THE UNIVERSITY OF AKRON
The Department of Mathematical Sciences

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calculus
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Limits

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1. Introduction

The notion of *limit* is one of the most basic and powerful concepts in all of mathematics. Differentiation and Integration, which comprise the core of study in calculus, are both creatures of the limit — the concept of *limit* is the foundation stone of calculus and as such is the basis of all that follows it.

It is extremely important that you get a good understanding of the notion of limit of a function if you have a desire to fully understand calculus at the entry level.

2. Motivating the Concept

Algebra is a static mathematical field — it cannot be used to analyze the dynamics of a moving object, for example. The mathematics of calculus does have the built-in capability of making this analysis. The major concept that allows us to make the transition from algebra (static) to calculus (dynamic) is the *limit of a function*.

In this section, we give a general discussion of limits wherein I try to give you an intuitive "feel" for limit. The remaining sections consist of applications of the limit concept to physical science and geometry: Instantaneous Velocity, Tangent of a Curve, and Rate of Change.

2.1. General Discussion of Limits

Let us begin our study of limits by examining a example meant to introduce the concept of a limit and to illustrate some basic numerical techniques.

Example 2.1. Consider the function

$$f(x) = \frac{\sin(2x)}{x}, \quad \text{for } x \neq 0,$$

Discuss the behavior of this function near the exceptional point of $x = 0$.

Solution: As you can see, this function is the ratio of two well-known functions; however, something strange goes on at $x = 0$. At $x = 0$ the numerator equals 0, and the denominator equals 0 as well, so $f(0)$ is an undefined quantity. But for $x \neq 0$, $f(x)$ *is* well-defined quantity no matter how close x is to 0! What goes on here? What is the behavior (or trend) of the function near $x = 0$?

Below you will find a table of numerical calculations, please review . . . and I'll see you on the other side of that table.

$y = \sin(2x)/x$							
x	1.0	0.5	0.1	0.05	0.01	0.005	0.001
y	0.09093	1.6829	1.9867	1.9967	1.9998	1.9999	1.9999

Did you make any observations concerning the contents of the table? My observations are as follows:

Observation 1: The given values of x in the table are getting *closer and closer* to 0. This is because of our declared interested in understanding what is going on at $x = 0$. We cannot put $x = 0$, so the next best thing is to "sneak up on 0."

Observation 2: As you follow the table in the y-row from left to right, you see that the y-value entries *seem* be getting *closer and closer* to 2.

Observation 3: Summary. As x, the independent variable (the one the user has control over), gets *closer and closer* to 0, the corresponding value y-value *seems* to be getting closer and closer to 2. In this case, we write

$$\lim_{x \to 0} \frac{\sin(2x)}{x} = 2 \tag{1}$$

The above (standard) notation summarizes must succinctly our observations. Example 2.1. ■

Exercise 2.1. What are the dangers of making empirical observations based on a table of numerical calculations—just as we did in Example 2.1?

Next up is a physical example of limit that will heat up our discussion.

Illustration 1. There is a fireplace with a raging fire therein. As you move closer to the fire source the distance, x, between you and the fireplace decreases. At any given distance, x, you feel heat on your face. Let the temperature on the surface of your facial skin be measured as $f(x)$. Thus,

$$\begin{aligned} x &= \text{distance to the fire} \\ f(x) &= \text{temperature on surface of your face.} \end{aligned}$$

Now as you continue to move closer and closer to the heat source (i.e. x gets closer and closer to 0), you feel increased heat on your face. The closer your get, the greater the sense of heat. Now you would not want to actually put $x = 0$ as then you would be in the fire (a no-no, reference: childhood), but yet as you get closer, you have a sense that the temperature on the surface of your face will continue to increase until it reaches the temperature of the fire! In this case we might say:

$$\lim_{x \to 0} f(x) = \text{temperature of fire.} \tag{2}$$

Thus, from the behavior (or trend) of the function near $x = 0$, we have tried to extrapolate the functional values beyond its domain of definition; hence, we make the assertion equation (2). (It is truly a good mathematical sentence that has a 'thus' and a 'hence' in it—including this one!) ■

In each of the examples above, we were interested in the limit of a functions $f(x)$ as x got closer and closer to 0. There is nothing special about x going to 0. More generally, we are interested in the limit of a function $f(x)$ as x gets closer and closer to a number a of interest.

Based on the above example we are ready to give two rough descriptions of the symbol:

$$\lim_{x \to a} f(x) = L \tag{3}$$

Pedestrians Pay Attention: A pedestrian description of equation (3) can be phrased as follows:

> "As x gets *closer and closer* to a, the corresponding value $f(x)$ gets *closer and closer* to L."

A refinement, or rephrasing, of my *Pedestrian description* of limit is

> The limit of a function connotes the study of the behavior (or trend) of the function in smaller and smaller neighborhoods around a target point $x = a$.

2.2. Instantaneous Velocity

This section discusses the physical notion of *instantaneous velocity*: Given that a particle is in motion, define/calculate the velocity of the particle at a given instant in time. For those who want to know more, Click here.

2.3. Tangent to a Curve

In this section we discuss the *Fundamental Problem of Calculus I*: Given a function and a point of the graph of the function, define and calculate the equation of the line tangent to the graph at the given point. Click here to learn more.

2.4. Rate of Change

Given that two variables, x and y, are related, $y = f(x)$, this section discusses ways of measuring how fast the variable y *changes* per unit change in the x variable. This turns out to be a generic interpretation of the derivative of the function f. Click here for more.

3. Calculating Limits

The Goal of the Section: To develop some basic mechanical skills for evaluating

$$\lim_{x \to a} f(x) = L \tag{1}$$

where, f is a function of x and a is a number. In the section entitled Working with the Definitions, we take a deeper, more rigorous look at limit. Meanwhile, we shall be content to develop a series of "in the field" techniques, most of which are obvious. *Emphasis* will be placed on *good reasoning*, and *good and proper notation*.

Throughout this section, our guide post for evaluating limits in this section will be the Pedestrian description.

Let us begin with two (as promised) obvious rules, which we state as *theorems*.

Theorem 3.1. *Let a and c be numbers, then*

$$\lim_{x \to a} c = c. \qquad \text{(Rule 1)}$$

Proof.

From the point of view of our Pedestrian description, this is clear: As x gets closer and closer to a, what does c get closer and closer to? (Here, c in interpreted as the constant function $f(x) = c$.)

Another obvious point about the limit concept is

Theorem 3.2. *For any number a,*

$$\lim_{x \to a} x = a. \qquad \text{(Rule 2)}$$

Proof.

An intuitively satisfying observation: As x gets closer and closer to a, what does x get closer and closer to?

Visual manifestations of these rules would be

$$\lim_{x \to -1} x = -1$$
$$\lim_{t \to 7} t = 7$$
$$\lim_{w \to 100} 100 = 100.$$

Needless to say, the first two rules are somewhat limited. We need to explore how this notion of limit "interacts" with the basic arithmetic operations. For example, If,

$$\lim_{x \to 2} x = 2 \qquad \triangleleft \text{ Rule 1}$$
$$\lim_{x \to 2} 7 = 7 \qquad \triangleleft \text{ Rule 2}$$

then, what is

$$\lim_{x\to 2} 7x = L_1$$
$$\lim_{x\to 2} (x+7) = L_2$$
$$\lim_{x\to 2} 7x^2 = L_3$$
$$\lim_{x\to 2} \frac{x+7}{x} = L_4$$

The answers should apparent. For example, to intuit the limit L_1, when $x \approx 2$, then it is reasonable to suggest that $7x \approx 14$; furthermore, the closer x is to 2, the closer $7x$ is to 14. Hence, we state that $L_1 = 14$. Similarly, to evaluate L_2 we would reason as follows: When $x \approx 2$, from our knowledge of our arithmetic system, we feel that $x + 7 \approx 2 + 7$, or $x \approx 9$. Again, we would reason that as x gets closer and closer to 2, $x + 7$ would get closer and closer to 9. Thus, $L_2 = 9$.

EXERCISE 3.1. Mimic the reasoning of the previous paragraph and determine the values of L_3 and L_4.

In the next section we formalize the ideas above so we don't have to go through a complex reasoning process every time.

3.1. The Algebra of Limits

In this section we formalize the relation the limit operation has with our arithmetic system. These two interact quite nicely.

Theorem 3.3. (Algebra of Limits Theorem) *Let f and g be functions and let a and c be number. Suppose*

$$\lim_{x \to a} f(x), \quad \textit{and} \quad \lim_{x \to a} g(x)$$

exist and are finite. Then,

(1) $\displaystyle \lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x);$ (2)

(2) $\displaystyle \lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x);$ (3)

(3) $\displaystyle \lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x);$ (4)

(4) $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, *provided,* $\displaystyle \lim_{x \to a} g(x) \neq 0.$ (5)

Proof.

Exercise 3.2. Why do you think I refer to this theorem as the *Algebra of Limits*? What does the word *Algebra* mean to you? (In a wider sense than simply high school or college algebra.)

Let's spend some time mastering the contents of the theorem on the algebra of limits. Keep in mind throughout, our building block rules of Rule 1 and Rule 2.

The following is a verbose presentation.

Example 3.1. (Skill Level 0): Calculate,

$$\lim_{x\to 3} x^2 \quad \lim_{x\to 3} x^3 \quad \lim_{x\to 3} x^4 \quad \lim_{x\to 3} x^5$$

Exercise 3.3. Utilizing the techniques of the Example 3.1, argue that $\lim_{x\to 3} x^6$.

Based on our experience with the previous examples and exercises we are prepared to generalize those results.

Theorem 3.4. (Continuity of Power Functions) *Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then*

$$\lim_{x \to a} x^n = a^n. \tag{6}$$

Proof.

Before continuing with more examples, perhaps I had better explain the term *continuity* used in the title bar of the last theorem. This is an important concept that will be developed more extensively on the article on continuous functions; however, in the interim, I shall be content to state the formal definition of the terminology so we can refer back to it.

Definition 3.5. Let f be a function having a domain $\mathrm{Dom}(f)$, and let $a \in \mathrm{Dom}(f)$. We say that f is *continuous at a* provided

$$\lim_{x \to a} f(x) = f(a). \tag{7}$$

Furthermore, a function f is called a *continuous function* if f is continuous at every point in its domain.

Definition Notes: There are some thoughts on the definition, given in the form of bulleted paragraphs.

- The phrase "f is continuous at $x = a$" is designated to those functions for which the evaluation of the limit problem is done simply by evaluating the function, f at the limiting point, a. This is the content of (7). *Not all limit problems can be evalulated this way.* In a sense, the continuous functions easiest kind of function to deal with (yet very important). Of course, we must *first prove* a given function is continuous before evaluating limits so easily.
- There is a difference between *continuous at* $x = a$, and just *continuous*: the former is a *local property*, and the latter is a *global property*. A given function can be continuous *at* one point but not *at* another. For example, the function

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 17 & \text{if} x = 0 \end{cases}$$

Now this is a rather artificial example but it serves the point. It turns out that

$$\lim_{x\to 2} f(x) = 2 = f(2)$$

but,

$$\lim_{x\to 0} f(x) = 0 \neq f(0)$$

This function is *continuous at* $x = 2$, but *not continuous*, or (*discontinuous*) *at* $x = 0$. Thus, the property of being continuous is relative to the particular point in its domain (hence is a *local property*). On the other hand, if a function is continuous at each point in its domain, we refer to that function as being *continuous* – at every point understood.

▪ The content of Theorem 3.4 is that a *power function* is a *continuous function*. Having made that designation, we can now evaluate limit problems involving *power functions* is the easiest possible way, e.g.

$$\lim_{x\to 101} x^{123} = 101^{123},$$

no thinking necessary! ■

Now let's continue this the development of the limit concept through examples and discussion.

EXAMPLE 3.2. (Skill Level 0): Calculate the limit

$$\lim_{x \to 2} (3x^2 - 2x + 1).$$

EXERCISE 3.4. Calculate $\lim_{x \to -2} (2x^3 + x^2 - 3x + 2)$. Be sure to delineate all steps using standard notation.

Using the previous examples and exercises to guide our reasoning, while keeping in mind Rule 1, Rule 2, and the Algebra of Limits Theorem, we can now state the following theorem.

Theorem 3.6. (Continuity of Polynomial Functions) *Let $p(x)$ be a polynomial and $a \in \mathbb{R}$. Then*

$$\boxed{\lim_{x \to a} p(x) = p(a).} \tag{8}$$

Proof.

This theorem means that polynomials are continuous functions,

Now the process if evaluating limits of polynomials can be accelerated.

EXAMPLE 3.3. Calculate $\lim_{x \to -2} (\frac{1}{2}x^4 - \frac{2}{3}x^3 - \frac{4}{3})$, and $\lim_{w \to 4} (2w^2 - 6w + 1)$.

EXERCISE 3.5. Calculate the limit: $\lim_{x \to -1} (2x^3 + 6x^2 + 3x - 4)^3$.

Warning: Think *before* you act. Think about the content of the of the theorem on Continuity of Polynomial Functions. What, in essence, is it saying?

Now, let's take a look at another example of a different type.

EXAMPLE 3.4. (Skill Level 0): Calculate the limit

$$\lim_{x \to -1} \frac{x^2 - 3x + 2}{2x^3 + x - 4}.$$

Here is another example for your study.

Example 3.5. Calculate the limit

$$\lim_{x \to 2} \frac{x^3 - x^2 - 4x + 3}{x^2 + 3x - 6}.$$

And, furthermore, use correct notation, and write out a well-organized solution.

Study the presentations given in your calculus book and the ones in these tutorials. As you do problems, *use correct notation*, organize your steps, think and plan how to present the solution, and, of course, use good mathematics (algebra and calculus). I assure you, it is more time consuming for me to *type out* the mathematics in these tutorials in an organized way than it is for you to *write out* your solutions in a good form. If I can take the trouble of using standard notation, you can too.

We have been illustrating the pattern of thought for handling quotients of two polynomials. I'm sure you have seen the pattern. Let's formalize our observations.

Theorem 3.7. (Continuity of Rational Functions) *Let f be a rational function, and let $a \in \mathrm{Dom}(f)$. Then*

$$\lim_{x \to a} f(x) = f(a). \tag{9}$$

Proof.

Theorem/ Notes: We make a couple of observations.

▪ This theorem is saying that a rational function is continuous at each point $x = a$ that belongs to the domain of f; in other words, f is a continuous functions.

▪ In the corresponding theorem for polynomials, Theorem 3.6, the domain of polynomial functions is all of $\mathbb{R}$. Rational functions may have a limited domain. A careful domain analysis need be done before declaring continuity at a point.

▪ The theorem on Continuity of Rational Functions supersedes the previous theorems of the same type (see Theorem 3.4 and Theorem 3.6). Power functions and polynomials are special cases of rational functions. ■

Continuing now with some examples.

Example 3.6. Calculate $\lim_{x \to 2} \dfrac{5x^2 + 8}{3x + 1}$.

The previous examples were of *Skill Level 0*, let's move to *Skill Level 1* shall we?

Example 3.7. (Skill Level 1) Calculate $\lim_{x \to 2} \dfrac{x - 2}{x^2 - x - 2}$.

Here's another example of the same type, but I'll put the solution elsewhere.

Example 3.8. (Skill Level 1): Evaluate the limit: $\lim_{x \to -1} \dfrac{x^2 + 4x + 3}{2x^2 + x - 1}$.

In the previous examples, there were hidden factors that "cause" the numerator and denominator to go to zero. Once these offending factors were located and cancelled out, we were back to a Skill Level 0 problem. Here is a rule you can go by ... at least for now.

Empirical Oberservation: At our level of play (Calculus I), when we are trying to calculate the limit of a ratio of two expressions, and the limit of both the numerator and denominator is zero, then the numerator and denominator have a common factor(s) that need to be cancelled

Now you try some.

Exercise 3.6. Calculate the limit: $\displaystyle\lim_{x\to 2} \frac{x^3 - x^2 - 2x}{x^2 + x - 6}$.

Exercise 3.7. Calculate the limit: $\displaystyle\lim_{x\to 1/3} \frac{3x^2 + 2x - 1}{3x - 1}$.

Exercise 3.8. Calculate the limit: $\displaystyle\lim_{t\to 1} \frac{t^{-2} - 2t^{-1} + 1}{t^{-1} - t^{-2}}$.

EXERCISE 3.9. Calculate the limit: $\lim_{x \to 3} \dfrac{x(x^2+1)^3}{2x^2+1}$.

3.2. The Limit of Composite Functions

In the article on functions, there was extensive discussion on composite functions. Throughout, we assume knowledge of composition of functions.

In the sections on the Algebra of Limits, we built up some ideas and techniques of evaluating the limit of rational functions. Even though the theorem of the *Algebra of Limits* applied quite generally to all functions, our techniques were limited to rational functions. Not all functions are rational functions, i.e. there are functions out there that are more complicated than mere ratios of polynomial functions.

This section addresses itself to limit problems involving functions "built-up" through composition.

Let's take an example to motivate our discussion.

EXAMPLE 3.9. Calculate $\lim_{x \to 3}(x^2+1)^{23}$.

EXERCISE 3.10. Can you reason this one out? Calculate

$$\lim_{x \to 1} \sqrt{17x^3 + 23x^2 + 9}.$$

Think it out before daring to look.

Theorem 3.8. (The Composite Limit Theorem) *Let f and g be functions that are compatible for composition, let $a \in \mathbb{R}$. Suppose,*

(1) $\lim_{x \to a} g(x)$ *exists, let* $b = \lim_{x \to a} g(x)$*;*
(2) $b \in \mathrm{Dom}(f)$, *and* $\lim_{y \to b} f(y) = f(b)$ *exists.*

Then

$$\lim_{x \to a} f(g(x)), \quad \textit{exists}$$

and,

$$\lim_{x \to a} f(g(x)) = f(b),$$

or,

$$\boxed{\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)).} \tag{10}$$

Proof.

Before we can apply this new limit result to problems that are beyond our current level, it is necessary to introduce a *continuity theorem* for *root functions.*

Theorem 3.9. (Continuity of the Root Function) *Let* $n \in \mathbb{N}$*. Define* $f(x) = \sqrt[n]{x}$*, for* $a \in \mathrm{Dom}(f)$*. Then*

$$\lim_{x \to a} f(x) = f(a)$$

or,

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \tag{11}$$

Proof.

Theorem Notes: The domain of the n^{th}-root function: For $f(x) = \sqrt[n]{x}$, we have

$$\begin{aligned} \mathrm{Dom}(f) &= \mathbb{R} && \text{if } n \text{ is odd} \\ \mathrm{Dom}(f) &= [\,0, \infty\,] && \text{if } n \text{ is even} \end{aligned}$$

∎

Corollary 3.10. *Suppose* $\lim_{x \to a} g(x)$ *exists, then*

$$\lim_{x \to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \to a} g(x)}, \tag{12}$$

provided that the number $b := \lim_{x \to a} g(x)$ *is within the domain of the* n^{th}*-root function.*

Proof.

Now armed with these theorems, we now have the *theoritical base* to solve more difficult problems

EXAMPLE 3.10. (Skill Level 0) Calculate the limit: $\lim_{x \to 2} \sqrt{x^2 + 3x}$

Here's another example of the same type.

EXAMPLE 3.11. Evaluate the limit: $\lim_{t \to -1} (t^3 - 5t + 1)^{2/3}$.

Exercise 3.11. Evaluate the limit: $\lim_{x \to 4} \sqrt[4]{x^3 - 2x}$.
Justify your steps by explicitly referencing your steps in the manner of these tutorials.

Exercise 3.12. Evaluate the limit: $\lim_{x \to 1} (2x^2 - 3x + 5)^{3/2}$. *Justify* your steps by explicitly referencing your steps in the manner of these tutorials.

From our experiences of the past examples and exercises we are ready to make an observation that will greatly accelerate solving limiting problems of the type we have been working on.

Theorem 3.11. (Continuity of Algebraic Functions) *Let f be an algebraic function, and let $a \in \mathrm{Dom}(f)$. Then*

$$\lim_{x \to a} f(x) = f(a).$$

Proof.

This means that algebraic functions are continuous at every point with their domain of definition. The domain of algebra functions can be limited by any root functions and quotients that make up its definition. A careful *domain analysis* must be made to determine whether a given point $x = a$ does or does not belong to the domain of the algebraic function of interest.

EXAMPLE 3.12. Evaluate the limit: $\lim\limits_{x \to 4} \dfrac{\sqrt[3]{x^2 - 3x - 3}}{\sqrt{x} + \sqrt[3]{2x}}$.

Now let's consider a little harder one.

EXAMPLE 3.13. (Skill Level 1.3) Calculate the limit

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 1} - 1}{x^2}.$$

Now, how about doing one yourself?

EXERCISE 3.13. Calculate the limit: $\lim\limits_{x \to 0} \dfrac{\sqrt{3x + 4} - 2}{x}$.

Exercise 3.14. Evaluate the limit: $\lim\limits_{x \to 2} \dfrac{\sqrt{x} - 2}{x - 4}$.

3.3. Other Tools: The Squeeze Theorem

Some limits problems are of such a nature that they defy easy evaluation of their limit using the standard techniques described earlier in this section. In this section we illustrate a useful method of evaluating limits. The technique involves *comparing* the given limit problem with other limit problems that are easier to evaluate.

The actual Squeeze Theorem is discussed elsewhere in more detail and rigor.

The Squeeze Method

The Problem: Evaluate $\lim\limits_{x \to a} f(x)$.

The Guess: It helps in this method to guess the limit. Let's say you guess that $\lim\limits_{x \to a} f(x) = L$. Where L is your guess — a specific numerical value.

The Method: You must construct two functions g and h having the following properties.

1. The functions g and h "bound" the function f:
$$g(x) \leq f(x) \leq h(x), \tag{13}$$
for all x close to a.
2. The functions g and h are created so that
$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L, \tag{14}$$
The common value of the limits in (14) is L, your guess hopefully.

Conclusion: Should you be able to carry out the game plan described above, then by the Squeeze Theorem, we are emboldened to conclude

$$\lim_{x \to a} f(x) = L.$$

This technique of at first difficult to master. It's the first of many methods in mathematics that require the creation of inequalities.

EXAMPLE 3.14. Argue that $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$.

4. Trigonometric Limits

Looking forward to our article on Differentiation, we now look at certain important and basic limits of trigonometric functions. Another reason we consider trigonometric functions is that they supply us with a very nice application of the "squeeze techniques."

We begin with an obvious limit statement — but mildly difficult to prove. These limits are important in the study of *continuity* of the trigonometric functions.

Theorem 4.1. *The following limits are obtained.*

$$\lim_{x\to 0} \sin(x) = 0 \qquad \lim_{x\to 0} \cos(x) = 1. \tag{1}$$

Proof.

The proof is referenced above, but let me sketch the critical steps. In the proof, we demonstrate the following inequality:

$$-x \leq \sin(x) \leq x \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}. \tag{2}$$

Obviously,

$$\lim_{x \to 0} (-x) = \lim_{x \to 0} x = 0. \tag{3}$$

We are now in position to apply the squeeze method. In that method, $g(x) = -x$, and $h(x) = x$. Line (2) represents condition (1) in the squeeze method, and equation (3) represents condition (2) in the squeeze method. Therefore, we can allowed to conclude

$$\lim_{x \to 0} \sin(x) = 0. \tag{4}$$

Of course the critical step is (2). This is proved in the proof.

Exercise 4.1. The limit $\lim_{x \to 0} \cos(x) = 1$ follows from (4). Show this. (Hint: $\cos(x) = \sqrt{1 - \sin^2(x)}$, for $-\pi/2 \leq x \leq \pi/2$; apply Corollary 3.10.)

Another set of limit problems,

$$\lim_{x\to 0} \frac{\sin(x)}{x} \qquad \lim_{x\to 0} \frac{\cos(x)-1}{x} \tag{5}$$

important to the discovery of the derivatives of the trigonometric functions, is the purpose of our next study.

EXAMPLE 4.1. Calculate numerically, the limits given in (5).

The results of EXAMPLE 4.1 would suggest that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x\to 0} \frac{\cos(x)-1}{x} = 0 \tag{6}$$

This is how many mathematical inquiries are pursued: Defining the problem; numerical investigations that would suggest analytical directions; and analytical solution to the basic problem.

Theorem 4.2. *The following limits are obtained.*

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0. \tag{7}$$

Proof.

The proof of this is referenced, but let me sketch the critical steps. It is shown in the proof that

$$\cos(x) \leq \frac{\sin(x)}{x} \leq \frac{1}{\cos(x)} \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

This is the critical inequality needed to "squeeze" $\sin(x)/x$. Now by Theorem 4.1, $\lim_{x\to 0} \cos(x) = 1$, thus,

$$\lim_{x\to 0} \cos(x) = \lim_{x\to 0} \frac{1}{\cos(x)} = 1.$$

Once again we are in the divine state to use the squeeze method, and conclude

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1. \tag{8}$$

Exercise 4.2. The limit

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$$

follows from (8) and (4). Show this.

(*Hint*: $(1 - \cos(x))(1 + \cos(x)) = \sin^2(x)$)

While we're at it, let me demonstrate some ideas that utilize (1) and (7). It is quite typical of mathematical thinking to take a basic set of results and use them to make more complicated calculations.

Example 4.2. Calculate $\displaystyle\lim_{x \to 0} \frac{\sin(4x)}{x}$.

Here is another example of the same type.

Example 4.3. Calculate $\displaystyle\lim_{x \to 0} \frac{\sin(x/5)}{x}$.

Exercise 4.3. Calculate $\displaystyle\lim_{x \to 0} \frac{\sin(99x)}{x}$.

Exercise 4.4. Calculate $\lim_{x \to 0} \frac{x}{\sin(99x)}$.

An important variation on (7) is the point of the next exercise.

Exercise 4.5. Show $\lim_{x \to 0} \frac{\tan(x)}{x} = 1$. (*Hint*: $\tan(x) = \frac{\sin(x)}{\cos(x)}$.)

Because $\tan(x)/x$ satisfies the same limit equation as does $\sin(x)/x$, the same techniques used for $\sin(x)/x$ are valid for $\tan(x)/x$.

Exercise 4.6. Calculate $\lim_{x \to 0} \frac{\tan(7x)}{x}$.

Another nail in this same coffin is the next example.

Example 4.4. Calculate $\lim_{x \to 0} \frac{\sin(4x)}{\sin(6x)}$.

Exercise 4.7. Calculate $\lim_{x \to 0} \frac{\sin(5x)}{\sin(3x)}$.

Exercise 4.8. Calculate $\lim_{x \to 0} \frac{\sin(x/5)}{\sin(3x)}$.

Exercise 4.9. Calculate $\lim_{x \to 0} \dfrac{\tan(8x)}{\tan(4x)}$. (Can you guess the answer?)

5. One-Sided Limits

The past few sections we have been examining the limit concept:

$$\lim_{x \to a} f(x) = L,$$

where L is either a number of the symbol $\pm\infty$. These kinds of limits are *bi-directional*, i.e. $x < a$ and close to a and $x > a$ and close to a. As an additional tool for analyzing the behavior of functions near a point $x = a$, we now take a one-sided or uni-directional approach. Indeed, the behavior of a function f to the left of a may be radically different from the behavior of f to the right of a.

5.1. The Left-Hand Limit

Let f be a function and a a point. We write

$$L = \lim_{x \to a^-} f(x)$$

if it is true that as x gets closer and closer to a, *and* $x < a$, $f(x)$ gets closer and closer to L. In this case we say that L is the left-hand limit of f, or the limit from the left of f, as x approaches a.

As you can see, this is the same *Pedestrian Description* of limit we have seen before, except for the qualification that $x < a$;consequently, you can expect the left-hand limit to function similarly to the (bi-directional) limit.

Take note that the limit notation is a little different. We have written $x \to a^-$. The superscript of $-$ connotes the limit from the left.

Here is an example along with some of standard techniques that are used with the one-sided limit concept.

Example 5.1. Consider the function:

$$f(x) = \begin{cases} x^2 & x \leq 2 \\ x^3 & x > 2. \end{cases}$$

Calculate: $\lim_{x \to 2^-} f(x)$.

Exercise 5.1. Define the function f by

$$f(x) = \begin{cases} 6x^2 - 3x + 1 & x < -1 \\ 3 - 3x^2 - 2x^3 & x \geq -1 \end{cases}$$

Evaluate $\lim_{x \to -1^-} f(x)$.

Exercise 5.2. Consider the function:

$$f(x) = \begin{cases} x^2 - 2x & x \leq -2 \\ 1 - 2x^3 & -2 < x \leq 1 \\ 7x - 1 & x > 1. \end{cases}$$

Calculate: $\lim_{x \to -2^-} f(x)$, and $\lim_{x \to 1^-} f(x)$.

Exercise 5.3. Let $a \in \mathbb{R}$. Prove that $\lim_{x \to a^-} |x| = |a|$.

The concept of left-hand limit applies equally well to *infinite limits*.

Example 5.2. Discuss: $\displaystyle\lim_{x \to 1^-} \frac{x}{x-1}$.

Exercise 5.4. Discuss: $\displaystyle\lim_{x \to -1^-} \frac{x^2 - x - 3}{1 + x}$.

Exercise 5.5. Discuss: $\displaystyle\lim_{x \to 1/2^-} \frac{1}{\sqrt{1 - 2x}}$.

Exercise 5.6. Discuss: $\displaystyle\lim_{x \to 2^-} \frac{x - 1}{6 - x - x^2}$.

5.2. The Right-Hand Limit

Let f be a function and a a point. We write

$$L = \lim_{x \to a^+} f(x)$$

if it is true that as x gets closer and closer to a, *and* $x > a$, $f(x)$ gets closer and closer to L. In this case we say that L is the right-hand limit of f, or the limit from the left of f, as x approaches a.

As you can see, this is the same *Pedestrian Description* of limit we have seen before, except for the qualification that $x > a$; consequently, you can expect the right-hand limit to function similarly to the (bi-directional) limit.

Take note that the limit notation is a little different. We have written $x \to a^+$. The superscript of $-$ connotes the limit from the right.

Let's examine the same examples as in the previous section . . . giving me a chance to practice my *cut and paste techniques*.

EXAMPLE 5.3. (Continued from EXAMPLE 5.1) Consider the function:

$$f(x) = \begin{cases} x^2 & x \le 2 \\ x^3 & x > 2. \end{cases}$$

Calculate: $\lim_{x \to 2^+} f(x)$.

EXERCISE 5.7. (Continued form EXERCISE 5.1) Define the function f by

$$f(x) = \begin{cases} 6x^2 - 3x + 1 & x < -1 \\ 3 - 3x^2 - 2x^3 & x \geq -1 \end{cases}$$

Evaluate: $\lim_{x \to -1^+} f(x)$.

EXERCISE 5.8. (Continued form EXERCISE 5.2) Consider the function:

$$f(x) = \begin{cases} x^2 - 2x & x \leq -2 \\ 1 - 2x^3 & -2 < x \leq 1 \\ 7x - 1 & x > 1. \end{cases}$$

Calculate: $\lim_{x \to -2^+} f(x)$, and $\lim_{x \to 1^+} f(x)$.

EXERCISE 5.9. (Continued from EXERCSE 5.3) Let $a \in \mathbb{R}$. Prove that $\lim_{x \to a^+} |x| = |a|$.

5.3. Two-sided and One-sided Limits Related

The relationship between these various types of limits is contained in the following theorem.

Theorem 5.1. Two-sided and One-sided Limits Related *Let f be a function and $a \in \mathbb{R}$.*

(1) *If $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, and in this case*

$$\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

(2) *If $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, and if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ exists, and*

$$\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

That was quite a mouthful.

Theorem Notes: We make several cogent remarks.

▪ All that is being said in the pseudo-intellectual terminology of the theorem is this: The two-sided limit exists if and only if (is equivalent to) the left-hand limit equals the right-hand limit. In this case, the value of the two-sided limit is the common value of the two-one sided limits.

▪ One of the major applications of this theorem is to prove that a two-sided limit *does not exist*. From the first part of the theorem if the two-sided limit exists, the two one-sided limits are equal to each other; therefore, if the two one-sided limits either do not exist, or if they do exist they differ from each other, we can deduce the two sided limit does not exist.

▪ In the previous section, One-Sided Limits, we have tacitly been using this theorem in our work there — sorry I didn't tell you!

Techniques for showing the nonexistence of a limit
If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ does not exist.

Let's apply this principle.

Example 5.4. Once again, consider the function,

$$f(x) = \begin{cases} x^2 & x \leq 2 \\ x^3 & x > 2. \end{cases}$$

Argue that $\lim\limits_{x\to 2} f(x)$ does not exist.

Exercise 5.10. (Continued form **Exercise 5.1**) Define the function f by

$$f(x) = \begin{cases} 6x^2 - 3x + 1 & x < -1 \\ 3 - 3x^2 - 2x^3 & x \geq -1 \end{cases}$$

Argue that $\lim_{x \to -1} f(x)$ does not exist.

EXERCISE 5.11. (Continued form EXERCISE 5.2) Define the function:

$$f(x) = \begin{cases} x^2 - 2x & x \leq -2 \\ 1 - 2x^3 & -2 < x \leq 1 \\ 7x - 1 & x > 1. \end{cases}$$

Argue that $\lim_{x \to -2^+} f(x)$ d.n.e., and $\lim_{x \to 1^+} f(x)$ d.n.e.

EXERCISE 5.12. Discuss: $\displaystyle \lim_{x \to 3^+} \frac{x-1}{(x-2)(x-3)}$.

EXERCISE 5.13. Prove: $\lim_{x \to a} |x| = |a|$, for any $a \in \mathbb{R}$.

The last exercise (EXERCISE 5.13 is of sufficient importance that we feel a need to elevate it to the status of a theorem.

Theorem 5.2. *For any $a \in \mathbb{R}$,*

$$\lim_{x \to a} |x| = |a|.$$

Proof. We cite Exercise 5.13.

Exercise 5.14. Calculate: $\lim_{x \to -1} |2x^2 + 3x - 5|$. (*Hint*: Review Theorem 3.8 and Theorem 5.2)

Exercise 5.15. Calculate $\lim_{t \to 4} \left| \dfrac{t-6}{\sqrt{2t+1}} \right|$.

6. Limits Involving Infinity

In this section we discuss two types of limit operations: (1) limits having infinite values; and (2) limits at infinity. Both of these two types of limits have a strong graphical interpretations.

6.1. Infinite Limits

There are other ways that a limit does not exist. Consider the following example to illustrate the kind of limiting problems of this section.

EXAMPLE 6.1. Discuss $\lim_{x \to 0} \dfrac{1}{x^2}$.

In this section we consider limits of the following form:

$$\lim_{x \to a} f(x) = +\infty \qquad \lim_{x \to a} f(x) = -\infty$$

These kind of limits are called *infinite limits*.

Here is a rough and ready description of these two limit concepts, the rigorous definitions are given later.

Pedestrian Description of $\lim_{x \to a} f(x) = +\infty$.
As x gets closer and closer to a, $f(x)$ gets larger and larger without bound.

Pedestrian Description of $\lim_{x \to a} f(x) = -\infty$.
As x gets closer and closer to a, $f(x)$ gets smaller and smaller in without bound.

EXERCISE 6.1. Contemplate the *Pedestrian Descriptions*, ponder the concept of infinite limits, and try to formulate what I mean by the phrase “without bound.”

6.2. Limits at Infinity

In this section we take up the meaning of the symbols

$$\lim_{x \to +\infty} f(x) = L \qquad \lim_{x \to -\infty} f(x) = L,$$

their calculation, and their geometric interpretation. Later, in another section, we take up the meaning of this type of limit at the definition level.

7. Some Limits Do Not Exist

Lest you think that all limits exist, we have devoted this section to limits problems wherein the limit does not exit.

Fundamentally, we classify two ways: undefined limits and infinite limits. The latter topic has already been discussed, and the former is taken up in this section.

7.1. Undefined Limits

When we write

$$\lim_{x \to a} f(x) = L,$$

what do we mean? From our Pedestrian Description, which is the *only* guide we have of limit until the theoretical definition, this means that as x gets closer and closer to a, $f(x)$ gets closer and closer to L.

When we say limit

$$\lim_{x \to a} f(x), \qquad \text{does not exist,}$$

we mean there *does not exist a number* L having the property that as x gets closer and closer to a, $f(x)$ is getting closer and closer to L.

One of our *implicit* understandings of limit is that a limit is *unique*. That is, if we calculate $\lim_{x\to 2} x^2 = 4$, then we understand that there is not some *other* limiting value for the *same* problem. Given that the limit of $\lim_{x\to 2} x^2 = 4$ is a correct answer, we would *reject* the statement $\lim_{x\to 2} x^2 = 6.2490203309$ as being a *false statement*.

Discerning the *nonexistance* of a limit puts more demand on our intuition than calculating a limit that exists. One technique that is used quite often to argue that a limit *does not exist* is to try to *deny the uniqueness of the limit*. Let's do an example to illustrate the idea before formalizing it.

EXAMPLE 7.1. Analyze the limit problem: $\displaystyle\lim_{x\to 0} \sin(\frac{1}{x})$.

EXAMPLE 7.2. Define the function $f(x) = \begin{cases} x, & x \le 2 \\ x^2, & x > 2 \end{cases}$. Analyze the limit $\displaystyle\lim_{x\to 2} f(x)$.

The use of One-Sided Limits represents a general method of handling piecewise defined functions, whether the limit exists or not.

In the past two examples we have exhibited a “standard technique” for proving that a limit does not exist, let's formalize it.

Technique For Arguing the Nonexistance of a Limit. Let $f(x)$ be a function and $a \in \mathbb{R}$. In order to argue that $\lim_{x \to a} f(x)$ does not exits, first find a sequnce of x's that approach a such that $f(x)$ gets closer and closer to a number L_1. Then find *another* sequence of x's that approach a such that $f(x)$ gets closer and closer to *another* number L_2. ($L_1 \neq L_2$.)

8. Working with the Definitions

In this section we introduce the precise definitions of limit, and learn how to use them.

Click Here to go there.

9. Presentation of the Theory

In this section we present the proof of the theorems stated in this article. The material is self-contained and can be read without any prior knowledge. The only requirement is an enquiring mind, logical mind, and a desire to know why things are the way they are. Click Here.

The *verbalization* of the equation

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

is phrased as follows: "The *limit* of a *sum* is the *sum* of the *limits*."

The *verbalization* of the equation

$$\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$$

is phrased as follows: "The *limit* of a *constant* times a *functions* is the *constant* times the *limit* of the function"

The *verbalization* of the equation

$$\lim_{x\to a}(f(x)g(x)) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$$

is phrased as follows: "The *limit* of a *product* is the *product* of the *limits*."

The *verbalization* of the equation

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)},\quad \lim_{x\to a}g(x)\neq 0$$

is phrased as follows: "The *limit* of a *quotient* is the *quotient* of the *limits*, provided that the limit of the denominator is *nonzero*."

Solutions to Exercises

2.1. In making out the table in EXAMPLE 2.1, I had to decide at what points to evaluate the function $f(x) = \sin(2x)/x$. I decided to use

$$x = 1.0, 0.5, 0.1, 0.05, 0.01, 0.005, 0.001.$$

What if the behavior of the function as observed using these values differs from the behavior of the function using another set of values? Our empirical observation that the function seems to tend to a value of 2 is a function of the particular points we calculated. It *may* be true, in fact, that there is no limit of this function!

Another point concerns numerics. As the x gets closer and closer to 0, we are dividing a number close to 0 ($\sin(2x)$) by another number (x) close to zero. In such a situation, your calculator may not give an accurate calculation. Thus, the numbers we are using to judge the trend in the function values may be unreliable. Exercise 2.1. ■

3.1. $L_3 = 28$ and $L_4 = 9/2$.

For the case of L_3, when $x \approx 2$, $x^2 = xx \approx (2)(2) = 4$. Now if $x^2 \approx 4$, then $7x^2 \approx (7)(4) = 28$. Thus, $L_3 = 28$.

The evaluation of L_4 is left to you, dear reader. Exercise 3.1. ■

3.2. *Algebra*, to me, connotes a *system of symbols* (in the case of college algebra, these symbols are oft times denoted by x, y, and z); a *system of operations* used with these symbols (in college algebra, these operations are addition, subtraction, multiplication, and division); and a *system of rules* telling us how these symbols interact with the operations.

Sounds familiar doesn't it. I have tried to insinuate exactly such jargon into my discussions on limits. In the case of limits, the system of symbols consist of letters f, g, and h (and so on) that represent functions and x, y, and z that represent the usual algebra quantities. The *Algebra of Limits Theorem* delineates how the symbols interact with certain operations: the operations of limit, addition, subtraction, multiplication, and division.

Exercise 3.2. ■

3.3. We use the techniques of Example 3.1:

$$\begin{aligned} \lim_{x \to 3} x^6 &= \lim_{x \to 3} (x x^5) \\ &= (\lim_{x \to 3} x)(\lim_{x \to 3} x^5) && \triangleleft \text{ by (4)} \\ &= (3)(3)^5 && \triangleleft \text{ by Example 3.1} \end{aligned}$$

Exercise 3.3. ■

3.4. I will not delineate the details on this answer. Just follow the solution pattern in Example 3.2. As you go through the details, it will re-enforce your reasoning processes. The bottom line:

$$\lim_{x \to -2} (2x^3 + x^2 - 3x + 2) = -4.$$

Let $f(x) = 2x^3 + x^2 - 3x + 2$. Then we make the summarize our calculation:

$$\lim_{x \to -2} f(x) = f(-2).$$

Thus, f is continuous at $x = -2$. Exercise 3.4. ■

3.5. The function $f(x) = (2x^3 + 6x^2 + 3x - 4)^3$ is nothing more than a polynomial; therefore, by the theorem above,

$$\lim_{x \to -1} f(x) = f(-1) = (-3)^3 = -27.$$

I hope you didn't multiply out the expression $(2x^3 + 6x^2 + 3x - 4)^3$ and then take the limit ... that's what I was warning you about!

See what I was referring to when I hinted in the statement of the problem to understand the content of the theorem. The function was a polynomial and so the limit is $f(-1)$. This was Skill Level 0.

Exercise 3.5. ■

3.6. We proceed along standard lines of inquiry. Begin by observing

$$\lim_{x\to 2}(x^3 - x^2 - 2x) = 2^3 - 2^2 - 2(2) = 0$$

and,

$$\lim_{x\to 2}(x^2 + x - 6) = 2^2 + 2 - 6 = 0$$

Because the limit of the denominator, the theorem on quotients does not apply, yet, my *Empirical Observation* remains valid. We should seek to *factor* the numerator and denominator.

$$\begin{aligned}\lim_{x\to 2}\frac{x^3 - x^2 - 2x}{x^2 + x - 6} &= \lim_{x\to 2}\frac{x(x-2)(x+1)}{(x-2)(x+3)} \\ &= \lim_{x\to 2}\frac{x(x+1)}{x+3} && \triangleleft \text{ Cancellation of Factors} \\ &= \lim_{x\to 2}\frac{2(3)}{5} && \triangleleft \text{ by (9)} \\ &= \frac{6}{5}.\end{aligned}$$

Exercise 3.6. ■

3.7. This is basically an exercise in factoring. Find the common factors and cancel them out. Then find the limit that was obscured by those cancelled factors. Use good techniques. I'm trusting you to show all details, even as I have in these tutorials.

$$\lim_{x \to 1/3} \frac{3x^2 + 2x - 1}{3x - 1} = \frac{4}{3}.$$

Exercise 3.7. ■

3.8. Mere algebraic subterfuge trying to obfuscate our solution. The first thing you *should* have thought of is to rid yourself of those dastardly *negative exponents*.

$$\begin{aligned}\frac{t^{-2}-2t^{-1}+1}{t^{-1}-t^{-2}} &= \frac{t^{-2}-2t^{-1}+1}{t^{-1}-t^{-2}}\,\frac{t^2}{t^2} \\ &= \frac{1-2t+t^2}{t-1} \\ &= \frac{t^2-2t+1}{t-1}\end{aligned}$$

Now apply the techniques of the previous examples. (*Answer*: 0).

Exercise 3.8. ■

3.9. See the solution to Exercise 3.5 for my thoughts and feelings on the subject. The answer is

$$\lim_{x \to 3} \frac{x(x^2+1)^3}{2x^2+1} = \frac{3{,}000}{19}.$$

Exercise 3.9. ■

3.10. When x is close to 1, the radicand, $17x^3 + 23x^2 + 9$ is close to 49. Now the square root of anything close to 49 is close to 7; therefore, when x is close to 1, $\sqrt{17x^3 + 23x^2 + 9}$ is close to 7.

$$\lim_{x \to 1} \sqrt{17x^3 + 23x^2 + 9} = 7.$$

Hey, this is very natural! Exercise 3.10. ■

3.11. The device for making the argument is Corollary 3.10. By that corollary,

$$\lim_{x\to 4} \sqrt[4]{x^3 - 2x} = \sqrt[4]{\lim_{x\to 4}(x^3 - 2x)} \qquad \triangleleft \text{ by Cor 3.10}$$
$$= \sqrt[4]{56}.$$

Make a conscience note of the fact that 56 lies within the domain of the 4^{th}-root function, i.e. $\sqrt[4]{56}$ is defined.

Final Note: Let $F(x) = \sqrt[4]{x^3 - 2x}$. Then this exercise shows us that

$$\lim_{x\to 4} F(x) = F(4).$$

Exercise 3.11. ■

3.12. Reason as follows:

$$\begin{aligned}
\lim_{x\to 1}(2x^2-3x+5)^{3/2} &= \lim_{x\to 1}[(2x^2-3x+5)^3]^{1/2} \\
&= [\lim_{x\to 1}(2x^2-3x+5)^3]^{1/2} && \triangleleft \text{ by } (12) \\
&= [(4)^3]^{1/2} && \triangleleft \text{ by } (8) \\
&= 8.
\end{aligned}$$

Final Note: If we give our function a name like $F(x)=(2x^2-3x+5)^{3/2}$, then we can observe that

$$\lim_{x\to 1} F(x) = F(1)$$

Exercise 3.12. ■

3.13. Simply follow the solution in EXAMPLE 3.13. Here are some details:

Auxiliary Algebraic Calculation:

$$\begin{aligned}\frac{\sqrt{3x+4}-2}{x} &= \frac{\sqrt{3x+4}-2}{x}\,\frac{\sqrt{3x+4}+2}{\sqrt{3x+4}+2}\\ &= \frac{3}{\sqrt{3x+4}+2}\end{aligned}$$

Now for the limit calculations

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{3x+4}-2}{x} &= \lim_{x\to 0}\frac{3}{\sqrt{3x+4}+2} && \triangleleft \text{ Aux. Calc.}\\ &= \frac{3}{4}\end{aligned}$$

Exercise 3.13. ■

3.14. The limit of the numerator and denominator are both zero. The Empirical Observation tells us that there is a common factor.

Preliminary Algebraic Step:

$$\begin{aligned}\frac{\sqrt{x}-2}{x-4} &= \frac{\sqrt{x}-2}{x-4}\,\frac{\sqrt{x}+2}{\sqrt{x}+2} \\ &= \frac{x-4}{(x-4)(\sqrt{x}+2)} \\ &= \frac{1}{\sqrt{x}+2}\end{aligned}$$

Finally,

$$\begin{aligned}\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} &= \lim_{x\to 4}\frac{1}{\sqrt{x}+2} \\ &= \frac{1}{4}\end{aligned}$$

Exercise 3.14. ■

4.1. Proceed as follows.

$$\begin{aligned}\lim_{x\to 0}\cos(x) &= \lim_{x\to 0}\sqrt{1-\sin^2(x)} \\ &= \sqrt{\lim_{x\to 0}(1-\sin^2(x))} && \triangleleft \text{ Lim. Comp.} \\ &= \sqrt{1-0} && \triangleleft \text{ by (4)} \\ &= 1\end{aligned}$$

Where we have applied Corollary 3.10. (The function g is that statement is $g(x) = 1 - \sin^2(x)$. By (4), $\lim_{x\to 0}(1-\sin^2(x)) = 1 - 0 = 1$.)

Exercise 4.1. ■

4.2. I'll sketch some details. Finish it off yourself.

$$\begin{aligned}\frac{1-\cos(x)}{x} &= \frac{1-\cos(x)}{x}\,\frac{1+\cos(x)}{1+\cos(x)}\\ &= \frac{1-\cos^2(x)}{x(1+\cos(x))}\\ &= \frac{\sin^2(x)}{x(1+\cos(x))}\\ &= \frac{\sin(x)}{x}\,\frac{\sin(x)}{1+\cos(x)}.\end{aligned}$$

Now take the limit. "The limit of a product is the product of the limits," etc., etc.

Exercise 4.2. ■

4.3. You should have used the same techniques as the two previous examples.

$$\lim_{x \to 0} \frac{\sin(99x}{x} = 99 \lim_{x \to 0} \frac{\sin(99x)}{99x} = 99.$$

That's not too hard.

Exercise 4.3. ■

4.4. The answer is $\frac{1}{99}$ silly! Of course you know

$$\lim_{x \to 0} \frac{x}{\sin(99x)} = \lim_{x \to 0} \frac{1}{\sin(99x)/x}.$$

The denominator of the latter limit is 99 by the previous exercise.

Exercise 4.4. ■

4.5. You should have proceeded as follows:

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(x)}{x} &= \lim_{x\to 0}\frac{\sin(x)}{x\cos(x)}\\ &= \lim_{x\to 0}\frac{\sin(x)}{x}\,\frac{1}{\cos(x)}\\ &= \lim_{x\to 0}\frac{\sin(x)}{x}\,\lim_{x\to 0}\frac{1}{\cos(x)}\\ &= (1)(1)\end{aligned}$$

The last two limit calculations follow from (7) and (1)

Exercise 4.5. ■

4.6. You should have called on your experience gained form the exercises and examples above.

$$\lim_{x \to 0} \frac{\tan(7x)}{x} = 7 \lim_{x \to 0} \frac{\tan(7x)}{7x} = 7(1) = \boxed{7.}$$

Exercise 4.6. ■

4.7.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(5x)}{\sin(3x)} &= \lim_{x\to 0}\frac{\sin(5x)/x}{\sin(3x)/x}\\ &= \lim_{x\to 0}\frac{5\ \sin(5x)/5x}{3\ \sin(3x)/3x}\\ &= \frac{5}{3}\lim_{x\to 0}\frac{\sin(5x)/5x}{\sin(3x)/3x}\\ &= \boxed{\frac{5}{3}.}\end{aligned}$$

Exercise 4.7. ■

4.8. This is the same as the previous exercise, only you have to be careful with your algebra — to not error!

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin(x/)}{\sin(3x)} &= \lim_{x\to 0}\frac{\sin(x/5)/x}{\sin(3x)/x} \\
&= \lim_{x\to 0}\frac{(1/5)\,\sin(x/5)/(x/5)}{3\,\sin(3x)/3x} \\
&= \frac{1/5}{3}\lim_{x\to 0}\frac{\sin(x/5)/(x/5)}{\sin(3x)/3x} \\
&= \boxed{\frac{1}{15}.}
\end{aligned}$$

See what I mean. You've got to be solid in algebra. Exercise 4.8. ■

4.9. Just go at it the same as you would if it were a sine divided by a sine!

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(8x)}{\tan(4x)} &= \lim_{x\to 0} \frac{\sin(8x)/x}{\sin(4x)/x} \\
&= \lim_{x\to 0} \frac{8\ \sin(8x)/8x}{4\ \sin(4x)/4x} \\
&= \frac{8}{4} \lim_{x\to 0} \frac{\sin(8x)/5x}{\sin(4x)/4x} \\
&= \boxed{2.}
\end{aligned}$$

Exercise 4.9. ■

5.1. We reason as before.

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 6x^2 - 3x + 1 \qquad \triangleleft \text{ since } x < -1$$
$$= 10 \qquad \triangleleft \text{ by (8)}$$

Exercise 5.1. ■

5.2. Merely two problems of the same type.

$$\begin{aligned}
\lim_{x\to -2^-} f(x) &= \lim_{x\to -2^-} (x^2 - 2x) && \triangleleft \text{ since } x < -2 \\
&= 4 && \triangleleft \text{ by (8)}
\end{aligned}$$

and,

$$\begin{aligned}
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} (1 - 2x^3) && \triangleleft \text{ since } -2 < x < 1 \\
&= -1 && \triangleleft \text{ by (8)}
\end{aligned}$$

For this last evaluation, $x < 1$ and close to 1. Well, if x is close to 1, it will eventually be larger than -2. So for the purposes of the evaluation of the function, we can assume that $-2 < x < 1$, hence for that case $f(x) = 1 - 2x^3$. See how it work? Exercise 5.2. ■

5.3. The first thing you have to remember is the meaning of the symbols, in particular,

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

I discern two cases for analysis. This case study is necessary to get rid of the absolute value.

Case 1: $a \leq 0$. Now, if x is close to a and $x < a$, then because $a \leq 0$ we conclude that $x < 0$ too. Thus,

$$|a| = -a \qquad |x| = -x.$$

Therefore,

$$\begin{aligned} \lim_{x \to a^-} |x| &= \lim_{x \to a^-} (-x) && \triangleleft \text{ since } x < 0 \\ &= -a && \triangleleft \text{ by (8)} \\ &= |a|. && \triangleleft \text{ since } a \leq 0 \end{aligned}$$

Case 2: $a > 0$. Left to the reader — that's you!

Exercise 5.3. ■

5.4. Let's summarize the limiting properties of numerator and denominator:

$$\lim_{x \to -1^-} (x^2 - x - 3) = -3$$
$$\lim_{x \to -1^-} (1 + x) = 0$$

This indicates an infinite limit: Numerator having a finite *nonzero* limit, and denominator a *zero* limit. The only question: Is the limit positive or negative infinity?

Sign Analysis: When $x < -1$ and close to -1, the numerator is close to -3 — so the numerator must be negative: $x^2 - x - 3 < 0$. When $x < -1$ and close to -1, the denominator is $x + 1 < 0$ (Note: $x < -1 \implies x + 1 < 0$). Thus,

$$x^2 - x - 3 < 0, \text{ and } x + 1 < 0 \implies \frac{x^2 - x - 3}{x + 1} > 0.$$

Conclusion: Thus, $\boxed{\lim_{x \to -1^-} \frac{x^2 - x - 3}{x + 1} = +\infty.}$

Graphically Speaking: This function has a vertical asymptote that *zips* off to positive infinity as x approaches -1 from the left.

Exercise 5.4. ■

5.5. The function under consideration is $f(x) = \dfrac{1}{\sqrt{1-2x}}$. Its domain is

$$\begin{aligned}\operatorname{Dom}(f) &= \{\, x \in \mathbb{R} \mid 1 - 2x > 0 \,\} \\ &= \{\, x \in \mathbb{R} \mid x < 1/2 \,\} \\ &= (\, -\infty, \frac{1}{2} \,)\end{aligned}$$

As you can see, it doesn't make sense to consider a two-sided limit for this problem at $x = 0$. This is one of the utilities of the one-sided limit concept — to be able to investigate the behavior of a function at a boundary point.

Obviously,

$$\begin{aligned}\lim_{x\to 1/2^-} \sqrt{1-2x} &= \sqrt{\lim_{x\to 1/2^-} (1-2x)} \\ &= 0.\end{aligned}$$

Thus the limit of the denominator is 0. The numerator is positive and the denominator is positive; therefore,

$$\boxed{\lim_{x \to 1/2^-} \frac{1}{\sqrt{1-2x}} = +\infty.}$$

Exercise 5.5. ■

5.6. Same as previous exercises:

$$\lim_{x \to 2^-} (x - 1) = 2 \qquad \triangleleft \text{ the numerator}$$

$$\lim_{x \to 2^-} (6 - x - x^2) = 0 \qquad \triangleleft \text{ the denominator}$$

The numerator is positive and approaching 2; the denominator is of unknown sign and approaching 0. One way of determining the sign of the denominator is to take a *test point*: Choose $x < 2$ and close to 2, say $x = 1.9$; the value of the denominator is then $6-(1.9)-(1.9)^2 > 0$.

Therefore, when $x < 2$ and close to 2,

$$\frac{x - 1}{6 - x - x^2} > 0.$$

Conclusion: $\boxed{\lim_{x \to 2^-} \frac{x - 1}{6 - x - x^2} = +\infty.}$

Graphically Speaking: This function has a vertical asymptote that *zips* off to positive infinity as x approaches 2 from the left.

Exercise 5.6. ■

5.7. We reason as before.

$$\begin{aligned} \lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} 3 - 3x^2 - 2x^3 && \triangleleft \text{ since } x > -1 \\ &= 2 && \triangleleft \text{ by } (8) \end{aligned}$$

Exercise 5.7. ■

5.8. Merely two problems of the same type.

$$\begin{aligned} \lim_{x \to -2^+} f(x) &= \lim_{x \to -2^+} 1 - 2x^3 && \triangleleft \text{ since } -2 < x < -1 \\ &= -15 && \triangleleft \text{ by } (8) \end{aligned}$$

For this last evaluation, $x > -2$ and close to -2. Well, if x is close to -2, it will eventually be smaller than -1. So for the purposes of the evaluation of the function, we can assume that $-2 < x < -1$, hence for that case $f(x) = 1 - 2x^3$.

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} 7x - 1 && \triangleleft \text{ since } x > 1 \\ &= 6 && \triangleleft \text{ by } (8) \end{aligned}$$

Exercise 5.8. ■

5.9. The first thing you have to remember is the meaning of the symbols, in particular,

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

I discern two cases for analysis. This case study is necessary to get rid of the absolute value.

Case 1: $a \geq 0$. Now, if x is close to a and $x > a$, then because $a \geq 0$ we conclude that $x > 0$ too. Thus,

$$|a| = a \qquad |x| = x.$$

Therefore,

$$\begin{aligned} \lim_{x \to a^+} |x| &= \lim_{x \to a^+} x && \triangleleft \text{ since } x > 0 \\ &= a && \triangleleft \text{ by (8)} \\ &= |a|. && \triangleleft \text{ since } a \geq 0 \end{aligned}$$

Case 2: $a < 0$. Left to the reader — that's you!

Exercise 5.9. ■

5.10. From earlier exercises,

$$\lim_{x\to -1^-} f(x) = 10 \neq 2 = \lim_{x\to -1^+} f(x).$$

Now call for Theorem 5.1. Exercise 5.10. ■

5.11. From earlier exercises,

$$\lim_{x\to -2^-} f(x) = 4 \neq -15 = \lim_{x\to -2^+} f(x),$$
$$\lim_{x\to 1^-} f(x) = -1 \neq 6 = \lim_{x\to 1^+} f(x).$$

Now call for Theorem 5.1. Exercise 5.11. ■

5.12. You should have begun your analysis by computing the limit of the numerator and denominator separately.

$$\lim_{x \to 3^+} (x - 1) = 2 \qquad \triangleleft \text{ the numerator}$$

$$\lim_{x \to 3^+} (x - 2)(x - 3) = 0 \qquad \triangleleft \text{ the denominator}$$

Sign Analysis: The numerator is positive when $x < 3$ and close to 3, since the limit of the numerator is $2 > 0$. The denominator is negative. We assume $x < 3$ and so close to 3 that $x > 2$. Thus, for the purpose of the sign analysis, we take $2 < x < 3$. The denominator then is

$$(x - 2)(x - 3) : (+)(-) = (-)$$

Conclusion: $\boxed{\displaystyle \lim_{x \to 3^+} \frac{x - 1}{(x - 2)(x - 3)} = -\infty.}$

Graphically Speaking: This function has a vertical asymptote to the right of $x = 3$ that plunges off to $-\infty$. Exercise 5.12. ■

5.13. Here, we simply invoke our (your?) memories. Put the following three facts together in an organized way to prove the result: Theorem 5.1, Exercise 5.3, and Exercise 5.9. Good Luck!

Exercise 5.13. ■

5.14. From Theorem 5.2), we have

$$\lim_{x \to a} |x| = |a| \qquad a \in \mathbb{R}.$$

The function $F(x) = |2x^2 + 3x - 5|$ is the composition of two functions: $f(x) = |x|$ and $g(x) = 2x^2 + 3x - 5$ such that $F = f \circ g$. Therefore, by Theorem 3.8, the following manipulation is legal:

$$\begin{aligned} \lim_{x \to -1} |2x^2 + 3x - 5| &= |\lim_{x \to -1} (2x^2 + 3x - 5)| \\ &= |-7| \\ &= 7 \end{aligned}$$

Exercise 5.14. ∎

5.15. Invoking the *continuity of absolute value*, and Theorem 3.8, we reason as follows:

$$\begin{aligned}\lim_{t\to4}\left|\frac{t-6}{\sqrt{2t+1}}\right| &= \left|\lim_{t\to4}\frac{t-6}{\sqrt{2t+1}}\right| \\ &= \left|\frac{4-6}{\sqrt{9}}\right| \\ &= \frac{2}{3}\end{aligned}$$

Exercise 5.15. ■

6.1. I address only the first. The phrase "$f(x)$ gets larger and larger" has an ambiguous meaning without the phrase "without bound." For example, for the function $f(x) = 1 - x^2$, as x gets closer and closer to 0, $f(x)$ gets larger and larger, in fact, $\lim_{x\to 0} f(x) = 1$. But this is not the sense that we truly mean when we say "f(x) gets larger and larger." In my little example, $f(x) = 1 - x^2$, even though $f(x)$ gets larger and larger, as x approaches 0, the values of f are not getting arbitrary large — they are bounded, i.e. $f(x) = 1 - x^2 \leq 1$. Here, we say the function f is bounded by 1 as x goes to 0.

When we say "$f(x)$ gets larger and larger without bound" we mean that $f(x)$ is getting larger and larger in such a way that there is no number M such that $f(x) \leq M$. That is, the values of f are not kept from "going off to infinity."

This concept can and will be defined later. Exercise 6.1. ■

Solutions to Examples

3.1. We solve this problem by building on the solution of each previous problem — a standard technique in mathematics.

$$\begin{aligned} \lim_{x\to 3} x^2 &= \lim_{x\to 3}(xx) \\ &= (\lim_{x\to 3} x)(\lim_{x\to 3} x) && \triangleleft \text{ by (4)} \\ &= (3)(3) && \triangleleft \text{ by Rule 2} \\ &= (3)^2. \end{aligned} \tag{S-1}$$

Next,

$$\begin{aligned} \lim_{x\to 3} x^3 &= \lim_{x\to 3}(xx^2) \\ &= (\lim_{x\to 3} x)(\lim_{x\to 3} x^2) && \triangleleft \text{ by (4)} \\ &= (3)(3)^2 && \triangleleft \text{ by Rule 2 and (S-1)} \\ &= (3)^3. \end{aligned} \tag{S-2}$$

Next, again,

$$\begin{aligned}\lim_{x\to 3} x^4 &= \lim_{x\to 3}(xx^3) \\ &= (\lim_{x\to 3} x)(\lim_{x\to 3} x^3) && \triangleleft \text{ by (4)} \\ &= (3)(3)^3 && \triangleleft \text{ by Rule 2 and (S-2)} \\ &= (3)^4. && \text{(S-3)}\end{aligned}$$

and finally,

$$\begin{aligned}\lim_{x\to 3} x^5 &= \lim_{x\to 3}(xx^4) \\ &= (\lim_{x\to 3} x)(\lim_{x\to 3} x^4) && \triangleleft \text{ by (4)} \\ &= (3)(3)^4 && \triangleleft \text{ by Rule 2 and (S-3)} \\ &= (3)^5. && \text{(S-4)}\end{aligned}$$

Summary of Results: $\lim_{x\to 3} x^n = 3^n$, for $n = 1, 2, 3, 4,$ and 5.

Example 3.1. ■

3.2. If we want to be logical and organized, we proceed by a series of steps.

$$\begin{aligned}
\lim_{x\to 2}(3x^2 - 2x + 1) &= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 2x + \lim_{x\to 2} 1 && \triangleleft \text{ by (2)} \\
&= 3\lim_{x\to 2} x^2 - 2\lim_{x\to 2} x + 1 && \triangleleft \text{ by (3) and Rule 1} \\
&= 3(2^2) - 2(2) + 1 && \triangleleft \text{ by (6) and Rule 2} \\
&= 9
\end{aligned}$$

Thus, $\boxed{\lim_{x\to 2}(3x^2 - 2x + 1) = 9.}$

Study this solution method. Take note of the use of the correct notation. As you do problems at home, use the proper notation! You should strive for understanding of the concepts, but also you should strive to become literate mathematically: That means you need to be able to read, write, and speak mathematics in a correct way.

Another point to be made in this example is the following one. Define $f(x) = 3x^2 - 2x + 1$. Our problem was to compute $\lim_{x\to 2} f(x)$, and the answer was $\lim_{x \to 2} f(x) = 9$. You probably have already observed that $9 = f(2)$. Thus, *for this example*,

$$\lim_{x \to 2} f(x) = f(2).$$

Hummmmm ... interesting. f is continuous at $x = 2$.

Example 3.2. ■

3.3. This is now the height of triviality:

$$\lim_{x\to -2}(\frac{1}{2}x^4 - \frac{2}{3}x^3 - \frac{4}{3}) = (\frac{1}{2}(-2)^4 - \frac{2}{3}(-2)^3 - \frac{4}{3}) = 12$$

$$\lim_{w\to 4}(2w^2 - 6w + 1) = 2(4)^2 - 6(4) + 1 = 9$$

Example 3.3. ■

3.4. The first thing your eyes should see is that this is a limit of a quotient of two functions. By (5), *the limit of a quotient is the quotient of the limits, provided the limit of the denominator is nonzero* — this is the basic content of (5). Therefore,

$$\begin{aligned}
\lim_{x\to -1} \frac{x^2 - 3x + 2}{2x^3 + x - 4} &= \frac{\lim\limits_{x\to -1}(x^2 - 3x + 2)}{\lim\limits_{x\to -1}(2x^3 + x - 4)} && \triangleleft \text{ by (5)} \\
&= \frac{(-1)^2 - 3(-1) + 2}{2(-1)^3 + (-1) - 4} && \triangleleft \text{ by (8)} \\
&= -\frac{6}{7}.
\end{aligned}$$

The limit of the denominator was

$$\lim_{x\to -1}(2x^3 + x - 4) = -7 \neq 0;$$

consequently, the invocation of (5) was indeed justified.

If we give the function a name

$$f(x) = \frac{x^2 - 3x + 2}{2x^3 + x - 4},$$

then we can summarize the limit calculation of this example as

$$\lim_{x \to -1} f(x) = f(-1),$$

since, as you can verify, $f(-1) = -6/7$.

Example 3.4. ■

3.5. The problem is

$$\lim_{x\to 2} \frac{x^3 - x^2 - 4x + 3}{x^2 + 3x - 6}.$$

This is the limit of a quotient. Thus

$$\begin{aligned}
\lim_{x\to 2} \frac{x^3 - x^2 - 4x + 3}{x^2 + 3x - 6} &= \frac{\lim\limits_{x\to 2}(x^3 - x^2 - 4x + 3)}{\lim\limits_{x\to 2}(x^2 + 3x - 6)} && \triangleleft \text{ by } (5) \\
&= \frac{2^3 - 2^2 - 4(2) + 3}{2^2 + 3(2) - 6} && \triangleleft \text{ by } (8) \\
&= -\frac{1}{4}.
\end{aligned}$$

Again take note of the observation:

$$\lim_{x\to 2} f(x) = f(2), \qquad f(x) = \frac{x^3 - x^2 - 4x + 3}{x^2 + 3x - 6}.$$

Thus, this function f is continuous at $x = 2$. Example 3.5. ■

3.6. The theorem on Continuity of Rational Functions streamlines the process considerably.

$$\lim_{x\to 2} \frac{5x^2+8}{3x+1} = \frac{5(2)^2+8}{3(2)+1} = \frac{28}{7} = 4.$$

Example 3.6. ■

3.7.

We have the limit of the quotient of two functions. From (5), *the limit of a quotient is the quotient of the limits, provided the limit of the denominator is nonzero.* The limit of the denominator

$$\lim_{x\to 2}(x^2 - x - 2) = 2^2 - 2 - 2 = 0.$$

Hence, Skill Level 1! Notice also, that the limit of the numerator is

$$\lim_{x\to 2}(x - 2) = 0.$$

A double whammy. We have the following situation: When $x \approx 2$, $(x-2)/(x^2 - x - 2)$ is the ratio of two small numbers. But all is not lost:

$$\begin{aligned}\lim_{x\to 2}\frac{x-2}{x^2-x-2} &= \lim_{x\to 2}\frac{x-2}{(x+1)(x-2)}\\ &= \lim_{x\to 2}\frac{1}{x+1}\\ &= \frac{1}{3}.\end{aligned}$$

You did see the common factor, didn't you?

Thus,

$$\boxed{\lim_{x \to 2} \frac{x-2}{x^2-x-2} = \frac{1}{3}.}$$

Example 3.7. ■

3.8. Again, the limit of the denominator is zero. What saves us is the fact that the limit of the numerator is zero too.

$$\lim_{x \to -1} (x^2 + 4x + 3) = (-1)^2 + 4(-1) + 3 = 0$$
$$\lim_{x \to -1} (2x^2 + x - 1) = 2(-1)^2 + (-1) - 1 = 0$$

As in EXAMPLE 3.7, the numerator and denominator share a common factor:

$$\begin{aligned} \lim_{x \to -1} \frac{x^2 + 4x + 3}{2x^2 + x - 1} &= \lim_{x \to -1} \frac{(x+1)(x+3)}{(x+1)(2x-1)} \\ &= \lim_{x \to -1} \frac{x+3}{2x-1} \\ &= \frac{2}{-3} \\ &= -\frac{2}{3} \end{aligned}$$

Example 3.8. ■

3.9. In one sense, this is an easy problem. We are taking the limit of a polynomial; the techniques in the section on the Algebra of Limits apply. In particular, (8) applies, or, for that matter, (9) does too. That having been said, we want to analyze this function as a composite. The function is

$$F(x) = (x^2 + 1)^{23}.$$

Now F is the composition of the functions:

$$f(x) = x^{23}$$

and,

$$g(x) = x^2 + 1$$

These definitions having been made, we obviously have

$$F(x) = f(g(x)).$$

The advantage of this point of view is that we have a "complicated function," that's F, that has been de-composed into two "simple functions," those are f and g.

From our previous work, when

$$x \approx 3, \text{ implies } g(x) \approx g(3) = 10$$

since,

$$\lim_{x \to 3} g(x) = g(3). \qquad \triangleleft \text{ By (8).}$$

But,

$$g(x) \approx 10, \text{ implies } f(g(x)) \approx f(10) = 10^{23}$$

since,

$$\lim_{x \to 10} f(x) = f(10). \qquad \triangleleft \text{ By (8).}$$

Thus,

$$x \approx 3, \text{ implies } f(g(x)) \approx 10^{23}.$$

That was a bit awkward. Basically what I am saying is when x is "close" to its limit of 3, $g(x)$ will be "close" to its limit of $g(3) = 10$. But f, in turn takes in the values of g, and these values of g are close to 10, so f operating on those values of g, i.e. $f(g(x))$, will be close to its limit of $f(10)$.

This is a heuristic way of calculating the limit of a composite function without the support of basic theory. Below you can find a statement of the theorem we need. The above reasoning is useful to get a “feel” for what is going on.

Example 3.9. ■

3.10. The function is $F(x) = \sqrt{x^2 + 3x}$ is a composition of two other functions: the outer function, $f(x) = \sqrt{x}$, and the inner function $g(x) = x^2 + 3x$. (Verify: $F = f \circ g$.) The essence of the Theorem 3.8 is that you can take the limit operation inside the outer function:

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

Do you see how the limit is moved to the inside (or to the right of) the function f? Now let's apply Theorem 3.8 to this problem:

$$\lim_{x \to 2} \sqrt{x^2 + 3x} = \sqrt{\lim_{x \to 2} (x^2 + 3x)},$$

provided $\lim_{x \to 2} x^2 + 3x$ exists (it does here), and the limiting value belongs to the domain of the outer function. In this case, the limit value is $\lim_{x \to 2} x^2 + 3x = 10$; and 10, in turn, being nonnegative falls within the domain of the outer function, the square root function.

Now that we have run through in our minds the justification for the steps we can finish up.

$$\begin{aligned}\lim_{x\to 2}\sqrt{x^2+3x} &= \sqrt{\lim_{x\to 2} x^2+3x},\\ &= \sqrt{10}.\end{aligned}$$

Thus,

$$\boxed{\lim_{x\to 2}\sqrt{x^2+3x} = \sqrt{10}.}$$

Final Note: Observe $\lim_{x\to 2} F(x) = F(2)$, where F is defined above as $F(x) = \sqrt{x^2+3x}$.

Example 3.10. ■

3.11. The function we are working with is $F(t) = (t^3 - 5t + 1)^{2/3}$. This function is the composition of two: the outer function, $f(t) = t^{2/3}$, and the inner function, $g(t) = t^3 - 5t + 1$. (Verify that $F = f \circ g$.)

$$\begin{aligned} \lim_{t \to -1} (t^3 - 5t + 1)^{2/3} &= \big(\lim_{t \to -1} (t^3 - 5t + 1)\big)^{2/3} \\ &= 5^{2/3}. \end{aligned}$$

Thus,

$$\boxed{\lim_{t \to -1} (t^3 - 5t + 1)^{2/3} = 5^{2/3}.}$$

The above method is the usual method utilized by students at this level of play; however, technically, we need to check on the conditions of Theorem 3.8. Take a quick look at the theorem. The first condition is O.K. since $g(t) = t^3 - 5t + 1$ and $\lim_{t \to -1} g(t) = 5$. The second condition requires the limit of the inner function be in the domain of the outer function: The number 5 is indeed inside the domain of

$f(t) = t^{2/3}$. Finally, the second condition requires that $\lim_{t\to 5} f(t) = f(5)$. Let's argue that.

$$\begin{aligned}
\lim_{t\to 5} f(t) &= \lim_{t\to 5} t^{2/3} \\
&= \lim_{t\to 5} t^{1/3}\, t^{1/3} \\
&= (\lim_{t\to 5} t^{1/3})(\lim_{t\to 5} t^{1/3}) && \triangleleft \text{ by } (4) \\
&= 5^{1/3}\, 5^{1/3} && \triangleleft \text{ by } (11) \\
&= 5^{2/3} \\
&= f(3)
\end{aligned}$$

Do you see how we use the theorems and techniques to investigate a mathematical point of interest?

Final Note: Note that $\lim_{t\to -1} F(t) = F(-1)$, where, F is defined above as $F(t) = (t^3 - 5t + 1)^{2/3}$.

Example 3.11. ■

3.12. We are still at Skill Level 0, the function involved here is an algebraic function. The limiting point is 4 (we are taking the limit as x goes to 4) is in the domain of the function (how do I know that?). Thus,

$$\begin{aligned}\lim_{x \to 4} \frac{\sqrt[3]{x^2 - 3x - 3}}{\sqrt{x} + \sqrt[3]{2x}} &= \frac{\sqrt[3]{4^2 - 3(4) - 3}}{\sqrt{4} + \sqrt[3]{2(4)}} && \triangleleft \text{ by Theorem 3.11} \\ &= \frac{1}{4}\end{aligned}$$

Note: The reason that I knew that 4 was in the domain of the function is that (1) for $x = 4$, all the roots could be computed and (2) the denominator did not evaluate to *zero* when I put $x = 4$. In fact, the domain of this function is $(\,0, \infty\,)$.

Example 3.12. ■

3.13. There are some tricky bits here. Notice that the numerator and denominator both go to zero. My Empirical Observation is still valid, we just have to work a little harder (Skill Level 1.3). To reveal the common factor, we use the *conjugate trick* from algebra.

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{x^2+1}-1}{x^2} &= \lim_{x \to 0} \frac{\sqrt{x^2+1}-1}{x^2} \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1} && \triangleleft \text{ conjugate} \\
&= \lim_{x \to 0} \frac{(x^2+1)-1}{x^2(\sqrt{x^2+1}+1)} \\
&= \lim_{x \to 0} \frac{x^2}{x^2(\sqrt{x^2+1}+1)} \\
&= \lim_{x \to 0} \frac{1}{\sqrt{x^2+1}+1} \\
&= \frac{1}{2}
\end{aligned}$$

Thus,

$$\boxed{\lim_{x \to 0} \frac{\sqrt{x^2+1}-1}{x^2} = \frac{1}{2}.}$$

Example 3.13. ■

3.14. If you were to examine the graph of the function $\sin(\frac{1}{x})$, you will see that is oscillates "wildly" as you plot the graph closer and closer to $x = 0$. Multiplication of $\sin(\frac{1}{x})$ tends to dampen this oscillation down. However how do we make a formal argument?

In the SQUEEZE METHOD, the function $f(x) = x^2 \sin(\frac{1}{x})$. This is our "target function." We want to squeeze f between two other functions (yet to be determined), g and h, such that both functions g and h go to zero. How do we create these two functions? Typically by *trial and error*!

Observe that

$$-1 \leq \sin(\tfrac{1}{x}) \leq 1 \qquad \forall x \neq 0.$$

Therefore, for any $x \neq 0$, if we multiply all sides of this double inequality relation by the nonnegative number x^2 we obtain

$$-x^2 \leq x^2 \sin(\tfrac{1}{x}) \leq x^2 \tag{S-5}$$

Think of the functions g and h to be

$$g(x) = -x^2 \qquad h(x) = x^2.$$

With this choice of function, (S-5) then states

$$g(x) \le f(x) \le h(x) \qquad \forall x.$$

We have then condition (1) ((13), i.e., we have squeeze f between two much simpler function g and h.

Observe also that

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}(-x^2) = 0$$
$$\lim_{x\to 0} h(x) = \lim_{x\to 0} x^2 = 0.$$

Thus,

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = 0.$$

This last equation is condition (2) of the Squeeze Method, equation (14).

We have satisfied conditions (1) and (2) of the Squeeze Method and are therefore entitled to make the conclusion,

$$\lim_{x\to 0} x^2 \sin(\tfrac{1}{x}) = 0.$$

Example Notes: Note that we were able to "get rid" of the most complex component, $\sin(\frac{1}{x})$, of the function f without "changing" the limit. This is quite often the goal of the manipulations: to eliminate the more complex components of the target function by *bounding* it by other, more easily manageable functions.

■ The meaning of the phrase "complex components" has variable meaning. Complex component may mean a ugly algebraic expression, or it may mean a "simple" expression whose limit is difficult to discern or does not exist.

■ In the case of this problem, the function $\sin(\frac{1}{x})$ is a simple function whose limit, as x approaches zero, does not exist. That being the case, it is difficult to argue rigorously that the limit is zero—we must eliminate it. The device used is the SQUEEZE METHOD. ■

Example 3.14. ■

4.1. The following tables need to be studied.

$y = \sin(x)/x$							
x	1.0	0.5	0.1	0.05	0.01	0.005	0.001
y	0.8415	0.9588	0.9983	0.9996	0.99998	0.99999	0.999999

$y = (\cos(x) - 1)/x$							
x	1.0	0.5	0.1	0.05	0.01	0.005	0.001
y	-0.4597	-0.2448	-0.04996	-0.02499	-0.00499	-0.0025	-0.0005

Notes: It appears $\sin(x)/x$ approaches 1 as x goes to 0; and it appears, though not as strongly, that $(\cos(x)-1)/x$ tends to 0 as x tends to 0.

- All entries on the second table are negative since $\cos(x) \leq 1$.

Example 4.1. ■

4.2. In (7), we have developed the basic limit result:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

A rough translation of this would is as follows: When you take the sine of a number getting closer and closer to 0, and divide by that exact same number, then the ratio of the two tend to 1. For example, (7) can be construed as saying

$$\lim_{x \to 0} \frac{\sin(4x)}{4x} = 1,$$

since we are dividing by the same quantity we are taking the sine of in the numerator, and ax x tends to 0, so does $4x$ as well.

With that discussion as background, let us make the required calculation:

$$\lim_{x \to 0} \frac{\sin(4x)}{x} = 4 \lim_{x \to 0} \frac{\sin(4x)}{4x} = 4(1) = \boxed{4,}$$

where we have multiplied and divided by 4 so that the argument of the sine function in the numerator exactly matched the denominator expression. This newly constructed limit is 1, by the discussion above; therefore, the limit of the original expression is 4.

Example 4.2. ■

4.3. As in the previous example, we insert a "fudge factor" so that the denominator matches the argument of the sine function.

$$\lim_{x \to 0} \frac{\sin(x/5)}{x} = (1/5) \lim_{x \to 0} \frac{\sin(x/5)}{x/5}$$

$$= \boxed{\frac{1}{5}.} \qquad \triangleleft \text{ by } (7)$$

See how that works?

Example 4.3. ■

4.4. The problem

$$\lim_{x \to 0} \frac{\sin(4x)}{\sin(6x)}$$

is the limit of a ratio of two sine functions both of which go to zero as x tends to zero, by (1). Consequently, we cannot see what the value of the limit is. We need a device that will reveal it for us.

Here is the trivially tricky trick: We observe that

$$\frac{\sin(4x)}{\sin(6x)} = \frac{\sin(4x)/x}{\sin(6x)/x}. \tag{S-6}$$

In other words, we divide the numerator and denominator by x. What does this get us? It gets us back to familiar territory, partner! We have seen that the ratios such as the ones in the numerator and the denominator in (S-6) have limits we can compute.

Therefore,

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(4x)}{\sin(6x)} &= \lim_{x\to 0}\frac{\sin(4x)/x}{\sin(6x)/x}\\ &= \lim_{x\to 0}\frac{4\ \sin(4x)/4x}{6\ \sin(6x)/6x}\\ &= \frac{4}{6}\lim_{x\to 0}\frac{\sin(4x)/4x}{\sin(6x)/6x}\\ &= \boxed{\frac{2}{3}.}\end{aligned}$$

Thus,

$$\lim_{x\to 0}\frac{\sin(4x)}{\sin(6x)} = \frac{2}{3}$$

Example 4.4. ■

5.1. The one-sided limit concept provides an important tool for working with piecewise defined functions. We reason as follows:

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} x^2 && \triangleleft \text{ since } x < 2 \\ &= 4 && \triangleleft \text{ by (8)} \end{aligned}$$

Because f is a piecewise defined function, f is a little more difficult to work with. The one-sided limit is very useful in this regard. Note that we simplify the limit problem by making the use of the assumption that $x < 2$ in this limit problem. Once we do this, we have a more conventional limit problem to evaluate. Example 5.1. ■

5.2. This is the limit of a quotient of two expressions. The limit of the numerator is $\lim_{x \to 1^-} x = 1 \ldots$ that's good; whereas the limit of the denominator is $\lim_{x \to 1^-}(x - 1) = 0 \ldots$ that's bad.

At this point it is important to do a *sign analysis* of the expression. When x is close to 1, and $x < 1$, we know that $x > 0$ (that's the numerator), and $x - 1 < 0$ (that's the denominator). Thus, when $x < 1$ and close to 1, the expression is $x/(1 - x) < 0$, is negative.

Therefore, when $x < 1$, and x is close to 1, $x/(1 - x)$ is a negative quantity, the numerator is close to 1, but the denominator is close to 0. We see in this way that $x/(1-x)$ is extremely small in the negative direction. The closer x gets to 1, with $x < 1$, the smaller $x/(1 - x)$ is *without bound.*

Conclusion: $\displaystyle \lim_{x \to 1^-} \frac{x}{1 - x} = -\infty.$

Graphically Speaking: This means that the graph of the function has a vertical asymptote at $x = 1$. As you approach 1 from the left-hand side, the graph plunges to $-\infty$.

Example 5.2. ■

5.3. The one-sided limit concept provides an important tool for working with piecewise defined functions. We reason as follows:

$$\begin{aligned} \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} x^3 && \triangleleft \text{ since } x > 2 \\ &= 8 && \triangleleft \text{ by (8)} \end{aligned}$$

Note that we simplify the limit problem by making the use of the assumption that $x > 2$ in this limit problem. Once we do this, we have a more conventional limit problem to evaluate. Example 5.3. ■

5.4. From Example 5.1 we have

$$\lim_{x \to 2^-} f(x) = 4,$$

and by Example 5.3 we have

$$\lim_{x \to 2^+} f(x) = 8.$$

Thus, the left-hand limit is different from the right-hand limit; therefore, by Theorem 5.1, the two-sided limit does not exist, i.e.

$$\lim_{x \to 2^+} f(x) \quad \text{d.n.e.}$$

That is very mechanical. Example 5.4. ■

6.1. We begin by using the methods of the previous section, in particular, we will try the technique of showing the nonexistence of limits. Choose a sequence of x's approaching 0.

$$a_n = \frac{1}{n}, \qquad n = 1, 2, 3, 4, \ldots .$$

Note that when n gets larger and larger, a_n gets closer and closer to 0. Now if we define $f(x) = 1/x^2$, we can now examine $f(a_n)$.

$$\begin{aligned} f(a_n) &= \frac{1}{a_n^2} & \text{but } a_n = \frac{1}{n} \\ &= \frac{1}{(1/n)^2} \\ &= n^2 & n = 1, 2, 3, 4, \ldots . \end{aligned}$$

Now, as n gets larger and larger, a_n gets closer and closer to 0, and $f(a_n) = n^2$ gets larger and larger without bound. That is, $f(a_n)$ is not even approaching a number. For this reason, we deduce that the limit does not exist (as a finite number).

However, unlike the previous examples seen in these notes, we cannot assign a numerical value to the limit problem; in this case we write

$$\lim_{x \to 0} \frac{1}{x^2} = +\infty.$$

A study of the graph of $f(x) = 1/x^2$, indicates the situation: The function f has a *vertical asymptote* at $x = 0$ that "zips" off to $+\infty$ — like every good Akron U. student. Example 6.1. ■

7.1. Can you imagine what the graph of this function looks like? For reference let $f(x) = \sin(\frac{1}{x})$.

Case 1: We begin by considering some x's getting closer and closer to 0, and study what the corresponding values of $f(x)$ do. The x's I want to choose are numbers of the form:

$$a_n = \frac{1}{2\pi n}, \qquad n = 1, 2, 3, 4, \ldots$$

Note that these values of x are getting closer and closer to 0 as you consider larger and larger values of n. Now, substituting these x's into the function we get

$$\begin{aligned} f(a_n) &= \sin(\frac{1}{a_n}) \qquad && n = 1, 2, 3, 4 \ldots \\ &= \sin(2\pi n) \qquad && n = 1, 2, 3, 4 \ldots \\ &= 0 \qquad && n = 1, 2, 3, 4 \ldots \end{aligned}$$

Thus, $f(a_n) = 0$, for $n = 1,\ 2,\ 3,\ 4,\ \ldots$. Thus as $x = a_n$ gets closer and closer to 0, $f(a_n)$ is getting closer and closer to 0 too — if fact, $f(a_n)$ is constantly 0.

Conclusion Case 1: It *appears* that the limit is 0.

Case 2: Now choose different x's getting closer and closer to 0, but this time with functional values having a different limit. Define,

$$b_n = \frac{2}{(2n-1)\pi}, \qquad n = 1, 2, 3, 4, \ldots$$

Notice again that $x = b_n$ is getting closer and closer to 0 as n gets larger and larger. Finally,

$$\begin{aligned} f(b_n) &= \sin(\frac{1}{b_n}) && n = 1, 2, 3, 4 \ldots \\ &= \sin(\frac{(2n-1)\pi}{2}) && n = 1, 2, 3, 4 \ldots \\ &= 1 && n = 1, 2, 3, 4 \ldots \end{aligned}$$

Thus, $f(b_n) = 1$, for $n = 1, 2, 3,4, \ldots$. Thus as $x = b_n$ gets closer and closer to 0, $f(b_n)$ is getting closer and closer to 1 — if fact $f(b_n)$ is constantly b.

Conclusion Case 2: It *appears* that the limit is 1.

Discussion: When we look at a one particular sequence of x's, the numbers a_n, the function appears to approach 0; however, if we look and another sequence of x's, the numbers b_n, the function appears to approach 1. From this we deduce that the *limit does not exist*! This is because *if the limit did exist*, it would be impossible for the function to be approaching *two* different values — that would deny the *uniqueness of the limit.*

Conclusion: $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist.

The Geometry: The function $f(x) = \sin(\frac{1}{x})$ crosses the x-axis infinitely many times close to $x = 0$. The numbers a_n are the x-intercepts of the function f. As you know, the sine function has a series of high points. This function f has infinitely many high points closer to 0.

The numbers b_n were chosen to be the x-coordinates of these high points.

Graphing the function often helps you to choose the *two* sets of x's that have different limiting characteristics.

Student Recommendation: Study how this argument goes, the reasoning and the notation.

Example 7.1. ■

7.2. This is a piecewise defined function. The function f equals one expression on the left side of $x = 2$, and another expression on the other side of $x = 2$. We argue that the limit does not exist by choosing a set of x's on each side.

Case Study 1: Choose a sequence of x's smaller than 3 yet getting closer and closer to 2. How can I use good mathematical notation to express this idea? Define a_n, a subscripting notation is used because the value of a will depend on n, by

$$a_n = 2 - \frac{1}{n}, \qquad n = 1, 2, 3, 4, \dots .$$

Note that a_n is getting closer and closer to 2 as n gets larger and larger. Now take our a_n and find their functional values.

$$\begin{aligned} f(a_n) &= a_n && \triangleleft \text{ since } a_n < 2 \\ &= 2 - \frac{1}{n} \end{aligned}$$

The value of n controls which a_n we are looking at. To make a_n get closer and closer to 2, we must let n get larger and larger. Now then n is very large, $a_n \approx 2$, and $f(a_n) \approx 2$, as well.

Conclusion Case 1: Thus, f seems to be approaching 2 as x approaches 2.

Case Study 2: Now take a sequence of x's to the right of $x = 2$ but getting closer and closer to 2, say

$$b_n = 2 + \frac{1}{n}, \qquad n = 1, 2, 3, 4, \dots .$$

Observe that b_n approaches 2, as n gets large. Then,

$$\begin{aligned} f(b_n) &= b_n^2 && \triangleleft \text{ since } b_n > 2 \\ &= (2 + \frac{1}{n})^2 \\ &= 4 + \frac{2}{n} + \frac{1}{n^2} \end{aligned}$$

Now when n is large, $b_n \approx 2$, and $f(b_n) \approx 4$.

Conclusion Case 2: f seems to be approaching 4 as x approaches 2.

Final Conclusion: The limit does not exist because the function appears to be approaching values of 2 *and* 4 as x approaches 2.

Example 7.2. ■