Solution for Chapter 18

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A.
Exercise 18.1 Derivation of MHD Equations [by Guodong Wang/99]
(a).

$$\mathbf{j}\times\mathbf{B} = \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} = -\frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{B}) = \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} - \frac{\nabla(B^2)}{2\mu_0} \quad (1)$$

$$-\nabla\cdot\mathbf{T}_M = -\nabla\cdot(\frac{B^2\mathbf{g}}{2\mu_0} - \frac{\mathbf{B}\otimes\mathbf{B}}{\mu_0}) = -\frac{\nabla(B^2)}{2\mu_0} + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}, \quad (2)$$

where we have used $\nabla\cdot\mathbf{B} = 0$. Compare Eqns. (1) and (2), we have

$$\mathbf{j}\times\mathbf{B} = -\nabla\cdot\mathbf{T}_M. \quad (3)$$

(b) First we have

$$\frac{\partial}{\partial t}(\frac{B^2}{2\mu_0}) + \nabla\cdot(\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}) = \frac{1}{\mu_0}[\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} + \nabla\cdot(\mathbf{E}\times\mathbf{B})] =$$
$$\frac{1}{\mu_0}[-\mathbf{B}\cdot(\nabla\times\mathbf{E}) + \nabla\cdot(\mathbf{E}\times\mathbf{B})] = -\frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B}) = -\mathbf{j}\cdot\mathbf{E} \quad (4)$$

Then similar to Eq. (BT-12.51)

$$(\mathbf{v}\cdot\nabla)P = \nabla\cdot(\rho\mathbf{v}h) + \frac{\partial(\rho u)}{\partial t} - \rho T\frac{ds}{dt} \quad (5)$$

Combining Eq. (4) and Eq. (5) and the mass conservation equation with $\mathbf{v}\cdot$ Eq. (BT 18.12), we obtain the energy conservation law

$$\frac{\partial}{\partial t}[(\frac{1}{2}v^2 + U + \Phi)\rho + \frac{B^2}{2\mu_0}] + \nabla\cdot[(\frac{1}{2}v^2 + h + \Phi)\rho\mathbf{v} + \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}] = \rho T\frac{ds}{dt} - \frac{j^2}{\kappa_e}. \quad (6)$$

(c).

$$\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s\mathbf{v}) = \frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s\mathbf{v}) - s\frac{\partial\rho}{\partial t} - s\nabla\cdot(\rho\mathbf{v})$$
$$= \rho\frac{\partial s}{\partial t} + \rho\mathbf{v}\cdot\nabla s = \rho\frac{ds}{dt} \quad (7)$$

In the case ds/dt is not zero, according to the first law

$$\rho T\frac{ds}{dt} = \mathbf{j}\cdot\mathbf{E}' = \frac{1}{\kappa_e\mu_0^2}(\nabla\times\mathbf{B})^2 = \frac{j^2}{\kappa_e}. \quad (8)$$

That is Eq. (BT-18.18). Combinging with Eq. (6), we get the energy conservation law (BT-18.17).

Exercise 18.2 Diffusion of Magnetic Field [by Xinkai Wu/02]

As we assume the plasma has sufficient inertia to remain at rest, we can set $\mathbf{v} = \mathbf{0}$ in all the equations.

(a) The sum of the rate of change of the manetic energy and that of heat production via Ohmic heating is given by

$$\frac{\partial}{\partial t}\frac{\mathbf{B}^2}{2\mu_0} + \mathbf{j}\cdot\mathbf{E} = \frac{1}{\mu_0^2\kappa_e}\left[\mathbf{B}\cdot\nabla^2\mathbf{B} + (\nabla\times\mathbf{B})\cdot(\nabla\times\mathbf{B})\right]$$

where we have used equation (18.9) for $\frac{\partial \mathbf{B}}{\partial t}$ and equations (18.6) and (18.7) for $\mathbf{j}$ and $\mathbf{E}$, respectively.

Now writing out the components in Cartesian coordinates explicitly and using $\nabla\cdot\mathbf{B} = 0$, we readily reduce this sum to

$$\frac{1}{\mu_0^2\kappa_e}\partial_k\left[B_j(\partial_k B_j - \partial_j B_k)\right] = -\nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\right)$$

which is a total derivative and vanishes upon integration over the whole space, namely the reduction of magnetic energy as the field decays is compensated by the Ohmic heating of the plasma.

(b) The magnetic field satisfies the diffusion equation:

$$\frac{\partial B_z}{\partial t} - \frac{1}{\mu_0\kappa_e}\nabla^2 B_z = 0$$

and the initial condition is

$$B_z(t=0) = B_0 \text{ for } \varpi < R, \text{ and 0 elsewhere}$$

with R being the radius of the cylinder. The boundary condition is given by

$$B_z(\varpi = R) = B_{z\ \text{outside}} = 0;\ B_z(\varpi = 0) = \text{finite}$$

where we have used the fact that since there is no surface current(we've turned off the solenoid current that generated the initial magnetization) B_z is continuous across the boundary. Also we approximate the magnetic field outside the cylinder as zero because we assume that the decay is slow(as a consequence of large κ_e) and thus EM radiation into the vacuum can be neglected. Solving the above boundary value problem is straightforward (see e.g. Mathews and Walker), and we find

$$B_z(\varpi, t) = \sum_{i=1}^{\infty} c_i e^{-\frac{\xi_i^2}{\mu_0\kappa_e R^2}t} J_0\left(\frac{\xi_i\varpi}{R}\right)$$

with $J_0(\xi)$ being the zeroth Bessel function, and ξ_i being the location of its ith zero, $0 < \xi_1 < \xi_2 < .....$ So for large time, when the field has decayed to a small fraction of its original value, all the higher (i.e. $i > 1$) modes have decayed away, and we have

$$B_z(\varpi, t) \approx c_1 e^{-\frac{\xi_1^2}{\mu_0 \kappa_e R^2} t} J_0(\frac{\xi_1 \varpi}{R}) = 1.6 B_0 e^{-\frac{2.4^2}{\mu_0 \kappa_e R^2} t} J_0(\frac{2.4 \varpi}{R})$$

where we have used $\xi_1 \approx 2.4$ and $c_1 \approx 1.6 B_0$.

B.

Exercise 18.3 The Earth's Bow Shock [by Xinkai Wu/00]

2. The Earth's Bow Shock

(a) The momentum flux $\sim \rho v^2$ while the magnetic pressure generated by the earth's dipole field $\sim B^2/2\mu_0 \sim (B_E^2/2\mu_0)(r_E/r)^6$, noticing that for a dipole field $B \sim r^{-3}$. Balancing these two terms and plugging in the numbers:

$B_E \sim 3 \times 10^{-5} T$, $r_E \sim 6 \times 10^6 m$

we get $r \sim 8.5 r_E \sim 5 \times 10^7 m$

(b) B&T eqn (18.23) gives: $E_1 + v_s B_1 = E_2 + v_s B_2$ and eqn (18.21) gives: $\rho_1(v_1 - v_s) = \rho_2(v_2 - v_s)$

In the infinite conductivity limit and applied to both sides of the shock front, eqn (18.5) gives:

$E_1 = -v_1 B_1$ and $E_2 = -v_2 B_2$

Combining the above equations we get

$B_2/B_1 = \rho_2/\rho_1 = (v_1 - v_s)/(v_2 - v_s)$

Namely the magnetic field strength will increase by the same ratio as the density on crossing the shock front.

Intuitively we expect the compression to decrease as the field is increased, because increasing the field means increasing the magnetic pressure, which will in turn resist compression. To be more rigorous, let's look at a limiting case of equation (18.24): When B gets very large, the magnetic pressure term dominates and this equation gives $B_1 \approx B_2$, i.e. $\rho_1 \approx \rho_2$., which means there's almost no compression.

Exercise 18.5 Force-Free Equilibria [by Guodong Wang/99]

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 = \nabla \cdot (\alpha \mathbf{B}) = \nabla \alpha \cdot \mathbf{B} \tag{9}$$

Where we haved used $\nabla \cdot \mathbf{B} = 0$.

Eq. (9) implies $\nabla \alpha \cdot \frac{\mathbf{B}}{B} = 0$, which means α is constant along the direction of $\mathbf{B}$. correspondingly α must be constant everywhere if the field lines travel everywhere.

C. Exercise 18.10 Rotating Magnetospheres [by Guodong Wang/02]

We use Cylindrical cooordinates throughout this problem.

(a).

$$0 = \partial_t \mathbf{B} = -\nabla \times \mathbf{E} \tag{10}$$

Since the system is axisymmetric, So $\partial_\phi \equiv 0$. Intergrating Eq. (10) along circle $\phi \in [0, 2\pi]$ for any given ϖ, z so that $\mathbf{r} = (\varpi, \phi, z)$, Then

$$\mathbf{E}_\phi \cdot 2\pi\varpi = 0 \qquad \Rightarrow \mathbf{E}_\phi = 0. \tag{11}$$

The magnetosphere is perfectly conducting, then

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} \tag{12}$$

so $\mathbf{v} = v\mathbf{e}_\phi$, we can write it as $\mathbf{v} = \mathbf{\Omega}(\mathbf{r}) \times \mathbf{r}$. If the magnetosphere's conducting fluid is rotating, it is obvious that Ω is its angular velocity.

(b).

$$0 = \partial_t \mathbf{B} = -\nabla \times \mathbf{E} = \nabla \times [(\Omega \times \mathbf{r}) \times \mathbf{B}] \tag{13}$$

Note that $\mathbf{\Omega} = (0, 0, \Omega)$, $\mathbf{\Omega} \times \mathbf{r} = (0, \varpi\Omega, 0)$, (Let $\phi = 0$ since the field is axisymmetric.)

$$\Rightarrow -\mathbf{E} = (\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{B} = (\varpi\Omega B_z, 0, -\varpi\Omega B_\varpi) \tag{14}$$

$$\Rightarrow 0 = \nabla \times [(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{B}] = [(\partial_z(\varpi\Omega B_z) + \partial_\varpi(\varpi\Omega B_\varpi)]\mathbf{e}_\phi \tag{15}$$

Combining Eq. (15) with $\nabla \cdot \mathbf{B} = \frac{1}{\varpi}\partial_\varpi(\varpi B_\varpi) + \partial_z B_z = 0$, we obtain

$$(\mathbf{B} \cdot \nabla)\mathbf{\Omega} = \mathbf{0}. \tag{16}$$

(c). At the surface,$v_n = 0$, $[\mathbf{E}_t] = 0$, $[\mathbf{B}_n] = 0$. So

$$\varpi\Omega^* B_n = E_t, \tag{17}$$

here the left side is the tangential electric field on the star's surface and the right side is the tangential electrice field in the magnetosphere and $B_n = \mathbf{B} \cdot \frac{\mathbf{r}}{r}$. Plugging in Eq. (14), we get

$$\varpi\Omega^* B_n = \varpi\Omega B_n \tag{18}$$

So $\Omega^* = \Omega$.

D.

Exercise 18.11 Solar Wind [by Xinkai Wu/02]

(a) Write the velocity as $\mathbf{v} = v_\phi \mathbf{e}_\phi + \mathbf{v}_P$, and the magnetic field as $\mathbf{B} = B_\phi \mathbf{e}_\phi + \mathbf{B}_P$, where the term with subscript P means the part of the vector that lies in the (r, θ) plane. Then we see that the ϕ-component of $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ comes purely from $\mathbf{v}_P \times \mathbf{B}_P$. The vanishing of this implies that $\mathbf{v}_P \propto \mathbf{B}_P \propto (\mathbf{B} - B_\phi \mathbf{e}_\phi)$. Absorbing the B_ϕ part into the $\mathbf{\Omega} \times \mathbf{r}$, we get

$$\mathbf{v} = \frac{\kappa \mathbf{B}}{\rho} + (\mathbf{\Omega} \times \mathbf{r})$$

Multiplying both sides of the above equation by ρ and then taking its divergence we get

$$\begin{aligned} 0 &= \nabla \cdot (\rho \mathbf{v}) = \nabla \cdot (\kappa \mathbf{B}) + \nabla \cdot (\rho \mathbf{\Omega} \times \mathbf{r}) \\ &= \mathbf{B} \cdot \nabla \kappa \end{aligned}$$

where we've used the mass conservation equation together with the stationarity, $\nabla \cdot \mathbf{B} = 0$, and $\nabla \cdot (\rho \mathbf{\Omega} \times \mathbf{r}) = 0$(since $\rho \mathbf{\Omega} \times \mathbf{r}$ only has ϕ-component and by axisymmetry derivative w.r.t. ϕ vanishes). Thus we seeκ is constant along a field line. And the proof for the constancy of $\mathbf{\Omega}$ is the same as that given in part (b) of Exercise 18.10.

(b) The divergence of a vector $\mathbf{C}$ in the spherical coordinates is given by $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 C_r)$ upon using axisymmetry. Taking $\mathbf{C}$ to be $\rho\mathbf{v}$ and using $\nabla \cdot (\rho \mathbf{v}) = 0$ we find that $\rho v_r r^2$ is a constant, while taking $\mathbf{C}$ to be $\mathbf{B}$ and using $\nabla \cdot \mathbf{B} = 0$ we find that $B_r r^2$ is a constant.

(c) Using the result of part (a), we have $v_r = \frac{\kappa}{\rho} B_r$, and $v_\phi = \frac{\kappa}{\rho} B_\phi + \Omega r$. Combining these we readily get

$$\frac{v_r}{v_\phi - \Omega r} = \frac{B_r}{B_\phi}$$

(d) The e.o.m. in the stationary case is given by

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\rho \nabla \Phi - \nabla P + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0}$$

Let's consider the ϕ-component of the above equation. Then the gravity and pressure terms have no contribution because of axisymmetry. And we find (using the connection coefficients in spherical coordinates we learned in Chapter 10),

$$\rho v_r \frac{1}{r}\frac{\partial}{\partial r}(r v_\phi) = \frac{B_r}{\mu_0 r}\frac{\partial}{\partial r}(r B_\phi)$$

Multiplying both sides of the above equation by r^3 and using the constancy of $\rho v_r r^2$ and $B_r r^2$ we find that

$$\Lambda \equiv r v_\phi - \frac{r B_r B_\phi}{\mu_0 \rho v_r}$$

is constant.

(e) Define the radial Alfven speed to be $a_r = \frac{B_r}{(\mu_0 \rho)^{1/2}}$, and the radial Alfven Mach number to be $M_A = \frac{v_r}{a_r}$. Then we have $\frac{1}{\mu_0 \rho} = \frac{a_r^2}{B_r^2} = \frac{v_r^2}{M_A^2 B_r^2}$. And thus we have

$$\begin{aligned} \Lambda &= r v_\phi - \frac{r B_r B_\phi}{v_r} \frac{v_r^2}{M_A^2 B_r^2} \\ &= r v_\phi - \frac{r B_\phi v_r}{B_r M_A^2} \\ &= r v_\phi - \frac{r v_\phi - \Omega r^2}{M_A^2} \end{aligned}$$

where to get to the last line we've used the result of part (c) to eliminate $\frac{v_r}{B_r}$.

Plugging the above expression for Λ into the r.h.s. of (18.87) we see that it indeed reduces to the l.h.s.: v_ϕ. Based on previous parts, we know $B_r \propto \frac{1}{r^2}$, $a_r \propto \frac{B_r}{\sqrt{\rho}} \propto \frac{1}{r^2\sqrt{\rho}}$, and thus

$$M_A = \frac{v_r}{a_r} \propto v_r r^2 \sqrt{\rho} \propto \frac{\rho v_r r^2}{\sqrt{\rho}} \propto \rho^{-1/2}$$

As ρ varies when one goes outward radially, M_A will eventually become unity at some critical radius r_c.

(f) Equation (18.87) tells us that at this critical r_c, $\Lambda = \Omega r_c^2$. Then the timescales for the sun to lose its mass and spin are

$$\begin{aligned} \tau_m &= \frac{m_{sun}}{dm/dt} \approx \frac{m_{sun}}{4\pi\rho v_r r^2} \\ \tau_L &= \frac{L}{dL/dt} \approx \frac{m_{sun}\Omega r_{sun}^2}{4\pi\rho v_r r^2 \Lambda} = \frac{m_{sun}\Omega r_{sun}^2}{4\pi\rho v_r r^2 \Omega r_c^2} = \left(\frac{r_{sun}}{r_c}\right)^2 \tau_m \end{aligned}$$

thus we see that the sun loses its spin faster than it loses its mass by a ratio of $(r_c/r_{sun})^2 \approx 400$.

(g) Plugging in the numbers, we find $\tau_c \sim 3\times10^{19}s \sim 10^3$billion years, which is much larger than the lifetime of the sun. So there's no need to worry about sun's stopping spinning (actually I don't know what harm it will bring about even if the sun does stop spinning)!

Exercise 18.6 Hartmann Flow [by Guodong Wang/99]

The force balance equation is given by equation (18.34)

$$\nabla P = \mathbf{j} \times \mathbf{B} + \eta \nabla^2 \mathbf{v}$$

the x component of which is, using $\mathbf{B} = B_0\mathbf{e}_z$ and $P = -Qx + p(z)$,

$$-Q = j_y B_0 + \eta \frac{d^2 v_x}{dz^2}$$

Now using equation (18.5) $\mathbf{j} = \kappa_e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ and $\mathbf{E} = E_0\mathbf{e}_y$ we find

$$j_y = \kappa_e(E_0 - B_0 v_x)$$

Combining the above two equations, we get

$$\frac{d^2 v_x}{dz^2} - \frac{\kappa_e B_0^2}{\eta} v_x = -\frac{(Q + \kappa_e B_0 E_0)}{\eta}$$

(b) A special solution to this equation is

$$v_0 = \frac{Q + \kappa_e B_0 E_0}{\kappa_e B_0^2}$$

and the general solution is thus given by

$$v_x = v_0 + C_1 cosh\left(\frac{Hz}{a}\right) + C_2 sinh\left(\frac{Hz}{a}\right)$$

where $H = B_0 a\left(\frac{\kappa_e}{\eta}\right)^{1/2}$. Using the boundary condition $v_x(z = \pm a) = 0$ we find

$$C_1 = \frac{-v_0}{cosh(H)}, \quad C_2 = 0$$

and thus

$$v_x = \frac{Q + \kappa_e B_0 E_0}{\kappa_e B_0^2}\left[1 - \frac{cosh(Hz/a)}{coshH}\right]$$