both ends of the string. Hence the boundary conditions force us to write

$$\frac{d^2T}{dt^2} + \omega^2\, T = 0$$

where ω^2 is real and positive. Similarly,

$$\frac{d^2X}{dx^2} + \omega^2 u^2\, X = 0$$

where we usually write

$$\omega^2 u^2(x) = k^2(x) > 0 .$$

If k is constant (*i.e.* the string has uniform density), the equation for $X(x)$ becomes easy to solve according to the boundary conditions

$$X(0) = X(l) = 0.$$

The general solution of

$$\frac{d^2X}{dx^2} + k^2\, X = 0$$

is

$$X(x) = A\sin(kx) + B\cos(kx) ;$$

from the condition at $x=0$ we see $B=0$, and from that at $x=l$ we see

$$kl = n\pi ,\ n=1, 2, \ldots$$

Thus the possible vibration frequencies are determined to be

$$\omega_n = uk_n = u\,\frac{n\pi}{l}$$

which are immediately seen to have the correct dimensionality,

$$[\omega_n] = t^{-1}.$$

Rayleigh-Ritz variational principle for eigenvalues

The equation

$$\frac{d^2X}{dx^2} + \omega^2 u^2(x)\, X = 0\ ,$$

subject to boundary conditions, poses an eigenvalue problem akin to that found in discrete vibrating systems (coupled oscillators). The Hamiltonian for a particular normal mode of vibration becomes

$$H \;=\; \int dx \left[\frac{1}{2}\,\mu(x)\, X^2(x)\, \dot{T}^2(t) \;+\; \frac{1}{2}\,\tau\, T^2(t) \left(\frac{dX(x,t)}{dx}\right)^2\right];$$

averaging over one cycle of the (sinusoidal) oscillation we have

$$\langle H \rangle \;=\; \frac{1}{2}\int_0^l dx \left[\frac{1}{2}\,\mu(x)\, X^2(x)\, \omega^2 \;+\; \frac{1}{2}\,\tau \left(\frac{dX(x,t)}{dx}\right)^2\right];$$

it is easy to see that we can obtain our differential equation for $X(x)$ by varying the functional

$$\int_0^l dx\;\; \tau \left(\frac{dX(x,t)}{dx}\right)^2$$

subject to the constraint

$$\int_0^l dx\; \mu(x)\, X^2(x) = \text{constant}\ .$$

That is, the frequency plays the role of the Lagrange multiplier. This leads us to the Rayleigh-Ritz variational principle,

$$\omega^2 \;<\; \frac{\displaystyle\int_0^l dx\;\; \tau \left(\frac{dX(x,t)}{dx}\right)^2}{\displaystyle\int_0^l dx\; \mu(x)\, X^2(x)}\;;$$

the fact that it is indeed an *upper* bound on the squared frequency is proved in standard texts on mathematical physics.

Let us illustrate: we know the solution for the lowest-frequency mode of a uniform string is

$$X_1(x) = A\sin\!\left(\frac{\pi x}{l}\right),$$

giving the exact lowest frequency $\omega_1^2 = \frac{\tau}{\mu}\frac{\pi^2}{l^2}$. The Rayleigh-Ritz bound may be used to estimate this frequency, since any function that satisfies the boundary conditions will yield a value larger than the exact one. To see how good this can be, take a trial function

$$X_{trial}(x) = x\,(l - x)$$

and substitute into the upper bound:

$$\omega^2 \;<\; \frac{\tau \int_0^l dx\,\left(l - 2x\right)^2}{\mu \int_0^l dx\, x^2 \left(l - x\right)^2} = \frac{\tau}{\mu}\,\frac{l^3 - 2l^3 + \frac{4}{3}l^3}{\frac{l^5}{3} - \frac{l^5}{2} + \frac{l^5}{5}} \;=\; \frac{\tau}{\mu}\frac{10}{l^2}\,;$$

the exact number,

$$\pi^2 = (3.14159265\ldots)^2,$$

is only slightly less than

$$10 = (3.16227766\ldots)^2\,,$$

i.e. the Rayleigh-Ritz estimate is pretty good.

2. <u>Sound propagation in gases</u>

In a gas we may normally ignore viscosity and gravity. Euler's equation, the equation of mass conservation, and a bit of thermodynamics are all we need to derive a wave equation for sound.

Imagine the gas has equilibrium pressure $\bar{p}$, density $\bar{\rho}$, and zero macroscopic velocity*. Then we may imagine a sound wave to involve small excursions from equilibrium δp, $\delta\rho$ and $\delta\vec{v}$. If we ignore terms of second or higher order in small quantities, we find

$$\bar{\rho}\,\frac{\partial \delta\vec{v}}{\partial t} = -\nabla\delta p$$

and

$$\frac{\partial\delta\rho}{\partial t} + \bar{\rho}\,\nabla\cdot\,\delta\vec{v} = 0\,;$$

taking the divergence of the first, and the time derivative of the second equation and eliminating $\delta\vec{v}$ between them, we obtain

* Of course, the molecules move quite rapidly, but randomly, so there is no overall flow.

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \nabla^2 \delta p \ .$$

We can go no further without a way to relate δp and $\delta\rho$. Such a relation follows from an *equation of state*—something obtained from knowledge of the thermodynamics of the gas. Newton assumed that the temperature remains constant during the fluctuations of density and pressure in a sound wave. That is, he assumed Boyle's Law,

$$d\left(pV\right) = 0,$$

which is easily seen to yield the relation

$$p = \bar{p}\left(\frac{\rho}{\bar{\rho}}\right).$$

In fact, Newton was wrong—the fluctuations are *adiabatic* (sometimes called isentropic), meaning no heat flows in or out of the small volumes we have been considering. Under these conditions,

$$p = \bar{p}\left(\frac{\rho}{\bar{\rho}}\right)^{\gamma}$$

where γ is the ratio of specific heats at constant pressure and volume:

$$\gamma = \frac{c_p}{c_V} .$$

Then the small excursions are related by

$$\delta p = \gamma \frac{\bar{p}}{\bar{\rho}} \delta\rho \ ,$$

which, when substituted into our previous equation, leads to

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \left(\gamma \frac{\bar{p}}{\bar{\rho}}\right) \nabla^2 \delta\rho \ ,$$

from which we identify the speed of sound,

$$u^2 = \left(\gamma \frac{\bar{p}}{\bar{\rho}}\right) .$$

Since $\gamma = \frac{5}{3}$ for monoatomic gases ($c_p = \frac{5}{2} R$, $c_V = \frac{3}{2} R$) and $\frac{7}{5}$ for diatomic ones ($c_p = \frac{7}{2} R$, $c_V = \frac{5}{2} R$), the observed speed of sound in air (a mixture of diatomic gases) is about 20% greater than that predicted by Newton's theory. Newton was convinced of the general correctness of his ideas, and proposed a number of fixes to explain the discrepancy—not very different from what theorists do today when their numbers are a bit off. This was by no means equivalent to scientific fraud, however—Broad and Wade, in thus accusing Newton of fraud[*], merely reveal their ignorance of the distinction between good and fraudulent science.

Lagrangian for sound waves

If we imagine a small displacement $\vec{\zeta}(\vec{x}, t)$ of a volume of gas, from its equilibrium position, we can immediately express the kinetic energy density as

$$\frac{1}{2}\rho\left(\frac{\partial}{\partial t}\vec{\zeta}(\vec{x}, t)\right)^2 ;$$

but what shall we take for the potential energy density? The pressure has the dimensions of energy density, but the pressure at the new position, $p\left(\vec{x}+\vec{\zeta}(\vec{x}, t)\right)$, cannot be right. Physically, we want the average of the new pressure with the pressure at the point $\vec{x}$, giving

$$L = \int d^3x \left[\frac{1}{2}\rho\left(\frac{\partial}{\partial t}\vec{\zeta}(\vec{x}, t)\right)^2 - \frac{1}{2}\left[p\left(\vec{x}+\vec{\zeta}(\vec{x}, t)\right)+p(\vec{x})\right]\right].$$

Expanding to leading order in $\vec{\zeta}(\vec{x}, t)$ we obtain

$$L = \int d^3x \left[\frac{1}{2}\rho\left(\frac{\partial}{\partial t}\vec{\zeta}(\vec{x}, t)\right)^2 - \frac{1}{2}\vec{\zeta}(\vec{x}, t)\cdot\nabla p\right] + \text{ constant} .$$

Now, using the adiabatic equation of state and the continuity equation as previously, we find

$$\nabla p = \gamma\frac{\bar{p}}{\bar{\rho}}\nabla\delta\rho$$

and

$$\frac{\partial}{\partial t}(\delta\rho + \bar{\rho}\,\nabla\cdot\vec{\zeta}) = 0 ,$$

or

$$\nabla\delta\rho = -\bar{\rho}\,\nabla\left(\nabla\cdot\vec{\zeta}\right).$$

Thus, neglecting surface terms,

$$-\int d^3x\ \vec{\zeta}(\vec{x}, t)\cdot\nabla p\ =\ \gamma\bar{p}\int d^3x\ \vec{\zeta}\cdot\nabla\left(\nabla\cdot\vec{\zeta}\right) \equiv -\gamma\bar{p}\int d^3x\ \left(\nabla\cdot\vec{\zeta}\right)^2 .$$

Therefore our Lagrangian becomes

$$L = \int d^3x \left[\frac{1}{2}\bar{\rho}\left(\frac{\partial}{\partial t}\vec{\zeta}(\vec{x}, t)\right)^2 - \frac{1}{2}\left(\nabla\cdot\vec{\zeta}(\vec{x}, t)\right)^2\right].$$

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* W. Broad and N. Wade, *Betrayers of the truth* (Simon and Schuster, New York, 1982).

It is worth noting that the equation of motion is

$$\bar{\rho}\,\frac{\partial^2\vec{\zeta}}{\partial t^2} - \gamma\bar{p}\,\nabla\left(\nabla\cdot\vec{\zeta}\,\right) = 0$$

so that we find a wave equation in the divergence of the displacement:

$$\bar{\rho}\,\frac{\partial^2}{\partial t^2}\left(\nabla\cdot\vec{\zeta}\,\right) - \gamma\bar{p}\,\nabla^2\left(\nabla\cdot\vec{\zeta}\,\right) = 0\,.$$

This says that only longitudinal sound waves propagate—transverse waves would involve the curl of the displacement, which is obviously independent of position, hence cannot describe a wave.