THE UNIVERSITY OF AKRON
The Department of Mathematical Sciences

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calculus
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Article: Integration

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Integration

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1. Introduction

Prerequisite: Limits, Continuity, Differentiation.

2. The Indefinite Integral

We begin, as always, with a definition.

Definition 2.1. Let f be a function defined over an interval (a, b). A function F is called an *indefinite integral*, or an *antiderivative*, of f over the interval (a, b) provided

$$F'(x) = f(x) \qquad \text{for all } x \in (a, b).$$

Definition Notes: At our level of play, the reference to the interval (a, b) is suppressed; consequently, we speak of F as an indefinite integral, or antiderivative, of f.

▪ An antiderivative of a function f is a *function*, F. This point must always be kept in mind: The antiderivative of a function is a function.

■ The term *antiderivative* is more descriptive of the concept than the term *indefinite integral*. An antiderivative of f is any function, F, whose derivative is f. The term indefinite integral comes from the important role it plays in Definite Integration. ■

Let's have a quick example to illustrate the definition of antiderivative.

Illustration 1. For the function $f(x) = 2x$, the function $F(x) = x^2$ is an antiderivative of f since $F'(x) = 2x = f(x)$, for all $x \in \mathbb{R}$.

Question. Can a function have more than one antiderivative? If the answer is 'yes,' in general, how many antiderivative does a given function have? (Use $f(x) = 2x$ as an example to help you reason.)

Let's look an elementary example before continuing.

EXAMPLE 2.1. Consider the function $f(x) = x^3$, find an antiderivative of f.

It is important that you understand the meaning of the term 'antiderivative' and the relationship between a function and its antiderivative; furthermore, the concept of antiderivative does not depend on the letters used to describe the functions and the variables. The next set of exercises is meant query you on the definition of antiderivative.

EXERCISE 2.1. Let h be a function of the variable t, write the definition of an antiderivative of h.

Review the reasoning of EXERCISE 2.1, as well as the definition of antiderivative before answering the following quiz questions.

Quiz.

1. Given two functions f and g, f is an antiderivative of g provided,

(a) $g'(x) = f(x)$ (b) $f'(x) = g(x)$

2. Given two function H and q, q is an antiderivative of H provided

(a) $q'(t) = H(t)$ (b) $H'(s) = q(s)$

3. Define a function $f(s) = 4s^3$ and another function $F(t) = t^4$, is F an antiderivative of f?

(a) Yes (b) No

End Quiz.

Exercise 2.2. Verify that an antiderivative of $f(x) = 16(4x+1)^3$ is the function $F(x) = (4x+1)^4$.

Checking your answer.
To determine whether a function g is an antiderivative or indefinite integral of another function, we simple differentiate the function g we think is the antiderivtive and determine if the result is equal to f. In symbols, g is an antiderivtive of f provided,

$$g'(x) = f(x) \qquad \text{for all } x.$$

This is simply the definition.

Exercise 2.3. Determine whether the function $f(t) = (t^2+1)^2$ is an antiderivative of $g(t) = 4t(t^2+1)$.

Exercise 2.4. Determine whether the function $H(s) = \cos(2s)$ is an antiderivative of the function $g(s) = 2\sin(2s)$.

Let's now continue developing some of the basic ideas of the antiderivative.

As we have seen in Example 2.1, once we have found one antiderivative of a given function, we have found *infinitely many* antiderivatives. More precisely, if F is an antiderivative of f then for any constant C, $F + C$ is also an antiderivative of f. A natural question to ask: Suppose F is an antiderivative of f, do there exist antiderivatives of f that are *not* of the form $F + C$? The answer is *no*.

Recall a corollary to the Mean Value Theorem which states that if F and G are two functions such that $F'(x) = G'(x)$ for all x in an interval I of numbers, then there exists a constant C such that $F(x) = G(x) + C$ for all x in the interval I.

Now, let's prove the answer to the question.

Theorem 2.2. *Let f be a function having antiderivative F over an interval I. If G is any other antiderivative of f over I, then there exists a constant C such that $F(x) = G(x) + C$ for all $x \in I$.*

Proof. F is an antiderivative of f means

$$F'(x) = f(x) \qquad \text{all } x \in I.$$

G is an antiderivative of f means

$$G'(x) = f(x) \qquad \text{all } x \in I.$$

Therefore we have

$$F'(x) = f(x) = G'(x) \qquad \text{all } x \in I.$$

By the corollary to the MEAN VALUE THEOREM we then have

$$F(x) = G(x) + C \qquad \text{all } x \in I,$$

for some constant C. □

Theorem Notes: This shows that once we find an antiderivative, F, of f, then we have found *all* antiderivatives. Any other antiderivative of f *must* have the form: $F(x) + C$.

▪ Let us agree on some terminology. If F is an antiderivative of f, then $F(x) + C$ will be referred to as the *general antiderivative* of f. ▪

In the next few sections, antidifferentiation formulas are developed. Before we get to that point, try to solve each of the following exercises. The trick is to imagine what the function F would have to look like in order for its derivative to be the given function f. Use your knowledge of the differentiation formulas to construct the function F. You can check your answers before "jumping" to the solutions.

EXERCISE 2.5. Find the general antiderivative of $f(x) = x^7$.

EXERCISE 2.6. Find the general antiderivative of $f(x) = 4x^5$.

EXERCISE 2.7. Find the general antiderivative of $f(x) = 4x^5 + x^7$.

EXERCISE 2.8. Find the general antiderivative of $f(x) = 3\cos(x)$.

EXERCISE 2.9. Find the general antiderivative of $f(x) = 3\cos(x) + 4\sin(x)$.

And finally, to illustrate that the ideas in this section are not dependent on the name of the function and the variable name, try this exercise.

EXERCISE 2.10. Find the general antiderivative of $h(t) = 4t^7 - 6t^2 + 10$

The next few paragraphs can be skipped over on first reading.

For those who want to know more. The next exercise is a natural question: Must a function always have an antiderivative? The answer is "no," in general. Don't look at the solution, yet. Think about this question, and as you progress through these notes and learn more about antidifferentiation, perhaps you can answer this question on your own. Be aware that their are *infinity many* examples, so even if you produce an example, it may not be the same as mine.

• EXERCISE 2.11. Give an example of a function f defined over the interval $(0, 1)$ such that f *does not* have an antiderivative over the interval $(0, 1)$. (See the commantary that follows the statement of this problem.)

Thoughts on this Exercise. You have to create a function, f, that is so "weird" that for any function F, $F'(x) \neq f(x)$ for at least one $x \in (0, 1)$. That seems simple enough. After you think of a candidate for f, the interesting part is to *prove* your example has no antiderivative.

To do this, you must use the definition of derivative, some "common sense," and . . . the Mean Value Theorem.

I've marked this exercise with a '•', meaning it is moderately difficult, and requires some abstract thinking, and "proof" construction.

As you learn more about the differentiation process, maybe you can imagine such a function f would look like. Keep this exercise in mind as you work through these tutorial. *Come back soon*! ■

For those who want to know more. If you've solved the last exercise, or studied it in detail, here is another exercise for you. The function in my solution to Exercise 2.11 had the property that antiderivatives existed for it over the interval $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$, but not over $(0, 1)$. Now, construct a function f defined on the interval $(0, 1)$ such that f has no antiderivative over *any* subinterval of $(0, 1)$; i.e., find a function f such that for any $(a, b) \subseteq (0, 1)$, f has no antiderivtive over $(a, b,)$. :-)

I've marked this exercise with a '•', meaning it is more difficult and can be skipped over on first reading; however, by making a significant modification of my example in Exercise 2.11. (*Hint*: The example that I have in mind only takes on the values of 0 and 1.)

• Exercise 2.12. Give an example of a function, f, defined over the interval $(0, 1)$ such that f *does not* have an antiderivative over any subinterval of $(0, 1)$.

2.1. The Indefinite Integral Notation

Notation. The notation, which is due to Leibniz, you will find rather unusual. Let f be a function of x, an indefinite integral of f, denoted by

$$\int f(x)\,dx, \tag{1}$$

is any function whose derivative is f; that is, (1) is a symbol that represents any *antiderivative* of f.

Notation Notes: The function, f, is called the *integrand*. Notice that the indefinite integral is a function — this is an important point to remember.

- The symbol dx is called the differential of x. See below for a discussion of the role of dx.
- The symbol x is called the *variable of integraton.* ■

Quick Response.

1. Consider the integral: $\int 3x^4\,dx$. Which of the following is the *integrand*?

(a) x^4 (b) $3x^4$ (c) 3 (d) n.o.t.

2. Consider the integral: $\int a\cos(z) + bz\,dz$. What is the *variable of integration*?

(a) a (b) b (c) x (d) z

3. What is the *integrand* of the integral in (2)?

(a) $\cos(z)$ (b) $a\cos(z)$ (c) $a\cos(z) + bz$ (d) n.o.t.

4. Complete the following phrase: An indefinite integral is

(a) an antiderivative of its integrand.
(b) the derivative of its integrand.
(c) an antiderivative of the derivative of its integrand.
(d) the derivative of the antiderivative of its integrand.

Reading the Notation. The notation (1) is read as "The integral of $f(x)$ with respect to x." The differential notation, dx, in this context, is read as "with respect to x."

For example, consider the equation:

$$\int 3x^2\,dx = x^3 + C.$$

This equation may be read from left to right as "the integral of $3x^2$ with respect to x is x^3 plus an arbitrary constant C." The function $3x^2$ is the integrand. You'll notice that the indefinite integral of $3x^2$ is indeed a function of x.

When the integral notation is used by itself,

$$\int \cos(x)\,dx,$$

it poses the question: “What is the integral (antiderivative) of the function $\cos(x)$ with respect to x?” The integral also provides a handy notation for presenting the answer:

$$\int \cos(x)\,dx = \sin(x) + C,$$

which reads: “the integral of $\cos(x)$ with respect to x is $\sin(x)$ plus any constant C.”

EXERCISE 2.13. Write out a sentence which will be a precise English translation of the equation

$$\int \sin(t)\,dt = -\cos(t) + C.$$

Before trying the next exercise, review the definition of the indefinite integral and the description of the indefinite integral notation.

EXERCISE 2.14. What is the evaluation of $\dfrac{d}{dx}\displaystyle\int f(x)\,dx$?

Exercise 2.15. Evaluate the expression: $\dfrac{d}{dx}\displaystyle\int (x+\sin(x))^{10}\,dx$.

Exercise 2.16. Evaluate the expression: $\dfrac{d}{ds}\displaystyle\int \tan^{12}(s)\,ds$.

The significance of the dx. For right now, the role dx plays will be three-fold. (We'll get another fold later.)

1. An indefinite integral is supposed to be a function, but a function of what variable? The differential notation that is incorporated into the integral answers this question.

For example, in the statement,

$$\int x^2\,dx$$

the dx indicates that this indefinite integral represents a function. The defining property of this function (of x) is that its derivative *with respect to* x is x^2, the integrand. (*Note*: $\int x^2\,dx = \frac{1}{3}x^3 + C$.)

The statement

$$\int t^3\,dt$$

represents a function of t (as indicated by the dt) such that if we differentiate this function *with respect to* t we obtain the integrand, t^3. (*Note*: $\int t^3\,dt = \frac{1}{4}t^4 + C$.)

The statement

$$\int zs^5\,ds$$

is a function of s. Here, the symbol z may represent a constant or another function; in any case, regardless of the meaning of the symbol z, the integral represents a function of s (because of the ds).

2. The symbol dx tells us what variable we are to consider as the variable of integration. Why is it important to know the variable of integration? Read on.

Without the "differential notation," the following symbolism is ambiguous

$$\int s\cos(t).$$

Are we to consider the integrand a function of s ($f(s) = s\cos(t)$) and integrate with respect to s, or should we consider the integrand a function of t ($f(t) = s\cos(t)$) and integrate with respect to t? Depending on what variable is the variable of integration, we would have totally different answers:

$$\int s\cos(t)\,ds = \frac{1}{2}s^2\cos(t) + C$$

$$\int s\cos(t)\,dt = s\sin(t) + C.$$

In each of the evaluations, we assumed all other algebraic symbols were constants. To verify the correctness of these equations, simply differentiate the right-hand sides of these equations to obtain the integrand; remember, "the derivative of an indefinite integral is its integrand." (Differentiate with respect to the variable indicated by the differential.)

It is true that in many situations we know the variable of integration from the context. There can really be no dispute that in the integral

$$\int x^2,$$

x is the variable of integration; however, mathematicians like to have their notation tightly wrapped and addressed so that can be absolutely no confusion as to the variable of integration—hence the use of the dx notation.

- By the way, evaluate $\displaystyle\int x^2$ please.

3. The symbol dx also acts as a delimiter. It helps us to define the integrand. The integrand is the function that lies between the $\int$ symbol and the dx symbol.

$$\int \underbrace{\cdots\cdots}_{\text{integrand}} dx$$

Without this delimiter, the integrand may not be identifiable. For example, in some applications we want to calculate in integral and add it to another function. Consider the following integral without the benefit of the dx symbol

$$\int x^3 + x^2 + x.$$

Now do we want to calculate the integral of x^3 and then add it to the function $x^2 + x$, or to we want to calculate the integral of $x^3 + x^2$ and add this result to x?

Perhaps it would always be understood from the context of the problem what is meant, but mathematician like things more exactly organized.

2.2. An Application: Velocity and Acceleration

It seems that a student is always asking, "What's this good for?" This seems to me to be a fair and reasonable question.

"Antidifferentiation is nothing more than the reverse process to differentiation. You have this fancy notation for an antiderivative that doesn't make sense, and you have this term 'indefinite integral,' and what's this '$+C$' bit?"

The *indefinite integral* has wide ranging applications, as does the *definite integral*, yet to be taken up. In this section we look at a simple application to the antiderivative, and see what the '$+C$' is all about.

There are many physical systems that must obey certain physical laws. Many of these physical laws are described by mathematical formula. By identifying the physical laws the system must obey, and writing these laws mathematically, sometimes we can solve the equations and obtain, as a result, extensive knowledge of the state of the system.

Analysis of Free Falling Body. Suppose you are standing on the ground with a rock in your hand. At some instant in time (your choice) you throw the rock vertically upward. When the rock leaves your hand, the rock is s_0 feet above the ground, and it is going at a velocity of v_0 ft/sec. It is our desire to have *total knowledge* of the motion of the rock.

The rock must obey a certain physical law: Due to gravity, the rock must accelerate towards the earth at a rate of $32\,\text{ft/sec}^2$. To maintain total abstraction, let g denote the acceleration due to gravity.

The motion of the rock must then satisfy,

$$a(t) = -g, \tag{2}$$

at any time t. Acceleration, then, is a constant function of time, t. I have appended a minus sign to indicate that the rock is accelerating downward toward the earth.

Recall that for a particle in motion, *acceleration is the rate at which velocity changes with respect to time*. In terms of Calculus concepts:

$$a(t) = \frac{d}{dt} v(t).$$

In the language of this article, this means that $v(t)$ is an indefinite integral, or antiderivative, of $a(t)$. Symbolically,

$$v(t) = \int a(t)\, dt. \tag{3}$$

But by (2), $a(t) = -g$. Substituting this into (3) we get,

$$v(t) = \int -g\, dt. \tag{4}$$

The right-hand side of (4) is the integral of a specific function; namely, the constant function equal to $-g$. We can conjure up an antiderivative of $-g$: A function (of t) whose derivative (with respect to t) is equal to $-g$. After many hours of meditation we reach the result,

$$v(t) = \int -g\,dt = -gt + C. \tag{5}$$

You can check for yourself that the derivative of $-gt + C$ is the integrand, $-g$. We have shown that the (unknown) velocity function must be for the form $-gt + C$, for some constant C. This doesn't do us much good unless we can put our little phalanges on this C.

The value of C can be obtained by substituting some of the information we have about our rock; in particular, at time $t = 0$, the rock has a velocity of v_0. Thus, from (5),

$$v_0 = v(0) = -g(0) + C$$

or,

$$\boxed{C = v_0.} \tag{6}$$

Now update our equation (5) using (6) to get,

$$\boxed{v(t) = -gt + v_0.} \tag{7}$$

We now have total knowledge of the velocity of the rock at any time t.

But let's continue. What is the position of the rock at any time t? Again, we have seen that *velocity is the rate at which position changes with respect to time.* Let $s(t)$ denote the height the rock is off the ground at time t, then we know

$$v(t) = \frac{d}{dt} s(t).$$

But this says that $s(t)$ is an indefinite integral, or antiderivative, of $v(t)$. In the language of indefinite integrals, this equation becomes

$$s(t) = \int v(t)\, dt. \tag{8}$$

But from (7), $v(t) = -gt + v_0$. Substituting this into (8) we get

$$s(t) = \int -gt + v_0 \, dt \tag{9}$$

The right-hand side is the indefinite of a concrete function, v_0 being a symbol for a numerical constant. This can be calculated:

$$\int -gt + v_0 \, dt = \frac{1}{2}gt^2 + v_0 t + C. \tag{10}$$

This can be verified by differentiating the right-hand side with respect to t to obtain the integrand. Putting this result into (9), we obtain

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + C. \tag{11}$$

Once again we have the enigmatic C. The equation (11) states the *functional form* of $s(t)$. We can't use this equation until we know C.

But again, at time $t = 0$, we have the information that the rock was at a height of s_0. Putting $t = 0$ in (11) we get

$$s_0 = s(0) = 0 + 0 + C$$

or,

$$\boxed{C = s_0.}$$

Now, update equation (11) to get

$$\boxed{s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0.} \tag{12}$$

Now we have total knowledge of the physical system.

Summary: A rock leaves your hand at time $t = 0$ at an initial height of s_0 and an initial velocity of v_0. Then, for any time t,

$$\begin{aligned} a(t) &= -g \\ v(t) &= -gt + v_0 \\ s(t) &= -\frac{1}{2}gt^2 + v_0 t + s_0. \end{aligned} \tag{13}$$

An example will illuminate the concepts.

Example 2.2. You throw a rock is vertically at a speed of 50 ft/sec, and the rock is initially 6 ft off the ground.

a. Find the equation (13) that specifies the height, $s(t)$ of the rock above the ground at time t.
b. How high is the rock off the ground 1 second after the rock leaves your hand?
c. How long before the rock hits the ground?
d. What is the velocity of the rock when it hits the ground?
e. At what time is the rock 6 feet above the ground?
f. What is the velocity of the rock when the rock is 6 feet off the ground?
g. How high does the rock go?

3. Some Basic Integration Formulas

We now turn to the task of developing form formula for evaluating indefinite integrals. Each rule must be *memorized*. It is important to *memorize and understand* these formulas because they will represent a *base of knowledge* upon with you can reason, solve problems, communicate with others, and expand to more complicated ideas without being encumbered.

■ Here's an expansion that last point, for those who want to know more.

Fundamentally, there are two types of integration formulas: specific formulas and general formulas. In the next two sections we discuss each of these types.

3.1. Specific Formulas

A *specific formula* for integration is an integral formula that actually solves an integral problem. In this section we identify a few of the more elementary ones.

The Integral of 0. The most elementary integral formula is

$$\boxed{\int 0\,dx = C.} \tag{1}$$

The integrand is 0.

EXERCISE 3.1. Refer to equation (1). The integrand is suppose to be a function of x, yet is state, and I quote myself, "The integrand is 0." But 0 is a number not a function, explain this paradox.

▪ Why is this formula true? Because the derivative of a constant is zero; therefore, any constant function C is an antiderivative of the identically 0 function.

The Power Rule. Let $r \in \mathbb{Q}$ be a rational number, $r \neq -1$, then by the Power Rule, we have

$$\frac{d}{dx}\frac{x^{r+1}}{r+1} = \frac{1}{r+1}\frac{d}{dx}x^{r+1}\frac{1}{r+1}(r+1)x^r = x^r.$$

This says that an antiderivative of x^r is $\dfrac{x^{r+1}}{r+1}$. In terms of the indefinite integral notation, we have

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C.$$

Let's elevate this formula.

Power Rule Junior Grade:
Let $r \in \mathbb{Q}$ be a rational number, $r \neq -1$, then

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C.$$

EXERCISE 3.2. Why do you think that we require $r \neq -1$ in the Power Rule?

At the beginning of this section, I remarked that specific integration formulas are formulas that actually solve integral problems. This is apparent from the *Power Rule Formula* The left-hand side is the statement of an integral problem, the right-hand side is the solution to same.

The use of the Power Rule depends on your ability to identify power functions. If you cannot recognize a power function, then you will not be able to apply the power rule.

Quick Response. Which of the following functions is a power function of x? (Or, simplifies to a power function!)

(a) 5^{x+1} (b) $(x+1)^x$ (c) $\dfrac{\sqrt{x}}{x^3}$ n.o.t.

Here are some quick visual examples with positive integer exponents.

$$\int x^2\,dx = \frac{x^3}{3} + C$$
$$\int x^3\,dx = \frac{x^4}{4} + C$$
$$\int x^{10}\,dx = \frac{x^{11}}{11} + C$$
$$\int t^{20}\,dt = \frac{t^{21}}{21} + C$$
$$\int w^8\,dw = \frac{w^9}{9} + C.$$

Now for some quick visual examples with negative integer exponents.

$$\int x^{-3}\,dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2}x^{-2} + C$$

$$\int t^{-5}\,dt = \frac{t^{-4}}{-4} + C = -\frac{1}{4}t^{-4} + C$$

$$\int w^{-23}\,dw = \frac{w^{-22}}{-22} + C = -\frac{1}{22}w^{-22} + C.$$

How about fractional exponents?

$$\int x^{2/3}\,dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}x^{5/3} + C$$

$$\int u^{-2/3}\,du = \frac{u^{1/3}}{1/3} + C = 3x^{1/3} + C$$

$$\int z^{-11/4}\,dz = \frac{z^{-7/4}}{-7/2} + C = -\frac{4}{7}x^{-7/4} + C.$$

Do you get the idea?

Important. In the above quick visual examples, I presented examples with a variety of *variables of integration*. The application of the power rule does *not* depend on the variable, x, that is used to write the formula down. The power rule is

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C, \qquad r \neq -1.$$

On the left-hand side of this equation, the key point is this: the *base function* of the power function being integrated is *exactly* the same as the *variable of integration* as defined by the dx. If these two do not match, then the power rule, as currently stated, *does not apply*! A simple example of this observation is the following:

$$\int (2x+1)^{1/2} \, dx.$$

We are integrating a power function, but the *base function*, $2x+1$, does *not* match the variable of integration, x, as defined by the dx symbolism. Therefore, this form of the power rule *does not apply*. Below you will find a more general power rule that we can use here.

To summarize:

> Key Point: When you are integrating a power function, in order for the power rule to apply, the base of the power function must be the same as the variable of integration.

EXERCISE 3.3. Calculate $\int x^{-3/4}\,dx$.

EXERCISE 3.4. Calculate $\int w^{7/3}\,dw$.

Here's a slight variation on the previous exercises, see if you can think your way through.

EXERCISE 3.5. Calculate $\int (2x)^4\,dx$.

Important. When applying the Power Rule, the *power function must be in the numerator*. Move the power function into the numerator, for the correct calculation of the exponent!

To illustrate this point, consider this ...

Example 3.1. Evaluate $\displaystyle\int \frac{1}{x^2}\,dx$.

Exercise 3.6. Evaluate $\displaystyle\int \frac{1}{\sqrt{u}}\,du$.

Exercise 3.7. Calculate $\displaystyle\int x^2\sqrt{x}\,dx$.

Exercise 3.8. Evaluate $\displaystyle\int \frac{\sqrt{t}}{t^3}\,dt$.

(*Hint*: Make integrand into a power function.)

Here's a poser.

Exercise 3.9. Calculate $\displaystyle\int dx$, and $\displaystyle\int du$.

Trigonometric Functions. There are six formulas for solving integrals involving trigonometric functions.

Trigonometric Integration Formulas: Junior Grade:

$$(1) \int \cos(x)\,dx = \sin(x) + C \qquad (3) \int \sec^2(x)\,dx = \tan(x) + C$$

$$(2) \int \sin(x)\,dx = -\cos(x) + C \qquad (4) \int \csc^2(x)\,dx = -\cot(x) + C$$

$$(5) \int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$(6) \int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

The student should verify these six formulas by differentiating the left-hand side of each formula, to obtain the integrand of the right-hand side. See the exercise below.

You'll note that there are essentially three formulas here; the other three are "co'ed" versions of the first three. Formulas (2), (4), and (6) can be constructed from (1), (3), and (5), by "co-ing" the functions and appending a negative sign to the answer. For example, formula (3) is

$$\int \sec^2(x)\,dx = \tan(x) + C.$$

Now if we "co'ed" the functions, and appended a negative sign to the answer we get

$$\int \csc^2(x)\,dx = -\cot(x) + C$$

This makes it very easy to remember these six (three) formulas.

These formulas *must* be memorized. There are two ways of remembering something: sitting down and muttering to yourself, repeating the formulas over and over again (not good); do many problems, each time you use these formulas, *verbalize* the formula — after awhile, you'll have them memorized.

The exercises are limited right now.

EXERCISE 3.10. Evaluate $\int \cos(t)\,dt$.

EXERCISE 3.11. Evaluate $\int \csc^2(s)\,dx$.

EXERCISE 3.12. Evaluate $\int \sin^2(x^3) + \cos^2(x^3)\,dx$.

EXERCISE 3.13. Verify formula (4).

This is the sum total of the specific integration formulas. In Calculus II, many more formulas of this type will be developed.

3.2. General Formulas

A *general formula* for integration is a formula that *transforms* the integral into *another* integral or integrals. General formulas *do not* solve an integral problem.

Homogeneous Property. The homogeneous property comes from the corresponding property for differentiation.

Homogeneous Property:
For any constant c and any function f, we have

$$\int cf(x)\,dx = c\int f(x)\,dx.$$

If we think of the left-hand side as the given integral problem, then the formula does not solve the problem; it merely transforms the problem into another integral problem (the right-hand side).

This substance of the Homogeneous Property is that *constants can be taken outside an intgral.*

EXAMPLE 3.2. Verify the Homogeneous Property.

EXERCISE 3.14. Evaluate $\int 4x^6\,dx$.

Exercise 3.15. Evaluate $\displaystyle\int 6t\sqrt{t}\,dt$

The Additive Property. A fundamental formula which, along with the Homogeneous Property, delineates the *algebraic stucture* of the indefinite integral.

The Additivity of the Integral:
Let f and g be functions, then

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

Again note that this formula does not solve integrals — it merely transforms the integral problem on the left-hand side into two integral problems on the right-hand side. Solving integrals is the role of the specific formulas.

The use of this formulas depends on your ability to *realize* that the integrand is the sum of several functions. That shouldn't be too difficult? We'll find out.

EXAMPLE 3.3. (Skill Level 0). Evaluate $\int 3x^4 + 6x^2\,dx$.

EXERCISE 3.16. (Skill Level 0). Evaluate $\int \frac{2}{3}x^6 - 8x^{12}\,dx$.

EXERCISE 3.17. Evaluate $\int 8\sec^2(x) - 6\sec(x)\tan(x)\,dx$.

Some integrals require you to manipulate the integrand algebraically *before* attempting to integrate. Next up are a few examples of these creatures.

EXERCISE 3.18. Evaluate $\int (t^4 - 4t^3)^2\,dt$.

EXERCISE 3.19. Evaluate $\int \left(w^3 - \frac{1}{w^2}\right)^2 dw$.

Exercise 3.20. Evaluate $\int (\sec(x) + \tan(x))^2 \, dx$. (Hint: Square it, and use the identity: $\sec^2(x) - \tan^2(x) = 1$.)

4. The Technique of Substitution

The formulas and techniques already developed are useful and important, but they are limited in their scope. For example, the power rule can solve

$$\int x^{1/2} \, dx,$$

but cannot solve the integral,

$$\int (2x + 1)^{1/2} \, dx.$$

Do you know why? If not, review the discussion presented earlier.

4.1. Developing the Idea: Substitution

If indefinite integration is the reverse operation to differentiation, then *substitution* is the reverse operation of the *Chain Rule*.

Let f and g be are differentiable and compatible for composition. Let F be an antiderivative of f; this means that $F'(u) = f(u)$. Since F is an antiderivative of f, we have

$$\int f(u)\,du = F(u) + C. \tag{1}$$

Now from the Chain Rule,

$$(F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x).$$

In the language of antiderivatives, this equation says that the left-hand side is an antiderivative of the right-hand side. Therefore,

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C. \tag{2}$$

If we now take the right-hand side of (1), and replace u with $g(x)$, we obtain

$$\int f(u)\,du = F(g(x)) + C. \tag{3}$$

Notice that the right-hand sides of (2) and (3) are identical; therefore, the left-hand sides are equivalent. What this means is that we can solve the integral in (2), by first solving the integral in (1), then replacing u with $g(x)$.

In fact, we equate (2) and (3) we obtain the classic *substitution of variable formula*:

$$\boxed{\int f(g(x))g'(x)\,dx = \int f(u)\,du,} \tag{4}$$

where, $u = g(x)$.

Equation (4) displays the principle of *substitution*. We can think of the left-hand side or the right-hand side as our target, or given, integral.

The principle of *substitution* states then that the given integral is equal to the integral on the other side of the equality.

Equation (4) is especially pleasing when we remember the concept of the differential:

$$\text{If } u = g(x), \text{ then } du = g'(x)\,dx.$$

Thus, if $\int f(g(x))g'(x)\,dx$ is our given integral, we can let $u = g(x)$ and so $du = g'(x)\,dx$. Now replacing these symbolisms into our given integral we obtain

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

On the other hand, if $\int f(u)\,du$ is our given integral, we can let $u = g(x)$ and so $du = g'(x)\,dx$. Now replacing these symbolisms into our given integral we obtain

$$\int f(u)\,du = \int f(g(x))g'(x)\,dx.$$

Before looking at an extensive collection of examples, let's highlight this technique.

The Technique of Substitution:
Let f and g be functions. Let $u = g(x)$ and $du = g'(x)\,dx$, then
$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

4.2. Learning the Technique of Substitution

Let's now examine the circumstances under which this principle can be applied and exhibit the standard techniques of implementing the formula.

EXAMPLE 4.1. Evaluate $\displaystyle\int (x+1)^{15}\,dx$.

This example hopefully gives you a vision of the potential use of the Substitution. The next examples will tend to expand your vision.

Example 4.2. Evaluate $\int (2x+1)^{15}\,dx$.

Now you try one.

Exercise 4.1. Evaluate $\int (3x+1)^{20}\,dx$.

(*Hint*: Consider the substitution $u = 3x + 1$.)

These examples and exercises were all the same. They were the integrals of power functions. The technique of substitution is quite general, and can be applied in a wide variety of problems.

Example 4.3. Evaluate $\int \cos(2x)\,dx$.

Exercise 4.2. Evaluate $\int \sec^2(3x)\,dx$. Use the substitution $u = 2x$.

In the next two sections, we create specialized formulas for integrating power functions and trigonometric functions. The new formulas are created using the substitution formula applied to abstractions of the examples and exercises we just finished.

4.3. The Generalized Power Rule

We can generalize the basic Power Rule using the technique of substitution.

Generalized Power Rule:
Let u be a function of some variable (perhaps x, or t, or s, or any variable), and let $r \in \mathbb{Q}$ be a rational number, then

$$\int u^r \, du = \frac{u^{r+1}}{r+1} + C \qquad r \neq -1.$$

The first thing you will notice is that this formula is *exactly* the same as our old Power Rule. The only difference is the choice of the letter to describe the formulas. Yes, that's true. But our interpretation of this letter is *different*: We are thinking of u as a function of some other variable, say, $u = f(x)$, and so $du = f'(x)\,dx$. In this case, the Generalized Power Rule actually becomes

$$\int [f(x)]^r\, f'(x)\,dx = \frac{[f(x)]^{r+1}}{r+1} + C.$$

As you can see, this gives us the ability to solve the integrals of more general power functions ... if the conditions are right.

Let's take a look at an example in light of this new formula.

Example 4.4. Evaluate $\displaystyle\int (5x-3)^9\,dx$.

Now, we raise the level of difficulty a little, but not discouragingly so.

Example 4.5. Evaluate $\displaystyle\int x(3x^2-5)^{3/4}\,dx$.

Strategy. When trying to use the Power Rule to solve an integral involving power functions, let u be the base function of your power function, if the power rule is to apply, the rest of the integrand must be directly proportional to the du. If not there are two courses: (1) Some integrands have several power functions in them, try another choice; (2) The power rule does not apply, use another formula, or try a technique.

Commentary on the Previous Example: In light of the Strategy, let's look at the solution to EXAMPLE 4.5. I let $u = 3x^2 - 5$, this was the base of the power function. I calculated du to be $du = 6x\,dx$. You'll notice that the rest of the integrand is directly proportional to the calculate value of du:

$$\int (3x^2 - 5)^{3/4}\, x\, dx.$$

I then factored in the constant of proportionality, 6, and compensated for the insertion of this factor, by factoring in $\frac{1}{6}$, outside the integral.

The overall effect is to multiply by $(6)(\frac{1}{6}) = 1$. No damage done, but a lot of good.

$$\frac{1}{6}\int (3x^2 - 5)^{3/4}\, 6x\, dx.$$

The result is that I can not affect the substitution:

$$\frac{1}{6}\int u^{3/4}\, du,$$

and I am home free.

Keeping the Strategy in mind, solve following exercise, please.

EXERCISE 4.3. Evaluate $\displaystyle\int x^8(6x^9 + 12)^{1/3}\, dx$.

The next example illustrate a case when the *Power Rule* does not apply. This case is just as important because you need to learn to recognize when the power rule does not apply, so you can move on to another solution method — rather than giving up.

Example 4.6. (Power Rule Does Not Apply.)
Evaluate $\int x^3(2x^3+1)^7\,dx$.

Here are some exercises that are solved directly by the *Power Rule*.

Exercise 4.4. Evaluate $\int \frac{x}{\sqrt{4-3x^2}}\,dx$. (Hint: Convert integrand to a power function in the numerator!)

All these problems are pretty much the same. You're words of advice for today are

Identification and Implementation!

Exercise 4.5. Evaluate $\int (3x^3-1)(3x^4-4x+1)^{1.45}\,dx$. (Hint: Keep a cool head, and follow the strategy for the power rule.)

Quiz. Which of the following integrals can be solved by the *Power Rule*, and which cannot. Before you begin you may want to review the strategy for the *Power Rule*.

1. $\int (x^2+1)^2\,dx.$ (a) Yes (b) No

2. $\int x(x^2+1)^2\,dx.$ (a) Yes (b) No

3. $\int \frac{x^2}{\sqrt{x^3+1}}.$ (a) Yes (b) No

4. $\int \frac{x-1}{(x^2-2x-1)^2}.$ (a) Yes (b) No

5. $\int \frac{x}{(x^3+1)^{100}}.$ (a) Yes (b) No

6. $\int (x-2)(x^2-3x-1)^{17}\,dx.$ (a) Yes (b) No

7. $\int \frac{x}{x^2+1}\,dx.$ (a) Yes (b) No

Passing Score: 5 out of 7.

Quiz Notes: The last answer was 'No' because the power function is $(x^2 + 1)^{-1}$. We let $u = x^2 + 1$ and so $du = 2x\,dx$, which is directly proportional to the rest of the integrand, so why is 'No' the correct answer? Because the value of the exponent of the power function is $r = -1$. This is the exceptional case to which the Power Rule does *not apply*. *Trick Question*! You have to be on your *Power Rule toes*! The case $r = -1$ is covered in *Calculus II*. ∎

End Quiz

Finally, let's have a ...

Period of Consolidation. Take a moment to consolidate your knowledge by listing out the major points of this *Generalized Power Rule* section.

4.4. Integration of Trig Functions

We have two sets of elementary integration formulas: The original Power Rule and the set of Trigonometric. In the previous section, we have generalized the old *Power Rule* using the *Technique of Substitution* to obtain a more general, more powerful Generalized Power Rule. Now we do the same for the *Trig formulas*.

Before presenting the list of new formulas, you might review an earlier example, Example 4.3, in which the technique of substitiution is utilized to analyze a *trigonometric integral*.

Trigonometric Integration Formulas:

Let u be a function of some independent variable, then

$$(1)\ \int \cos(u)\,du = \sin(u) + C \qquad (3)\ \int \sec^2(u)\,du = \tan(u) + C$$

$$(2)\ \int \sin(u)\,du = -\cos(u) + C \qquad (4)\ \int \csc^2(u)\,du = -\cot(u) + C$$

$$(5)\ \int \sec(u)\tan(u)\,du = \sec(u) + C$$

$$(6)\ \int \csc(u)\cot(u)\,du = -\csc(u) + C$$

Here is an example with extensive discussion concerning the background thinking that should be going on.

Example 4.7. Evaluate $\displaystyle\int \sin(5x)\,dx$.

Strategy. Given that you have an integral to be solved that involve any of the trigonometric function types sin, cos, $\sec^2$, $\csc^2$, $\sec\tan$, or $\csc\cot$, then the Trig. formulas *might apply*. To verify that one of the *trig formulas* apply, let u be equal to the argument of the trigonometric function. The rest of the integral must be *directly proportional to the du*, the differential of u. If this is so, then the formula applies, if not, the formula does not apply.

Example 4.8. Evaluate $\displaystyle\int x\sec(x^2)\tan(x^2)\,dx$.

Exercise 4.6. Evaluate $\displaystyle\int \sec^2(4x)\,dx$.

Exercise 4.7. Evaluate $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$.

Example 4.9. (Trig. Formulas do not apply.) Evaluate $\displaystyle\int x\cos(x)\,dx$.

This next exercise may be a bit tricky.

Exercise 4.8. Evaluate $\displaystyle\int x\sec(x^2)\tan(x^2)\sec^2(\sec(x^2))\,dx$.
(*Hint*: Follow the strategy.)

5. Substitution: Two Attitudes

The use we have made of the technique of substitution could be described as *formula checking*. In all the examples and exercises given so far, we have had an integral, and we have solved that integral using a formula. Before we can use a particular formula we must check whether it applies. The checking process is carried out by the device of *substitution*.

5.1. Formula Checking

Given that we have a integral problem:

$$\int x(3x^2+4)^{1/3}\,dx. \tag{1}$$

We decide to *try* to solve this integral using the *Power Rule*:

$$\int u^r\,du = \frac{u^{r+1}}{r+1} + C \qquad r \neq -1. \tag{2}$$

Does this formula successfully solve the given problem, (1)? The way we see this is to set up a *correspondence* between the given integral, (1), and the selected formula. The formal mechanism for setting up this correspondence is *substitution*. In the formula (2), the u is the base of the power function; therefore, *if* (2) is going to solve (1), then u *must* be the base of the power function in (1). This is why we would naturally say,

$$\text{Let } u = 3x^2 + 4, \text{ and so } du = 6x\,dx. \tag{3}$$

Rearrange the order of our integral so that the power function is listed first (as that is the way it is written in the formula we are trying to use).

$$\int (3x^2 + 4)^{1/3}\, x\,dx \tag{4}$$

We notice that everything in the formula integral, (2), following the power function is the du of the integral; therefore, if the formula is to apply, everything after our power function in our integral (4) must be the du part. We notice that the expression that follows the power function in (4) is *directly proportional* to the du, as calculated in (3).

Insert now, the appropriate fudge factor,

$$\frac{1}{6}\int (3x^2+4)^{1/3}\,6x\,dx. \tag{5}$$

Now, all the component parts of the formula integral (2) match up with our integral (5): the integral in (5) has the form u raised to a power times the differential of that u. We know now that the *Power Rule* does apply, but to make it absolutely clear, we can go ahead and make the substitution:

$$\begin{aligned}\int x(3x^2+4)^{1/3}\,dx &= \frac{1}{6}\int (3x^2+4)^{1/3}\,6x\,dx \\ &= \frac{1}{6}\int u^{1/3}\,du.\end{aligned}$$

Now, we *really can* see that the *Power Rule* is applicable and we can go on to evaluate the integral using that rule.

The formal substitution into the integral really isn't necessary:

$$\begin{aligned} \int x(3x^2+4)^{1/3}\,dx &= \frac{1}{6}\int \underbrace{(3x^2+4)^{1/3}}_{u^r}\,\underbrace{6x\,dx}_{du} \\ &= \frac{1}{6}\frac{3}{4}(3x^2+4)^{4/3}+C \\ &= \frac{1}{8}(3x^2+4)^{4/3}+C \end{aligned} \tag{5}$$

Here, rather than making the substitution, I just invoked the *Power Rule*: "raise the function to one greater power, and divide by that power."

Can you see how substitution is used to check whether a given formula can solve a given integral? Once the determination has been made, the actual formal substitution need not even take place, see (5).

The next example exhibits how *substitution* can be used to show that a given formula *does not* solve a given integral.

Example 5.1. Argue that the Power Rule does not solve the integral $\int x(x^3 + 1)^{100}\, dx$.

Example 5.2. Verify, through substitution, that $\int \cos(2x)\, dx$ can be solve using (1). Solve the integral without making the substitution.

Exercise 5.1. Verify, through substitution, that $\int \sec^2(3x^2)\, x\, dx$ can be solve using (3). Solve the integral without making the substitution.

5.2. True Substitution of Variables

The technique has more powerful uses than simple formula checking. It can be used in the spirit of true substitution of variables. The substitution formula is

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du. \tag{6}$$

Then using this equation for formula checking, we usually use it from left to right; that is, we think of our given integral as the left-hand side, make a substitution, to obtain the right-hand side. Let me pull some trick photography on you: In (6), interchange the roles of x and u, and move each integral to the opposite side of the equation. If you followed that description, you will get the equation,

$$\int f(x)\,dx = \int f(g(u))g'(u)\,du. \tag{7}$$

Again think of the left-hand side as the given integral. This will be our working formula for these paragraphs.

True substitution of variables is performed with a different attitude than in formula checking. In formula checking, we had a definite formula in mind and we used substitution to check whether it applied to our problem integral. In true substitution of variables, we have no such formula in mind; in fact, we really don't know what to do or how to solve the problem.

When in doubt, substitute!

Typically, when you have an integral

$$\int f(x)\,dx \tag{8}$$

and you choose to make a substitution, the *form of the substitution* is likely to look like

$$\text{Let } x = g(u), \text{ and } dx = g'(u)\,dx, \tag{9}$$

where g is some appropriately chosen function. Given this choice, then it is a simple matter formally substitute (9) into (8) to obtain,

$$\int f(x)\,dx = \int f(g(u))g'(u)\,du \tag{10}$$

which is our working substitution formula (7).

Substitution Strategy. Quit often, we choose a substitution $x = g(u)$ that tends to *simplify* the function $f(x)$. This *simplification* of $f(x)$

comes at the expense of making the differential part of the integral more *complicated.*

$$f(x) \longrightarrow \underbrace{f(g(u))}_{\text{simpler}} \qquad dx \longrightarrow \underbrace{g'(u)\,du}_{\text{more complex}}$$

It is hoped that the overall effect is a successful continuation of the problem ultimately to solution.

So much for abstractions and general principles, let's focus on an example.

Example 5.3. Evaluate $\displaystyle\int x(x+1)^{100}\,dx$.

Example 5.4. Evaluate $\displaystyle\int x^2(2x+1)^{1/2}\,dx$.

Try this exercise on for size.

Exercise 5.2. Evaluate $\displaystyle\int x(3x-2)^{-4}\,dx$.

(*Hint*: Let $u = 3x - 2$.)

Exercise 5.3. Evaluate $\displaystyle\int x^2(6x+1)^{-1/2}\,dx$.

(*Hint*: Let $u = 6x + 1$.)

Generalizations. All these integral problems are all of the same type:

$$\int x^n(ax+b)^m\,dx,$$

where $n \in \mathbb{N}$ is a small positive integer. The change of variables

$$\text{Let } u = ax + b. \text{ Then, } x = \frac{1}{a}(u-b) \text{ and } dx = \frac{1}{a}\,du.$$

This substitution will work nicely.

There are a number of situations where a substitution of variables is productive. These will be surveyed in *Calculus II*.

6. Strategies for Integration

Often when a student looks at an integral problem, such as this one,

$$\int x^2 \sec(3x^3) \tan(3x^3)\, dx, \tag{1}$$

the student takes one look and says, "I don't know how to solve it!" The problem lies not in the difficulty level of the integral, but is the unfocused thinking of the student.

In this section I lay out some thoughts on the subject.

Keys to Success. Here are the keys to successfully solving integrals at the *Calculus* level.

1. A definite and precise knowledge of the integral formulas and how they are applied.

2. A definite and precise knowledge of the techniques used to manipulate integrals.

3. Acquisition of a history of problem solving.

4. The ability to learn from problem solving.

5. A developed pattern of thinking for analyzing integral problems.

6.1. Knowledge of the Integral Formulas

The best way to have knowledge of the integral formulas is by using them — many times. As you use them, verbalize them: "The integral of the sine of some function times the differential of that function is the minus the cosine of the function." Verbalizations are supplied throughout these files. As you verbalize the formulas, you will in turn hear them. It is the hearing yourself say the formula *as you use them* that enables you to remember them: You can remember yourself saying the formula — as a result, just listen to yourself.

Knowledge of the formulas implies the ability of *recognize* them. For example, if you have a knowledge of the formulas, then you would know that the integral in (1) can be solved by one of the basic formulas:

$$\int \sec(u)\tan(u)\,du = \sec(u) + C.$$

Whereas, this integral

$$\int x^2 \sec(3x^3)\cot(3x^3)\,dx$$

cannot be solved by any of the basic formulas.

Knowledge of the formulas means that when you look at these integrals

$$\int \frac{x}{\sqrt{x^2+1}} \qquad \int \sin^4(x)\cos(x)\,dx$$

$$\int (w^3+4)^{3/2} w^2\,du \qquad \int \frac{(\sqrt{x}+1)^{20}}{\sqrt{x}}\,dx$$

as *all the same problem*: Same in the sense that they can all be solved by the Power Rule.

6.2. Knowledge of the Techniques

A integration technique in any process or activity that *transforms* your integral problem into another integral problem. The idea is to try to solve the new integral problem. There are two types of techniques.

Two Types of Techniques

1. The application of a formula in which the integral symbol appears on both sides of the equation; for example,

$$\int cf(x)\,dx = c\int f(x)\,dx$$
$$\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$
$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du \qquad u = g(x)$$

Each of these general formulas can be looked on a techniques; they transform your problem into another problem. In *Calculus II* we develop more techniques.

2. Direct manipulation of the *integrand*. You can use *algebra* to transform the integrand or, perhaps, *trigonometric identities*. The heavy use if trigonometric identities will be delayed until *Calculus II*, but algebraic manipulation of the integrand is always in order.

6.3. Obtain a History of Problem Solving

At this level of play, integration is really quite simple: You *know* the problem is solvable, and there are only a *finite number* of formulas and techniques you can use (where the 'finite number' is 'small'); therefore, you just have to keep at it — It'll come . . . eventually.

> *Do not give up*: Each time you successfully solve a problem you are learning something, you are acquiring a history of problem solving, you increase your confidence that you can solve the next problem.

Do not treat each problem as an unique problem you have never seen before; actually, the kinds of problems you see is extremely limited — but *disguised*! If you solve a hundred problems using the power rule, then you have not solved one hundred distinct problems — you've basically solved the same problem over and over again with different 'u's'.

Your job is to pull off the disguise to see the true identity of the problem. The problem,

$$\int x \sin(x^2) \cos(\cos(x^2)) \sin^3(\cos(x^2))\, dx \tag{2}$$

is a nasty looking one, but actually, it is just the

$$-\frac{1}{2} \int u^3\, du$$

where $u = \sin(\cos(x^2))$. Do you see now how simple the integral in (2) really is?

EXERCISE 6.1. Solve the integral in (2)

Be like the Moray Eel. It is said that once the moray eel locks onto its victim with its mighty jaws, it will not let its victim go until the victim yields (dies — sorry). You must be a moray eel, your victim is any problem in mathematics. Clamp onto your victim and hold on. Don't let your victim go until it yields. It'll wiggle and jerk. It'll strain and struggle. *Don't let it go until it capitulates*! You are the master, the problem *must submit to you*!

6.4. Learn from Problem Solving

As you solve problems, it would really be nice if you could learn from your experiences.

6.5. Patterned Thought: The Butterfly Method

When you look at a integral problem, how should you think? Well, of course, you are at liberty to think anyway you wish — as long as it works for you. However, if do you lack a disciplined pattern of thought, I would put forth my one suggestions: the Butterfly Method.

Consider the following listing of formulas and techniques:

Integral Formulas and Techniques

Specific Formulas	**Techniques**
Power Rule	Homogeneity
Trig. (1)	Additivity
Trig. (2)	Substitution
Trig. (3)	**Algebraic Manipulation**
Trig. (4)	**Trigonometric Manipulation**
Trig. (5)	
Trig. (6)	

Butterfly Method

Problem. Solve $\int f(x)\,dx$.

Begin.

1. Beginning at the top of the left-hand column, labeled **Specific Formulas**, go down the list. For each formula in the list, determine

whether that formula solves the **Problem**. Use the formula checking technique here.

2. If successful, you are done, **Go to End**, else, **Go to Step 3**.

3. Beginning at the top of the right-hand column, labeled **Techniques**, go down the list. Choose a technique and apply it. Applying one or more techniques does not solve the **Problem**; what it does is to create one or more new integral problems. Now **Go to Step 1** and apply the Butterfly Method to *each* of unsolved integral problems.

End.

Butterfly Notes: The first formula in the list of **Specific Formulas** is the *Power Rule*; this is the *first formula you check*. The *Power Rule* can solve a variety of diversely different looking integrals. Use formula checking and use the Power Rule Strategy outlined earlier. Never overlook the *Power Rule*.

■ Many formulas in the **Specific Formula** list can be eliminated immediately. For example, if the integral does *not* have trig functions in it, obviously, the only specific formula that could possibly apply is the *Power Rule*. Of course, in *Calculus II*, we obtain more **Specific Formulas**, but until then, this simplified thinking is valid.

■ Even if the integrand involves trigonometric functions, test out the *Power Rule*. When testing whether one of the trig integral formulas apply, use the Trig Strategy.

■ At the *Calculus I* level, the **Techniques** used are fairly obvious: Manipulate the integrand algebraically, separate the integrand, if possible, using the Additive Property and factor out any constants using the Homogeneity Property. Manipulation by trigonometric identities is an option you'll see more often in *Calculus II*; the same is true for true substitution. The substitution technique is just formula checking.

■

We now present a series of examples to illustrate the Butterfly Method of solving indefinite integrals. Following those, is a series of exercises for the user — that's you.

Example 6.1. Evaluate $\displaystyle\int x^3(x^4+3)^{1/3}\,dx$.

Example 6.2. Evaluate $\displaystyle\int x^2(x^2+1)^2\,dx$.

Example 6.3. Evaluate $\displaystyle\int \frac{\sqrt{x}+1}{\sqrt{x}}\,dx.$

Exercise 6.2. The previous example can be solved another way. Can you?

Exercise 6.3. The last exercise and the last example yielded two seemingly different answers for the same integral. Resolve this apparent ambiguity. (*Hint*: Review Theorem 2.2)

Example 6.4. Evaluate $\displaystyle\int \frac{x^2}{\sqrt{x+1}}\,dx.$

The above examples have concentrated exclusively on integrands that were algebraic functions, here's a couple of examples involving trigonometric functions.

Example 6.5. Evaluate $\displaystyle\int \frac{\csc^2(\frac{1}{x})}{x^2}\,dx.$

One last example, and I'll turn it over to you.

Example 6.6. Evaluate $\displaystyle\int \frac{x\sin(x^2)}{\sqrt{\cos(x^2)}}\,dx.$

Power Rule Junior Grade:
Let $r \in \mathbb{Q}$ be a rational number, $r \neq -1$, then

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C.$$

The integral of x raised to a power, is x raised to one greater power, divided by that greater power ... plus an arbitrary constant.

Homogeneous Property:
For any constant c and any function f, we have

$$\int cf(x)\,dx = c\int f(x)\,dx.$$

The integral of a constant times a function is that contant times the integral of the function.

The Additivity of the Integral:
Let f and g be functions, then

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

The integral of the sum of two functions is the sum of the integrals of each.

Generalized Power Rule:
Let u be a function of some variable, and let $r \in \mathbb{Q}$ be a rational number, then

$$\int u^r \, du = \frac{u^{r+1}}{r+1} + C \qquad r \neq -1.$$

The integral of u raised to a power times the differential of u is the base function u raised to one greater power, divided by that greater power ... plus an arbitrary constant.

The Generalized Power Rule

a. The Generalized Power Rule itself.

b. Recognition of a power function.

c. General Strategy. Can I get the du?

d. Use of fudge factors. Yes, put in fudge factors.

e. Recognition when *Power Rule* does not apply. In this case try another technique.

f. Keep an eye out for the exceptional case $r = -1$. In this case, survive *Calculus I* and go into *Calculus II* to see the solution. Good Knowledge! (Not luck!)

Solutions to Exercises

2.1. A function H is said to be an antiderivative of h provided $H'(t) = h(t)$ for all t.

Exercise 2.1. ■

2.2. We verify this by differentiation. From the definition of anti-derivative, all we must do is check whether

$$F'(x) = f(x).$$

Well,

$$\begin{aligned} F'(x) &= \frac{d}{dx}(4x+1)^4 \\ &= 4(4x+1)^3 \frac{d}{dx}(4x+1) && \triangleleft \text{ power rule} \\ &= 4(4x+1)^3(4) \\ &= 16(4x+1)^3 \\ &= f(x) \end{aligned}$$

Exercise 2.2. ■

2.3. The answer is "Yes." I've switched letter on you, I hope that didn't confuse you—it's the ideas not the letters; comprehend the meaning of the ideas, don't be letter dependent.

Anyway, f is an antiderivative of g since

$$\begin{aligned} f'(t) &= \frac{d}{dt}(t^2+1)^2 \\ &= 2(t^2+1)\frac{d}{dt}(t^2+1) && \triangleleft \text{ power rule} \\ &= 2(t^2+1)(2t) && \triangleleft \text{ power rule} \\ &= 4t(t^2+1) \\ &= g(t). \end{aligned}$$

Exercise 2.3. ■

2.4. The answer is “No.” To see why, simply differentiate the function that is postulated to be an antiderivative of the other function. Indeed,

$$H'(s) = \frac{d}{ds}\cos(2s) = -\sin(2s)\frac{d2s}{ds} = -2\sin(2s)$$

Observe that the derivative of H is not the same as g:

$$H'(s) = -2\sin(2s) \neq 2\sin(2s) = g(s).$$

Therefore, we are entitled to say that H is *not* an antiderivative of g.

Exercise 2.4. ■

2.5. $F(x) = \frac{1}{8}x^8 + C$. (Check the answer by differentiating F. It should be true that $F'(x) = f(x)$)

Exercise 2.5. ■

2.6. $F(x) = \frac{4}{6}x^6 + C = \frac{2}{3}x^6 + C$. (Check the answer by differentiating F. It should be true that $F'(x) = f(x)$)

Exercise 2.6. ■

2.7. $F(x) = \frac{2}{3}x^6 + \frac{1}{8}x^8 + C$. (Check the answer by differentiating F. It should be true that $F'(x) = f(x)$.)

Exercise 2.7. ■

2.8. $F(x) = 3\sin(x) + C$. (Check the answer by differentiating F. It should be true that $F'(x) = f(x)$) Exercise 2.8. ■

2.9. $F(x) = 3\sin(x) - 4\cos(x) + C$. (Check the answer by differentiating F. It should be true that $F'(x) = f(x)$) Exercise 2.9. ■

2.10. $H(t) = \frac{1}{2}t^8 - 2t^3 + 10t + K$, where K is a constant. (Check the answer by differentiating H. It should be true that $H'(t) = h(t)$)

Exercise 2.10. ■

2.11. Define the function f by

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

Now for the interesting part—trying to explain why no antiderivative of f over the interval $(0, 1)$ exists.

Suppose such a function F did exist; that is suppose F is a function such that $F'(x) = f(x)$ for all $x \in (0, 1)$. Let's calculate the right-hand derivative of F as $x = \frac{1}{2}$ and see something weird happen!

Right-hand derivative:

$$\begin{aligned} 0 = f(\tfrac{1}{2}) &= F'(\tfrac{1}{2}) = F'_+(\tfrac{1}{2}) \\ &= \lim_{h \to 0^+} \frac{F(\frac{1}{2} + h) - F(\frac{1}{2})}{h} \end{aligned} \tag{A-1}$$

For $h > 0$, by the MEAN VALUE THEOREM, there is a number c_h, $\frac{1}{2} < c_h < \frac{1}{2} + h$ such that

$$\frac{F(\frac{1}{2}+h) - F(\frac{1}{2})}{h} = F'(c_h) = f(c_h) = 1. \tag{A-2}$$

The last equaltiy follows since $c_h > \frac{1}{2}$. (The notation c_h is designed to suggest that the value of 'c' given to us by the MEAN VALUE THEOREM depends on the interval $(\,\frac{1}{2}, \frac{1}{2} + h\,)$. The latter interval is ever changing since we are taking the limit as $h \to 0^+$; hence the value of c_h changes with h.)

Thus, from (A-1) and (A-2), it follows

$$0 = f(\tfrac{1}{2}) = F'_+(\tfrac{1}{2}) = \lim_{h \to 0^+} f(c_h) = \lim_{h \to 0^+} 1 = 1.$$

Oops! $0 = 1$—definitely a *contradiction*! A contradiction has insinuated itself into our logical system. How could that have happened? It comes from the assumption that an antiderivative for f existed! An antiderivative does not exist!.

Exercise Notes: The function f does have an antiderivative over the interval $(\,0, \frac{1}{2}\,)$ *and* f has an antiderivative over the interval $(\,\frac{1}{2}, 1\,)$, but not a single function F that is an antiderivative over the whole interval $(\,0, 1\,)$.

■ Find the antiderivative of f over the interval $(\,0, \frac{1}{2}\,)$ and find the antiderivative of f over the interval $(\,\frac{1}{2}, 1\,)$.

■ Go through the above 'proof' and *justify* each equality by citing definitions and theorems—make sure that you know 'the why' of each step. ■

Exercise 2.11. ■

2.12. One such example is the "salt and pepper function." Define the function f, for $x \in (0, 1)$, by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$

Note: There is nothing special about the interval $(0, 1)$.

The next question: How to prove that no antiderivative exists for this function?

This is the second challanging part of this problem. Try to do it yourself. Use the tools of *Calculus I* to make an argument. *Hint*: Assume there is a function F such that $F'(x) = f(x)$ for all $x \in (0, 1)$ and try to get a contradiction. You will find the MEAN VALUE THEOREM quit useful.

Proof that f has no antiderivative.

Exercise 2.12. ■

2.13. The integral of $\sin(t)$ with respect to t is $-\cos(t)$ plus an arbitrary constant C.

Exercise 2.13. ■

2.14. Here we are playing mind games with you. The symbol

$$\int f(x)\,dx, \tag{A-3}$$

by the description of the notation, represents an antiderivative of f. An antiderivativeof f is any function whose derivative is f; therefore the derivative of (A-3) is f, i.e., in symbolics,

$$\frac{d}{dx}\int f(x)\,dx = f(x).$$

"The derivative of an indefinite integral is the integrand."

Notice that you were not asked to evaluate the integral, but to differentiate it. This could be done without even without precise knowledge of the definition of the integrand function. Exercise 2.14. ■

2.15. Because of the presence of the symbol dx, we know that the variable of integration is x. This means that the indefinite integral

$$\int (x + \sin(x))^{10}\, dx \tag{A-4}$$

is considered to be a function of x. The integral (A-4) represents any function (of x) whose derivative (with respect to x) is the integrand (which is $(x + \sin(x))^{10}$). Therefore,

$$\frac{d}{dx} \int (x + \sin(x))^{10}\, dx = (x + \sin(x))^{10}.$$

Exercise 2.15. ■

2.16. An indefinite integral is an antiderivative of its integrand:

$$\frac{d}{ds}\int \tan^{12}(s)\,ds = \tan^{12}(s).$$

Exercise 2.16. ■

3.1. You should have deduced: By your phrase, "The integrand is 0," you obviously mean, Sir, that the integrand is the zero function defined by, if memory serves, $f(x) = 0$, for all x. Exercise 3.1. ■

3.2. The condition that $r \neq -1$ is necessary in order to avoid dividing by 0.

Exercise 3.2. ■

3.3. Notice that the base of the power function is x, the same as the variable of integration, as defined by dx.

$$\int x^{-3/4}\,dx = \frac{x^{1/4}}{1/4} + C = \boxed{4x^{1/4} + C.}$$

Division by $1/4$ is the same as multiplication by 4. Exercise 3.3. ■

3.4. The problem is to calculate

$$\int w^{7/3}\,dw.$$

The base function, w, is the same as the variable of integration, w, as determined by the differential dw. The power rule can safely applied:

$$\int w^{7/3}\,dw = \frac{w^{10/3}}{10/3} + C = \boxed{\frac{3}{10}w^{10/3} + C.}$$

Division by 10/3 is the same as multiplication by 3/10.

Exercise 3.4. ■

3.5. Now here's a bit of a spanner in the works! The given integral problem,

$$\int (2x)^4 \, dx,$$

is the integral of a power function; however, the base of the power function $(2x)^4$ is $2x$ which *does not match* the variable of integration, x, as defined by the dx. In this simple case, we can easily remove the spanner. Proceed as follows:

$$\begin{aligned} \int (2x)^4 \, dx &= \int 2^4 \, x^4 \, dx \\ &= 2^4 \int x^4 \, dx \\ &= 16 \frac{x^5}{5} + C \qquad \triangleleft \text{ Power Rule} \\ &= \frac{16}{5} x^5 + C. \end{aligned}$$

When we applied the *power rule*, we did so to the problem of integrating x^4. Now the base of this power function matches the variable of integration.

Exercise 3.5. ■

3.6. I'll follow my own advice. Hope you did too. We proceed in a methodical and organized way. Note that $\sqrt{u} = u^{1/2}$.

$$\begin{aligned} \int \frac{1}{u^{1/2}}\,du &= \int u^{-1/2}\,du \\ &= \frac{u^{1/2}}{1/2} + C \qquad \triangleleft \text{ Power Rule} \\ &= 2\sqrt{u} + C. \end{aligned}$$

Notice that the base of the power function $u^{-1/2}$ is u, the same as the variable of integration, as defined by du. Exercise 3.6. ■

3.7. The integrand is $f(x) = x^2\sqrt{x}$. We cannot integrate this function as it is now expressed because it is not written as a power function. We must do that

$$f(x) = x^2\sqrt{x} = x^2x^{1/2} = x^{5/2}.$$

Thus,

$$\begin{aligned}\int x^2\sqrt{x}\,dx &= \int x^{5/2}\,dx \\ &= \frac{x^{7/2}}{7/2} + C \\ &= \frac{2}{7}x^{7/2} + C \\ &= \frac{2}{7}x^3\sqrt{x} + C\end{aligned}$$

We have a limited number of formulas to evaluate an integral; therefore, we must sometimes manipulate the integrand so that the problem fits into one of our formulas. Practically, the only formula we have is the Power Rule so we must try to make the integrand into a power function.

Exercise 3.7. ∎

3.8. The only way we can solve this problem is if the integrand is a power function. It is ... trust me; I know the person who made this problem up!

$$\begin{aligned}
\int \frac{\sqrt{t}}{t^3}\,dt &= \int \frac{t^{1/2}}{t^3}\,dt \\
&= \int t^{-5/2}\,dt \\
&= \frac{t^{-3/2}}{-3/2} + C \qquad \triangleleft \text{ Power Rule} \\
&= -\frac{2}{3}t^{-3/2} + C \\
&= -\frac{2}{3t\sqrt{t}} + C
\end{aligned}$$

Here is an important point: When you use the Power Rule, your power function *must be in the numerator*.

The ideas and techniques do not depend on the variable of integration.

Exercise 3.8. ■

3.9. Just apply the Power Rule for the case $r = 0$.

$$\int dx = x + C$$

$$\int du = u + C.$$

Stare at these equations. One gets the feeling that the *Int* symbol cancels out the d to get x and u.

$$\int dz = z + C$$

$$\int dw = w + C.$$

Exercise 3.9. ■

3.10. We use (1),

$$\int \cos(t)\, dt = \sin(t) + C.$$

The formulas are independent of the choice of the symbol to denote the *variable of integration*. Exercise 3.10. ■

3.11. We use (4),

$$\int \csc^2(s)\,ds = -\cot(s) + C.$$

The formulas are independent of the choice of the symbol to denote the *variable of integration.* Exercise 3.11. ■

3.12. None of the formulas apply. This is, in fact, a trick question. You should have realized that

$$\sin^2(x^3) + \cos^2(x^3) = 1,$$

thus,

$$\int \sin^2(x^3) + \cos^2(x^3)\,dx = \int 1\,dx = \int dx = x + C,$$

by the Power Rule.

Exercise 3.12. ■

3.13. Formula (4) *claims* that

$$\int \csc^2(x)\,dx = -\cot(x) + C.$$

The right-hand side is supposed to be an antiderivative of the integrand.

$$\frac{d}{dx}\big(-\cot(x) + C\big) = -\frac{d}{dx}\cot(x) = -\big(-\csc^2(x)\big) = \csc^2(x),$$

where we have used the fact that the derivative of a constant term, C, is zero, so we dropped it out of the calculations early; and the trig differentiation formulas (6). Thus the derivative of the *answer* is the integrand; this means that the answer is, indeed, an antiderivative of the integrand.

This is how you verify an integration formula. Exercise 3.13. ■

3.14. We use good notation and techniques:

$$\begin{aligned}\int 4x^6\,dx &= 4\int x^6\,dx && \triangleleft \text{ Homog. Prop.}\\ &= 4\frac{x^7}{7} + C && \triangleleft \text{ Power Rule}\\ &= \frac{4}{7}x^7 + C.\end{aligned}$$

Above is the proper presentation and thinking. You should consciously think the thoughts that justify each step — that will reenforce the rules.

Exercise 3.14. ■

3.15. The height of triviality. We concentrate, therefore, on *style*.

$$\begin{aligned} \int 6t\sqrt{t}\,dt &= 6\int t\sqrt{t}\,dt && \triangleleft \text{ Homogen. Prop.} \\ &= 6\int t^{3/2}\,dt \\ &= 6\frac{2}{5}t^{5/2} + C && \triangleleft \text{ Power Rule} \\ &= \frac{12}{5}t^2\sqrt{t} + C. \end{aligned}$$

I have left the answer in the same radical notation in which the original problem was posed. 'Nuff said. Exercise 3.15. ■

3.16. We utilize our tool kit of techniques.

$$\begin{aligned}
\int \frac{2}{3}x^6 - 8x^{12}\,dx &= \int \frac{2}{3}x^6\,dx - \int 8x^{12}\,dx && \triangleleft \text{ Additive Prop.}\\
&= \frac{2}{3}\int x^6\,dx - 8\int x^{12}\,dx && \triangleleft \text{ Homogen. Prop.}\\
&= \frac{2}{3}\frac{x^7}{7} - 8\frac{x^{13}}{13} + C && \triangleleft \text{ Power Rule}\\
&= \frac{2}{21}x^7 - \frac{8}{13}x^{13} + C.
\end{aligned}$$

I hope you used good techniques.

Exercise 3.16. ■

3.17. We use standard techniques,

$$\begin{aligned}
\int 8\sec^2(x) - 6\sec(x)\tan(x)\,dx \\
&= \int 8\sec^2(x)\,dx - \int 6\sec(x)\tan(x)\,dx && \triangleleft \text{ Additive Prop.} \\
&= 8\int \sec^2(x)\,dx - 6\int \sec(x)\tan(x)\,dx && \triangleleft \text{ Homogen. Prop.} \\
&= 8\tan(x) - 6\sec(x) + C && \triangleleft \text{ Trig. (3) \& (5)}
\end{aligned}$$

All these demonstrations are alike!

Exercise 3.17. ■

3.18. None of our specific integral formulas apply immediately: The integrand is not a power function, the integrand does not involve trigonometric functions. These are the types of functions we can integrate.

Whenever the specific integration formulas do not apply, we must transform the problem into another problem or problems using the general formulas, or by directly manipulating the integrand, then applying the general formulas. We elect the latter.

The Integrand:

$$(t^4 - 4t^3)^2 = t^8 - 8t^7 + 16t^3,$$

where, I have squared the binomial by *verbalizing*: *The square of a sum is the square of the first plus twice the product of the first and second, plus the square of the second.*

Thus

$$\begin{aligned}\int (t^4 - 4t^3)^2 \, dt &= \int t^8 - 8t^7 + 16t^3 \, dt \\ &= \frac{1}{9}t^9 - t^8 + 4t^4 + C\end{aligned}$$

Exercise 3.18. ■

3.19. You should not have encountered any technical difficulties preventing the successful completion of this problem.

The Integrand:
$$\left(w^3 - \frac{1}{w^2}\right)^2 = (w^3 - w^{-2})^2$$
$$= w^6 - 2w + w^{-4}$$

Evaluation:
$$\int \left(w^3 - \frac{1}{w^2}\right)^2 dw = \int w^6 - 2w + w^{-4}\, dw$$
$$= \frac{1}{7}w^7 - w^2 + \frac{w^{-3}}{-3} + C$$
$$= \frac{1}{7}w^7 - w^2 - \frac{1}{3}w^{-3} + C$$
$$= \frac{1}{7}w^7 - w^2 - \frac{1}{3w^3} + C$$

Exercise 3.19. ■

3.20. We must square the integrand.

$$(\sec(x) + \tan(x))^2 = \sec^2(x) + 2\sec(x)\tan(x) + \tan^2(x).$$

Keeping in mind we want to integrate the above function, we realize that the integrals of the first and second terms are exact integral formulas; the third term, $\tan^2(x)$ is a problem. However,

$$\sec^2(x) - \tan^2(x) = 1$$

or,

$$\tan^2(x) = \sec^2(x) - 1.$$

We can integrate the constant 1, and we can integrate the function $\sec^2(x)$. I leave the rest of the demonstration to you.

Answer:

$$\int (\sec(x) + \tan(x))^2 \, dx = 2(\tan(x) + \sec(x)) - x + C.$$

Exercise 3.20. ■

4.1. If $u = 3x + 1$, then $du = 3\,dx$, or $dx = \frac{1}{3}\,du$. Thus,

$$\begin{aligned}
\int (3x+1)^{20}\,dx &= \int u^{20}\frac{1}{3}\,du && \triangleleft \text{ Substitution} \\
&= \frac{1}{3}\int u^{20}\,du && \triangleleft \text{ Homogen.} \\
&= \frac{1}{3}\frac{1}{21}u^{21} + C && \triangleleft \text{ Power Rule} \\
&= \frac{1}{63}(3x+1)^{21} + C && \triangleleft \text{ since } u = 3x+1
\end{aligned}$$

Exercise 4.1. ■

4.2. Let $u = 2x$, so $du = 2\,dx$, or $dx = \frac{1}{2}\,du$. Then

$$\begin{aligned}
\int \sec^2(3x)\,dx &= \int \sec^2(u)\frac{1}{3}\,du && \triangleleft \text{ Substitution} \\
&= \frac{1}{3}\int \sec^2(u)\,du \\
&= \frac{1}{3}\tan(u) + C && \triangleleft \text{ Trig. (3)} \\
&= \frac{1}{3}\tan(3x) + C && \triangleleft \text{ since } u = 3x
\end{aligned}$$

Thus,

$$\boxed{\int \sec^2(3x)\,dx = \frac{1}{3}\tan(3x) + C.}$$

Did you check your answer *before* reading the solution?

Exercise 4.2. ■

4.3. If we want to solve the integral,

$$\int x^8 (6x^9 + 12)^{1/3}\, dx,$$

using the Power Rule, then we must choose u to base of a power function. The rest of the integrand must be directly proportional to the du of your chosen u.

Let $u = 6x^9 + 12$, then $du = 54x^8\, dx$. Now, taking the integral and rearranging the integrand,

$$\int (6x^9 + 12)^{1/3}\, x^8\, dx,$$

we see that the $x^8\,dx$ is directly proportional to the du. Success! Continuing now,

$$\begin{aligned}
\int (6x^9+12)^{1/3}\,x^8\,dx \\
&= \frac{1}{54}\int (6x^9+12)^{1/3}\,54x^8\,dx && \triangleleft \text{ insert fudge factors} \\
&= \frac{1}{54}\int u^{1/3}\,du && \triangleleft \text{ substitution} \\
&= \frac{1}{54}\frac{u^{4/3}}{4/3}+C && \triangleleft \text{ Power Rule} \\
&= \frac{1}{54}\frac{3}{4}(6x^9+12)^{4/3}+C && \triangleleft \text{ re-subsititute} \\
&= \frac{1}{72}(6x^9+12)^{4/3}+C
\end{aligned}$$

I hope your arrived at the conclusion:

$$\boxed{\int (6x^9 + 12)^{1/3}\, x^8\, dx = \frac{1}{72}(6x^9 + 12)^{4/3} + C}$$

By the way, let us agree that the insertion of the constant of proportionality into the integral be referred to as the "fudge factor."

Exercise 4.3. ■

4.4. We proceed along standard lines,

$$\int \frac{x}{\sqrt{4-3x^2}}\,dx = \int (4-3x^2)^{-1/2}\,x\,dx$$

Let $u = 4 - 3x^2$, $du = -6x\,dx$. The *Power Rule* is applicable since every thing left over after the power function part, $(4-3x^2)^{-1/2}$ is directly proportional to the du. All we have to do is insert our fudge factors:

$$\begin{aligned}
&\int (4-3x^2)^{-1/2}\,x\,dx \\
&\quad = -\frac{1}{6}\int (4-3x^2)^{-1/2}\,(-6x)\,dx && \triangleleft \text{ fudge factors} \\
&\quad = -\frac{1}{6}\int u^{-1/2}\,du && \triangleleft \text{ Substitituion} \\
&\quad = -\frac{1}{6}\,\frac{u^{1/2}}{1/2} + C && \triangleleft \text{ Power Rule} \\
&\quad = -2\frac{1}{6}(4-3x^2)^{1/2} + C && \triangleleft \text{ re-substitution}
\end{aligned}$$

$$= -\frac{1}{3}\sqrt{4-3x^2} + C$$

Exercise 4.4. ■

4.5. Let

$$u = 3x^4 - 4x + 1$$
$$du = 12x^3 - 4\,dx$$

or,

$$du = 4(3x^3 - 1)\,dx$$

Write the power function first,

$$\int (3x^4 - 4x + 1)^{1.45}\,(3x^3 - 1)\,dx.$$

Is the rest of the integrand, following the power function, directly proportional to the du? *Yes*! The *Power Rule* applies, and we are home free,

$$\begin{aligned}
\int (3x^4 - 4x + 1)^{1.45}\,(3x^3 - 1)\,dx \\
&= \frac{1}{4}\int (3x^4 - 4x + 1)^{1.45}\,4(3x^3 - 1)\,dx && \triangleleft \text{ fudge} \\
&= \frac{1}{4}\int u^{1.45}\,du && \triangleleft \text{ Sub.} \\
&= \frac{1}{4}\frac{u^{2.45}}{2.45} + C && \triangleleft \text{ Power Rule} \\
&= \frac{.25}{2.45}(3x^4 - 4x + 1)^{2.45} + C && \triangleleft \text{ re-sub.} \\
&= \frac{5}{49}(3x^4 - 4x + 1)^{2.45} + C
\end{aligned}$$

This problem is the same as the previous problems. The only difference is a more complicated base function u, which lead to a more complicated du. If you kept a cool head and followed the strategy you should have come out fine. Exercise 4.5. ■

4.6. Let $u = 4x$, $du = 4\,dx$. Then,

$$\begin{aligned}
\int \sec^2(4x)\,dx &= \frac{1}{4}\int \sec^2(4x)\,4\,dx && \triangleleft \text{ rearrange and fudge} \\
&= \frac{1}{4}\int \sec^2(u)\,du && \triangleleft \text{ substitute} \\
&= \frac{1}{4}\tan(u) + C && \triangleleft \text{ Trig. (3)} \\
&= \frac{1}{4}\tan(4x) + C && \triangleleft \text{ re-substitute}
\end{aligned}$$

Exercise 4.6. ■

4.7. You were asked to integrate the *sine of some function of* x: try the sine formula.

Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$. Then

$$\begin{aligned}
\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx &= \int \sin\sqrt{x}\,\frac{1}{\sqrt{x}}\,dx && \triangleleft \text{ re-arrange integrand}\\
&= 2\int \sin\sqrt{x}\,\frac{1}{2\sqrt{x}}\,dx && \triangleleft \text{ fudge}\\
&= 2\int \sin(u)\,du && \triangleleft \text{ substitution}\\
&= -2\cos(u) + C && \triangleleft \text{ Trig. (2)}\\
&= -2\cos\sqrt{x} + C && \triangleleft \text{ re-substitute}
\end{aligned}$$

Exercise 4.7. ■

4.8. Here, there are two possibilities for u.

First Analysis: We have a $\sec(x^2)\tan(x^2)$ combo; this is a form that appears in Trig. (5). In this case we are forced to say: Let $u = x^2$, since this is the argument of the trig functions, and $du = 2x\,dx$. Looking at our integral,

$$\begin{aligned}
&\int x\sec(x^2)\tan(x^2)\sec^2(\sec(x^2))\,dx \\
&\qquad = \int x\sec(x^2)\tan(x^2)\sec^2(\sec(x^2))\,x\,dx \\
&\qquad = \frac{1}{2}\int \sec(x^2)\tan(x^2)\sec^2(\sec(x^2))\,2x\,dx.
\end{aligned}$$

Thus, we can get our du, but we still have stuff left over. The factor $\sec^2(\sec(x^2))$ is left unaccounted for. Therefore, this attempt at using the Trig. formulas does not work.

Rather than throwing down our pencils and giving up, we'll try again.

Second Analysis: We have a $\sec^2$ function with a complicated argument $\sec(x^2)$. Let's try to use Trig. (3). In that case, we are forced to let u be equal to the argument of the sec^2 function. Let

$$\begin{aligned} u &= \sec(x^2) \\ du &= \sec(x^2)\tan(x^2)\,2x\,dx. \end{aligned}$$

Let's re-arrange our integral in an esthetically pleasing way:

$$\int x\sec(x^2)\tan(x^2)\sec^2(\sec(x^2))\,dx$$

$$= \int \sec^2(\sec(x^2))\ \sec(x^2)\tan(x^2)\,x\,dx.$$

Notice, everything following the $\sec^2$ factor is directly proportional to our du. Therefore, Trig. (3) will apply.

$$\begin{aligned}
&\int x \sec(x^2) \tan(x^2) \sec^2(\sec(x^2))\, dx \\
&\qquad = \frac{1}{2} \int \sec^2(\sec(x^2))\ \sec(x^2) \tan(x^2)\, 2x\, dx && \triangleleft \text{ fudge it} \\
&\qquad = \frac{1}{2} \int \sec^2(u)\, du && \triangleleft \text{ substitution} \\
&\qquad = \frac{1}{2} \tan(u) + C && \triangleleft \text{ Trig. (3)} \\
&\qquad = \frac{1}{2} \tan(\sec(x^2)) + C && \triangleleft \text{ re-substitute}
\end{aligned}$$

Despite the ugliness of the original problem, the given integral was just

$$\int \sec^2(u)\, du,$$

we just had to find the correct u. Exercise 4.8. ■

5.1. The referenced formula is

$$\int \sec^2(u)\,du = \tan(u) + C. \tag{A-5}$$

Our given integral is

$$\int \sec^2(3x^2)\ dx. \tag{A-6}$$

Now, in the formula (A-5), the u is the argument of the secant. If (A-5) is to solve our given integral, then we are forced to say

$$\text{Let } u = 3x^2 \text{ and } du = 6x\,dx.$$

If the formula (A-5) is to apply, everything following the $\sec^2(3x^2)$ must by the du. We don't have the du, but what we do have is off by a multiplicative constant — that good enough.

$$\int \sec^2(3x^2)\,x\,dx = \frac{1}{6}\int \sec^2(3x^2)\,6x\,dx$$

All the parts of the given integral are properly lined up with the corresponding parts of our chosen integral formula. (The correspondence being setup by the device of substitution.) Therefore,

$$\int \sec^2(3x^2)\, x, dx = \frac{1}{6} \int \underbrace{\sec^2(3x^2)}_{\sec^2(u)}\, \underbrace{6x\, dx}_{du}$$

$$= \boxed{\frac{1}{6} \tan(3x^2) + C.}$$

There is no real need to make the substitution. Exercise 5.1. ■

5.2. As suggested in the *Hint*:

$$\text{Let } u = 3x - 2, \text{ or } x = \frac{1}{3}(u+2), \text{ and so } dx = \frac{1}{3}\,du,$$

Take our integral now and replace the x's with the u's, *and* substitute for dx — very important!

$$\begin{aligned}
\int x(3x-2)^{-4}\,dx & \\
&= \int \frac{1}{3}(u+2)u^{-4}\,\frac{1}{3}\,du \\
&= \frac{1}{9}\int u^{-3} + 2u^{-4}\,du \\
&= \frac{1}{9}\left(-\frac{1}{2}u^{-2} - \frac{2}{3}u^{-3}\right) + C \\
&= \frac{1}{9}\left(-\frac{1}{2}(3x-2)^{-2} - \frac{2}{3}(3x-2)^{-3}\right) + C
\end{aligned}$$

Same problem as my two examples previously.

Exercise 5.2. ■

5.3. Again, we can check that the power rule does not solve the integral:

$$\int x^2 (6x+1)^{1/2}\,dx.$$

If you tried the suggested substitution, and followed the previous examples, you should be reading what you already know. So,

$$\text{Let } u = 6x + 1. \text{ Thus, } x = \frac{1}{6}(u-1), \text{ and so } dx = \frac{1}{6}\,dx.$$

The purpose of this substitution is to shift the binomial expression $(6x + 1)$ from underneath the $-1/2$ power, and move a new binomial expression to the squared term. Let's see if this, if fact, happens:

$$\begin{aligned}
\int x^2(6x+1)^{-1/2}\,dx \\
&= \int \frac{1}{36}(u-1)^2 u^{-1/2}\,\frac{1}{6}\,du \qquad \triangleleft \text{ substitution} \\
&= \frac{1}{216}\int (u^2-2u+1)u^{-1/2}\,du \\
&= \frac{1}{216}\int u^{3/2}-2u^{1/2}+u^{-1/2}\,du \\
&= \frac{1}{216}\left(\frac{2}{5}u^{5/2}-\frac{4}{3}u^{3/2}+2u^{1/2}\right)+C \\
&= \frac{1}{108}u^{1/2}\left(\frac{1}{5}u^2-\frac{2}{3}u+1\right)+C \\
&= \frac{1}{108}(6x+1)^{1/2}\left(\frac{1}{5}(6x+1)^2-\frac{2}{3}(6x+1)+1\right)+C
\end{aligned}$$

Exercise 5.3. ■

6.1. Let $u = \sin(\cos(x^2))$. Then

$$\begin{aligned} u &= \sin(\cos(x^2)) \\ du &= \cos(\cos(x^2))(-\sin(x^2))(2x)\,dx \\ &= -2x\sin(x^2)\cos(\cos(x^2))\,dx \end{aligned}$$

Thus,

$$\begin{aligned} &\int x\sin(x^2)\cos(\cos(x^2))\sin^3(\cos(x^2))\,dx \\ &\quad= \int \sin^3(\cos(x^2))\,x\sin(x^2)\cos(\cos(x^2))\,dx \\ &\quad= -\frac{1}{2}\int \sin^3(\cos(x^2))\,(-2x)\sin(x^2)\cos(\cos(x^2))\,dx \\ &\quad= -\frac{1}{2}\int u^3\,du \\ &\quad= -\frac{1}{2}\frac{1}{4}u^4 + C \end{aligned}$$

$$= \boxed{-\frac{1}{8}\sin^4(\cos(x^2)) + C.}$$

Exercise 6.1. ■

6.2. We manipulate algebraically the integrand.

$$\frac{\sqrt{x}+1}{\sqrt{x}} = 1 + \frac{1}{\sqrt{x}} = 1 + x^{-1/2}.$$

The power rule can be applied — the ball is in your court.

Exercise 6.2. ■

6.3. Let's summarize the results.

EXAMPLE 6.3:

$$\int \frac{\sqrt{x}+1}{\sqrt{x}}\,dx = (\sqrt{x}+1)^2 + C.$$

EXERCISE 6.2:

$$\int \frac{\sqrt{x}+1}{\sqrt{x}}\,dx = x + 2x^{1/2} + C.$$

If *both* of these "answers" are correct, they should *both* be antiderivatives of the integrand. By Theorem 2.2, these two "answers" should differ by a constant (actually, the way I phrased it in the theorem was that one function is equal to the other plus a constant). Let's check it out:

$$\begin{aligned}(\sqrt{x}+1)^2 - (x + 2x^{1/2}) &= (x + 2\sqrt{x} + 1) - (x + 2\sqrt{x}) \\ &= 1\end{aligned}$$

The two "answers" indeed differ by a constant — completely consistent with general theory. Thank goodness. $\mathfrak{DP}$

Exercise 6.3. ■

Solutions to Examples

2.1. Define $F(x) = \frac{1}{4}x^4$. Then by the rules of differentiation, $F'(x) = x^3 = f(x)$; therefore, by Definition 2.1, we are entitled to say that F is an antiderivative, or that F is an indefinite integral, of f.

Notice that we could have defined $F(x) = \frac{1}{4}x^4 + 1$, then this "new" function F would still be an antiderivative of f since $F'(x) = f(x)$ as well. (This is because the derivative of a constant term is 0.)

More generally, and function of the form $F(x) = \frac{1}{4}x^4 + C$, there C is any constant is an antiderivative of f. Example 2.1. ■

2.2. The rock has an initial velocity of $v_0 = 50$ (feet per second) and is initially, $s_0 = 6$ (feet) off the ground. Therefore,

$$s_0 = 6 \qquad v_0 = 50. \tag{S-1}$$

Solution to (a): Find the equation (13) that specifies the height $s(t)$ of the rock above the ground at time t.

We know from (13) that

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0.$$

Update this equation using the data in (S-1):

$$s(t) = -\frac{1}{2}gt^2 + 50t + 6.$$

Because the scale of measurement is in the English system of measurement, we know that $g = 32\,\text{ft/sec}^2$. Substituting this in we get

$$\boxed{s(t) = -16t^2 + 50t + 6.} \tag{S-2}$$

Solution to (b): How high is the rock off the ground 5 seconds after the rock leaves your hand?

We are being asked to take our mathematical model (S-2) and substitute $t = 1$ into it.

$$s(1) = -16(1)^2 + 50(1) + 6$$

$$\boxed{s(5) = 40\,\text{feet}}$$

Solution to (c): How long before the rock hits the ground?

The rock hits the ground when the distance off the ground is 0. The function $s(t)$ is the distance of the rock from the ground; therefore, we put

$$s(t) = 0.$$

This equation asks the question: At what time t is $s(t) = 0$? Replace $s(t)$ with (S-2):

$$-16t^2 + 50t + 6 = 0.$$

This is a second degree polynomial put equal to zero: This is a job for the *quadratic formula*:

$$\begin{aligned} t &= \frac{-50 \pm \sqrt{50^2 - 4(-16)(5)}}{-32} \\ &= -\frac{-50 \pm \sqrt{2820}}{32} \\ &= \frac{50 \pm \sqrt{2820}}{32} \end{aligned}$$

There are two solutions. We are interested in the one where $t > 0$ — since the rock is thrown at time $t = 0$, it will hit the ground sometime after time 0. Whipping out my calculator and choosing the positive solution we get,

$$\boxed{t \approx 3.322\,\text{sec.}} \tag{S-3}$$

Here, it have used the symbol $\approx$ to indicate that my calculated value of t is only *approximate*; it is accurate to 3 decimal places.

Solution to (d): What is the velocity of the rock when it hits the ground?

The velocity of the rock at any time t is $v(t) = \dfrac{ds}{dt}$.

$$s(t) = -16t^2 + 50t + 6 \qquad \triangleleft \text{ from (S-2)}$$
$$v(t) = -32t + 50 \tag{S-4}$$

Taking our velocity expression for the velocity of the rock, (S-4), and putting in the time value, (S-3), of when the rock hit the ground we get,

$$v(3.222) \approx -32(3.222) + 50 = -53.104.$$

Thus,

$$\boxed{v(3.222) \approx -53.104\,\text{ft/sec}}$$

Again, the $\approx$ means “approximately equal to,” this notation is called for in the velocity calculation because the value of t inserted was only approximate. The interpretation of the negative velocity, is that the rock is going 53.104 ft/sec *downward¿*

Solution to (e): At what time is the rock 6 feet above the ground?

This problem asks the question: For what value of t is it true that

$$s(t) = 6?$$

Or,

$$-16t^2 + 50t + 6 = 6.$$

This is just an exercise in solving equations.

$$\begin{aligned} -16t^2 + 50t + 6 &= 6 \\ -16t^2 + 50t &= 0 \\ 2t(25 - 8t) &= 9 \end{aligned}$$

Therefore, $t = 0$ or $t = 25/8$. At time $t = 0$ the rock is at 6 feet when it left your hand — that makes sense. The rock goes up, then comes down. Eventually, it attains a height of 6 feet again — at time

$$\boxed{t = 25/8 = 3\tfrac{1}{8}.} \qquad \text{(S-5)}$$

seconds after it leaves you hand.

Solution to (f): What is the velocity of the rock when the rock is 6 feet off the ground?

The velocity equation is $v(t) = -32t + 50$. From the previous part, the time when the rock reaches a height of 6 feet is $t = 25/8$, (S-5); therefore,

$$v(25/8) = -32\frac{25}{8} + 50 = -50$$

$$\boxed{v(25/8) = -50\,\text{ft/sec.}}$$

The interpretation of the *negative sign* is that the rock is moving *downward* when it attains a height of 6 feet.

Solution to (g): How high does the rock go?

We can determine how high the rock goes by making this simple observation: When the rock reaches its highest point, its velocity is 0.

We then ask ourselves the question: At what time is $v(t) = 0$?

$$\begin{aligned} v(t) &= 0 \\ -32t + 50 &= 0 \\ t &= \frac{-50}{-32} \\ t &= \frac{25}{16}. \end{aligned}$$

How high now is the rock with the time is $t = 25/16$? From (S-2)

$$\begin{aligned} s(26/16) &= -16(25/16)^2 + 50(25/16) + 5 \\ &= \frac{721}{16}. \end{aligned}$$

Thus,

$$\boxed{s(26/16) = \frac{721}{16} \text{ feet}}$$

is how high the rock goes. Example 2.2. ■

3.1.

$$\begin{aligned}\int \frac{1}{x^2}\,dx &= \int x^{-2}\,dx \\ &= \frac{x^{-1}}{-1} + C && \triangleleft \text{ Power Rule} \\ &= -\frac{1}{x} + C.\end{aligned}$$

Notice the base, x, of the power function, x^{-2}, is x, the variable of integration. Example 3.1. ■

3.2. We must argue that the right-hand side of

$$\int cf(x)\,dx = c\int f(x)\,dx.$$

is an antiderivative of the integrand of the left-hand side. Indeed,

$$\frac{d}{dx}c\int f(x)\,dx = c\frac{d}{dx}\int f(x)\,dx \tag{S-6}$$

$$= cf(x). \tag{S-7}$$

The equality in line (S-6) comes from the Homogeneous Property for differentiation. The equality of line (S-7) comes from the definition of the symbolism. The symbol $\int f(x)\,dx$ stands for any function whose derivative is $f(x)$; consequently, if we differentiate it, we get $f(x)$. See Exercise 2.14 for more details — if you have forgotten them.

Example 3.2. ■

3.3. As always, we use good techniques.

$$\begin{aligned}
\int 3x^4 + 6x^2\,dx &= \int 3x^4\,dx + \int 6x^2\,dx && \triangleleft \text{ Additive Prop.}\\
&= 3\int x^4\,dx + 6\int x^2\,dx && \triangleleft \text{ Homogen. Prop.}\\
&= 3\frac{x^5}{5} + 6\frac{x^3}{3} + C && \triangleleft \text{ Power Rule}\\
&= \frac{3}{5}x^5 + 2x^3 + C.
\end{aligned}$$

Do you see how we are slowly building up a kit of tools to handle integration problems?

There is a temptation to skip many of these steps, but I would advise against such a course. At first, methodically, go through all the steps, let the proper thinking flow through your brain a number of times before embarking on the potentially more dangerous course of skipping steps.

Example 3.3. ■

4.1. The integral,

$$\int (x+1)^{15}\,dx \tag{S-8}$$

can be evaluated using our current techniques: multiply out the integrand and integrate each term separately using the *Power Rule*. Good Luck! But I don't want to do it that way.

I'll make a substitution. Let $u = x + 1$ and so $du = dx$. Now formally substituting these into the given integral we obtain,

$$\int (x+1)^{15}\,dx = \int u^{15}\,du.$$

This new integral can be solved by the basic *Power Rule*,

$$\begin{aligned} \int (x+1)^{15}\,dx &= \int u^{15}\,du && \triangleleft \text{ Substitution} \\ &= \frac{1}{16}u^{16} + C && \triangleleft \text{ Power Rule} \\ &= \frac{1}{16}(x+1)^{16} + C && \triangleleft \text{ since } u = x+1 \end{aligned}$$

Example 4.1. ■

4.2. The integral problem

$$\int (2x+1)^{15}\,dx, \tag{S-9}$$

is conceptually the same as the previous problem: It is a degree one polynomial raised to a large degree. Make a substitution: Let

$$u = 2x + 1,$$

then,

$$du = 2\,dx.$$

We want to make the substitution. The strategy is to replace the variable x *and* the differential dx with the new variable u and du. Note that

$$du = 2\,dx \implies dx = \frac{1}{2}\,du.$$

Now let's substitute the pair,

$$u = 2x + 1$$

$$du = \frac{1}{2}\,dx$$

into (S-9):

$$\begin{aligned}
\int (2x+1)^{15}\,dx &= \int u^{15}\frac{1}{2}\,du && \triangleleft \text{ Substitution}\\
&= \frac{1}{2}\int u^{15}\,du && \triangleleft \text{ Homogen}\\
&= \frac{1}{2}\frac{1}{16}u^{16}+C && \triangleleft \text{ Power Rule}\\
&= \frac{1}{32}(2x+1)^{16}+C && \triangleleft \text{ since} u = 2x+1
\end{aligned}$$

Thus,

$$\boxed{\int (2x+1)^{15}\,dx = \frac{1}{32}(2x+1)^{16}+C.}$$

Example 4.2. ■

4.3. You'll notice that our cosine integral formula (1) does not apply. That formula states

$$\int \cos(x)\,dx = \sin(x) + C.$$

The choice of the variable of integration is unimportant. The key point is that the argument of the cosine function x matches the dx; by *match* I mean that the argument of x is the variable of integration, as defined by the differential dx. In our problem

$$\int \cos(2x)\,dx, \tag{S-10}$$

the argument of the cosine function $2x$ does not match the dx; that is, we are not taking the cosine of the variable of integration, but the cosine of twice the variable of integration; therefore, the formula does not apply. However, we can *make* is apply using the technique of *substitution*. Let

$$u = 2x$$
$$du = 2\,dx$$

or,

$$u = \frac{1}{2}\,du$$

Now substituting these equations into (S-10) to get,

$$\int \cos(2x)\,dx = \int \cos(u)\frac{1}{2}\,du \qquad \triangleleft \text{ Substitution}$$

Notice now that the argument of the cosine function in our *new integral* is u, which exactly matches the du. Continuing now,

$$\begin{aligned} \int \cos(2x)\,dx &= \int \cos(u)\frac{1}{2}\,du && \triangleleft \text{ Substitution} \\ &= \frac{1}{2}\int \cos(u)\,du \\ &= \frac{1}{2}\sin(u) + C && \triangleleft \text{ Trig. (1)} \\ &= \frac{1}{2}\sin(2x) + C && \triangleleft \text{ since } u = 2x \end{aligned}$$

Thus,

$$\boxed{\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C.}$$

You can always *check your answer* by differentiating the answer to obtained the integrand:

Check:

$$\frac{d}{dx}\frac{1}{2}\sin(2x) + C = \frac{1}{2}\cos(2x)\frac{d}{dx}2x = \cos(2x).$$

Checked!

Example 4.3. ■

4.4. The problem is

$$\int (5x - 3)^9 \, dx. \tag{S-11}$$

The first observation is that this is the integral of a function raised to a fixed power: a power function. The Power Rule for integration is designed to integration power functions; therefore, we investigate the power rule.

Look at the new *Generalized Power Rule*,

$$\int u^r \, du = \frac{u^{r+1}}{r+1} + C \qquad r \neq -1.$$

and compare it with your integral (S-11). We want to use the power rule to solve our problem. To do this, you have to set up a correspondence between your problem and the power rule; this correspondence is setup through the technique of substitution.

What should u in the Power Rule be. Just look at the rule. The variable u is the *base of the power function*. Therefore, in our problem, (S-11), we would set u equal to the base of the power function. Let

$$\begin{aligned} u &= 5x - 3 \\ du &= 5\,dx \end{aligned}$$

and so,

$$dx = \frac{1}{5}\,dx$$

We now take our integral, and substitute for x *and* for dx.

$$\begin{aligned} \int (5x-3)^9\,dx &= \int u^9\,\frac{1}{5}\,du && \triangleleft \text{ Substitution} \\ &= \frac{1}{5}\,\frac{u^{10}}{10} + C && \triangleleft \text{ Power Rule} \\ &= \frac{1}{50}(5x-3)^{10} + C && \triangleleft \text{ resubstitute} \end{aligned}$$

Thus,

$$\boxed{\int (5x-3)^9\,dx = \frac{1}{50}(5x-3)^{10} + C.}$$

Example 4.4. ■

4.5. Now the integrand of the problem

$$\int x(3x^2 - 5)^{3/4}\, dx,$$

consists of two factors: x and $(3x^2 - 5)^{3/4}$. Both are power functions, the latter one more complicated than the former. We are determined to use the Power Rule. How should we assign the value of u in the formula? Keeping in mind that in the power rule formula, u is the *base* of the power function, we try letting

$$u = 3x^2 - 5$$
$$du = 6x\, dx$$

In order to affect the substitution, we must get rid of all x's and the dx, replacing them with our new variable u and our new du. We can get rid of the $(3x^2 - 5)^{3/4}$ with $u^{3/4}$. But what about the left over x

and dx? There are several ways of handling this situation; here is one such.

$$\begin{aligned}
\int x(3x^2-5)^{3/4}\,dx &= \int (3x^2-5)^{3/4}\,x\,dx && \triangleleft \text{ rearrange integrand} \\
&= \frac{1}{6}\int (3x^2-5)^{3/4}\,6x\,dx && \triangleleft \text{ cleverly insert 1} \\
&= \frac{1}{6}\int u^{3/4}\,du && \triangleleft \text{ sub. for } u, du \\
&= \frac{1}{6}\frac{u^{7/4}}{7/4} + C && \triangleleft \text{ Power Rule} \\
&= \frac{1}{6}\frac{4}{7}u^{7/4} + C \\
&= \frac{2}{21}(3x^2-5)^{7/4} + C. && \triangleleft \text{ resubstitute}
\end{aligned}$$

Thus,

$$\boxed{\int x(3x^2-5)^{3/4}\,dx = \frac{2}{21}(3x^2-5)^{7/4} + C.}$$

The student should assure the self of the student that the answer is correct. Differentiate the answer to obtain the integrand.

Example 4.5. ■

4.6. The problem is to integrate

$$\int x^3(2x^3+1)^7\,dx.$$

This integral looks like the several ones already seen. The strategy is to set u equal to the base of a power function. We have two power functions; the one to try first is the more complicated of the two.

Let,

$$u = 2x^3 + 1$$
$$du = 6x^2\,dx$$

Now examining and rearranging the integral we get,

$$\int x^3(2x^3+1)^7\,dx = \int x(\underbrace{2x^3+1}_{u})^7\ \underbrace{x^2\,dx.}_{k\,du}$$

As you can see, our du calculation is $du = 6x^2\,dx$. To make the 'du' we need an $x^2\,dx$ which we have by breaking x^3 into xx^2 and moving the x^2 over next to the dx. But, we *still have an x left over*! This means that this integral *cannot be solved by the Power Rule*! For the *Power Rule* to apply, your entire integrand must be either part of the u^r or part of the du; we have an x that belongs to neither.

Normally, we would not set our minds to pondering how to solve this integral. You can solve this integral: Multiply everything out to obtain a polynomial, and integrate each term. Good luck, son. DPS

Example 4.6. ■

4.7. I look at the problem,

$$\int \sin(5x)\,dx, \tag{S-12}$$

and I see that we are asked to integrate the *sine of some function of* x. We have a formula for integrating the sine of some function of an independent variable, x in this case.

$$\int \sin(u)\,du = -\cos(u) + C.$$

I reason as follows: If I am going to use this formula to solve problem (S-12), then u must be $5x$; this is because in the formula, u is the argument of the sine function (i.e. u is the quantity that we are taking the sine *of*).

Let, therefore,

$$u = 5x$$
$$du = 5\,dx.$$

Substituting this into our problem, (S-12),

$$\begin{aligned}
\int \sin(5x)\,dx &= \frac{1}{5}\int \sin(5x)\,5\,dx && \triangleleft \text{ fudge factors} \\
&= \frac{1}{5}\int \sin(u)\,du && \triangleleft \text{ substitution} \\
&= -\frac{1}{5}\cos(u) + C && \triangleleft \text{ Trig. (2)} \\
&= -\frac{1}{5}\cos(5x) + C && \triangleleft \text{ re-substitute}
\end{aligned}$$

Thus,

$$\boxed{\int \sin(5x)\,dx = -\frac{1}{5}\cos(5x) + C.}$$

Example 4.7. ■

4.8. We have the integral of the $\sec\tan$ function with a common argument of x^2. Try to use Trig. (5):

$$\int \sec(u)\tan(u)\,du = \sec(u) + C.$$

Our problem is

$$\int x\sec(x^2)\tan(x^2)\,dx.$$

Following my own advice in the strategy, let

$$u = x^2$$

$$du = 2x\,dx$$

Now, is the rest of the integrand directly proportional to du? Yes.

$$\begin{aligned}
\int x\sec(x^2)\tan(x^2)\,dx &= \int \sec(x^2)\tan(x^2)\,x\,dx \\
&= \frac{1}{2}\int \sec(x^2)\tan(x^2)\,2x\,dx && \triangleleft \text{ insert fudge} \\
&= \frac{1}{2}\int \sec(u)\tan(u)\,du && \triangleleft \text{ substitution} \\
&= \frac{1}{2}\sec(u) + C && \triangleleft \text{ Trig. (5)} \\
&= \frac{1}{2}\sec(x^2) + C && \triangleleft \text{ re-substitute}
\end{aligned}$$

Thus,

$$\boxed{\int x\sec(x^2)\tan(x^2)\,dx = \frac{1}{2}\sec(x^2) + C.}$$

Verify the answer through differentiation. Example 4.8. ■

4.9. If we let u be the argument of the trigonometric function, the du must be directly proportional to the rest of the integrand: this is the Trig. Strategy.

In this problem,

$$\int x\cos(x)\,dx,$$

we would naturally let $u = x$ and so $du = dx$. We have an x left over: du is not directly proportional to the rest of the integrand.

This problem cannot be solved by any of the Trig. formulas. In *Calculus II*, we get some techniques that solve this integral; meanwhile, you can verify that

$$\int x\cos(x)\,dx = x\sin(x) + \cos(x) + C,$$

by differentiating the right-hand side to obtain the integrand.

Example 4.9. ■

5.1. We reason as above. The power rule is

$$\int u^r\,du = \frac{u^{r+1}}{r+1} + C \qquad r \neq -1.$$

and the given integral is

$$\int (x^3+1)^{100}\,x\,dx. \tag{S-13}$$

No if the formula is to solve the above integral, we are forced to say

$$\text{Let } u = x^3+1, \text{ and } du = 3x^2\,dx.$$

In order for the power rule to apply, everything following the power function must be part of the du. The du is $3x^2\,dx$. In our integral, (S-13), we need *at least* an x^2 — but we have *only* an x. *We cannot, therefore, get the du.* The expression that follows the power function in our integral (S-13) is not directly proportional to the calculated value of du. Example 5.1. ■

5.2. The referenced formula is

$$\int \cos(u)\,du = \sin(u) + C. \tag{S-14}$$

Our given integral is

$$\int \cos(2x)\,dx. \tag{S-15}$$

Now, in the formula (S-14), the u is the expression we are taking the cosine of (we say that u is the argument of the cosine). If (S-14) is to solve our given integral, then we are forced to say

$$\text{Let } u = 2x \text{ and } du = 2\,dx.$$

If the formula (S-14) is to apply, everything following the cosine must by the du; at the bear minimum, what follows the cosine must be directly proportional the du. Staring at the given integral for many hours, you make the following move,

$$\int \cos(2x)\,dx = \frac{1}{2}\int \cos(2x)\,2\,dx$$

All the parts of the given integral are properly lined up with the corresponding parts of our chosen integral formula. (The correspondence being setup by the device of substitution.) Therefore,

$$\int \cos(2x)\,dx = \frac{1}{2}\int \underbrace{\cos(2x)}_{\cos(u)}\,\underbrace{2\,dx}_{du}$$

$$= \boxed{\frac{1}{2}\cos(2x) + C.}$$

There is no real need to make the substitution.

Example 5.2. ■

5.3. The given integral is

$$\int x(x+1)^{100}\,dx.$$

The function f in (8) is $f(x) = x(x+1)^{100}$. This integral cannot be solved by a simple application of the power rule. (Use the techniques of formula checking.) We can solve this integral using the power rule by *multiplying everything out*! No way!

Alternately, we can do a substitution of variables. One natural substitution is to define a new variable u so as to simplify the $(x+1)^{100}$ part of the integrand. To do this, we could let $u = x+1$, then $(x+1)^{100}$ becomes u^{100} — now that's a simplification. Let's complete the substitution and see what we get,

$$u = x + 1 \text{ and } du = dx$$

or

$$x = u - 1 \text{ and } dx = du. \tag{S-17}$$

The latter set of equations, (S-17), is a set of equations of the recommended form (9).

Take our integral and get rid of all x's and dx's using (S-17):

$$\int x(x+1)^{100}\,dx = \int (u-1)u^{100}\,du$$

This last integral can be solved!

$$\begin{aligned}
&\int x(x+1)^{100}\,dx \\
&\quad = \int (u-1)u^{100}\,du && \triangleleft \begin{cases} x = u-1 \\ dx = du \end{cases} \\
&\quad = \int u^{101} - u^{100}\,du && \triangleleft \text{ multiply out} \\
&\quad = \frac{u^{102}}{102} - \frac{u^{101}}{101} + C && \triangleleft \text{ Power Rule} \\
&\quad = \frac{1}{102}(x+1)^{102} - \frac{1}{101}(x+1)^{101} + C
\end{aligned}$$

In the last we returned to our original variable x, by resubstituting: $u = x + 1$.

Example Notes: Notice the effect of this substitution. We transferred the binomial from the factor that had large power to the factor that had small power. This made it practical to "multiply out" the integrand. You can check that this is a correct answer by differentiating the answer. ∎

Algebra Fanatics: If there are any left, the answer can be simplified slightly by factoring out $(x+1)^{101}$, and combining the left-overs. The final, simplified answer is

$$\boxed{\int x(x+1)^{100}\,dx = \frac{1}{(101)(102)}(101x-1)(x+1)^{101} + C.}$$

Waiter! Check please! Example 5.3. ∎

5.4. The given integral

$$\int x^2(2x+1)^{1/2}\,dx,$$

cannot be solve using any of the integral formulas — surprise! Try a substitution.

$$\text{Let } u = 2x+1, \text{ or } x = \frac{1}{2}(u-1), \text{ and } dx = \frac{1}{2}\,du.$$

Substitute these into our integral, see what we get,

$$\begin{aligned}
\int x^2(2x+1)^{1/2}\,dx \\
&= \int \frac{1}{4}(u-1)^2 u^{1/2}\,\frac{1}{2}\,du \\
&= \frac{1}{8}\int (u^2-2u+1)u^{1/2}\,du \\
&= \frac{1}{8}\int u^{5/2}-2u^{3/2}+u^{1/2}\,du \\
&= \frac{1}{8}\left(\frac{2}{7}u^{7/2}-\frac{4}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right)+C \\
&= \frac{1}{8}\left(\frac{2}{7}(2x+1)^{7/2}-\frac{4}{5}(2x+1)^{5/2}+\frac{2}{3}(2x+1)^{3/2}\right)+C
\end{aligned}$$

Some additional simplification is possible — I'll leave that to you.

Example 5.4. ■

6.1. Look that the problem

$$\int x^3(x^4+3)^{1/3}\,dx.$$

Begin with the first formula in the left-hand column. That first formula is the Power Rule. Does the power rule solve this problem?

Formula Checking for Power Rule: The power rule addresses integrals of the form

$$\int u^r\,du.$$

Taking into consideration the Power Rule Strategy, there are two choices for u: $u=x$ (because we have a x^3 factor in our integral), or $u=x^4+3$ (because we have a $(x^4+3)^{1/3}$). Both of these choices are candidates for the u^r part in the power rule. Given the choice, the rest of the integrand must be the du. I mentally take note that the derivative of $x^4 + 3$ is directly proportional to the x^3 factor. Through this observation, I determine to say,

$$\text{Let } u = x^4 + 3, \text{ and so } du = 4x^3\,dx.$$

Our integral becomes,

$$\int x^3(x^4+3)^{1/3}\,dx = \frac{1}{4}\int \underbrace{(x^4+3)^{1/3}}_{u^r}\,\underbrace{4x^3\,dx}_{du}$$

We conclude that the Power Rule will solve this problem. Let's solve it!

Evaluation:

$$\begin{aligned}\int x^3(x^4+3)^{1/3}\,dx &= \frac{1}{4}\int (x^4+3)^{1/3}\,4x^3\,dx \\ &= \frac{1}{4}\frac{3}{4}(x^4+3)^{4/3} + C && \triangleleft \text{ Power Rule} \\ &= \boxed{\frac{3}{8}(x^4+3)^{4/3} + C.}\end{aligned}$$

Where we have not bothered to make the substitution — a waste of electronic bytes — I just applied the power rule. Example 6.1. ■

6.2. We start with the first formula in the left-most column of the table of Integral Formulas and Techniques. The first formula on that list is the *Power Rule*. Does the power rule solve this problem?

$$\textit{The Problem}: \int x^2 (x^2+1)^2\, dx. \tag{S-18}$$

Formula Checking for Power Rule: Again, we have two choices for u: $u = x$ and $u = x^2 + 1$.

$$(1)\text{ Let } u = x, \text{ then } du = dx.$$

The $u^r = x^2$. Is it true that everything after the u^r is directly proportional to du? *Ans*: **No**. Therefore, the power rule does not apply *for this choice of* u.

Try again.

$$(2)\text{ Let } u = x^2 + 1, \text{ then } du = 2x\, dx.$$

Rearrange the integral,

$$\int (x^2+1)^2\, x^2\, dx.$$

The $u^r = (x^2 + 1)1/2$. Is it true that everything after the u^r is directly proportional to du? *Ans*: **No**. Therefore, the power rule does not apply *for this choice of* u.

I conclude that the *Power Rule* does not solve this problem.

Continue: We still want to solve the problem, right? Continue on down the list of formulas. We skip over the rest of them because they all have trig functions in them. Our integral does not have any trig functions in it. We move over to the techniques column.

Apply a Technique: The original integral (S-18) has no multiplicative constants to factor out (that's the first technique on the list), the integrand has no terms in it so we cannot separate using additively (that's the second technique on the list). The remaining choices are to apply a true substitution of variables or manipulate the integrand algebraically.

As a pseudo-rule, substitution is a technique that we apply as a last resort. I'll choose to manipulate the integrand algebraically. The only thing that can be done is to *expand* the integrand.

$$x^2(x^2+1)^2 = x^2(x^4+2x^2+1) = x^6+2x^4+x^2.$$

Thus,

$$\begin{aligned}\int (x^2+1)^2\,x^2\,dx &= \int x^6+2x^4+x^2\,dx \\ &= \boxed{\frac{1}{7}x^7+\frac{1}{5}x^5+\frac{1}{3}x^3+C.}\end{aligned}$$

Once we determined our course of attack, we follow through. Expanding the integrand yielded a polynomial, to which we applied the Power Rule.

The point of this problem is that it illustrates a "natural" pattern of thinking. Methodically, we go down the list of integral formulas. We check each candidate using formula checking technique. Having reached the bottom of the list of integral formulas, we go over to the other column (I'm referring to the two-column list Integral Formulas and Techniques.). Once we applied a technique (algebra: expanding out the integrand), we separated integrals and factored out constants (though these simple steps are were not explicitly shown), and finally we jumped back to the left-hand column to apply the *Power Rule* — the first formula on our list to *each* of the resulting integrals. This is the Butterfly Method.

It's as simple as that! Example 6.2. ∎

6.3. We apply the Butterfly Method. We go to the first formula on the left-hand column of the list of Integral Formulas and Techniques. That formula is the *Power Rule*. Does the *Power Rule* solve this problem?

The Problem: $\displaystyle\int \frac{\sqrt{x}+1}{\sqrt{x}}\,dx.$

Formula Checking for Power Rule: We must choose the u in the formula

$$\int u^r\,du = \frac{u^{r+1}}{r+1} \qquad r \neq -1.$$

The *only choice* for u is $u = \sqrt{x}+1$. (Why?)

$$\text{let } u = \sqrt{x+1}, \text{and so } du = \frac{1}{2\sqrt{x}}\,dx.$$

Keeping in mind the Power Rule Strategy, rewrite the given integral as follows,

$$\int \frac{\sqrt{x}+1}{\sqrt{x}}\,dx = \int (\sqrt{x}+1)\,\frac{1}{\sqrt{x}}\,dx.$$

Is it true that everything that follows the u^r factor (here $r = 1$) is directly proportional to the calculated value of du? *Ans*: **Yes**. Therefore, the *Power Rule* solves this problem.

Evaluation:

$$\begin{aligned} \int \frac{\sqrt{x}+1}{\sqrt{x}}\,dx &= \int (\sqrt{x}+1)\,\frac{1}{\sqrt{x}}\,dx \\ &= 2\int \underbrace{(\sqrt{x}+1)}_{u^r}\,\underbrace{\frac{1}{2\sqrt{x}}\,dx}_{du} && \triangleleft \text{ insert fudge factor} \\ &= 2\frac{1}{2}(\sqrt{x}+1)^2 + C && \triangleleft \text{ Power Rule} \\ &= \boxed{(\sqrt{x}+1)^2 + C.} \end{aligned}$$

Butterfly works again. Example 6.3. ■

6.4. We proceed along standard lines of inquiry: The Butterfly pattern of thought.

Can the integral

$$\int \frac{x^2}{\sqrt{x+1}}\,dx \tag{S-19}$$

be solved by the *Power Rule*? We must first ask ourselves. Put it in the proper form:

$$\int (x+1)^{-1/2}\,x^2\,dx$$

If we think of $u = x + 1$, then $du = dx$. We have our du, but we have the x^2 unaccounted for. Therefore, the *Power Rule* does not apply at this time.

The other integral formulas in the list do not apply as we have no trig functions.

We must, therefore, do a *technique*. Homogeneity and Additivity cannot be used at this time. That leaves substitution or some direct manipulation of the integrand.

In terms of direct manipulation, there is nothing much to do: the integrand is a fairly compact expression. No wiggle room.

I'm left with the only alternative: Substitution. Now, this will be a true substitution of variables, not merely formula checking. (We don't have an integral formula to check.)

As for a substitution choice, that square root in the denominator of our problem (6) is the problem child. I'll use substitution to *make it go away*! You can usually make part of the integrand go away — at the expense of another part of the integrand. Let's hope the price is not too high.

The Substitution: Let $u = \sqrt{x+1}$, so

$$x = u^2 - 1 \text{ and } dx = 2u\,du,$$

where I have defined my new variable u to be the offending expression (so it will now become u), and then I solved for x in order to calculate dx.

Substitute In:

$$\int \frac{x^2}{\sqrt{x+1}}\,dx = \int \frac{(u^2-1)^2}{u}\,2u\,du$$
$$= 2\int (u^2-1)^2\,du.$$

The integral we get after substitution and simplification is the integral of a *polynomial* in the variable u — solvable! Theoretically done! All we need to do is to multiply out (a technique), separate integrals (a technique), and apply the power rule to each term (an integral formula).

Evaluation:

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x+1}}\,dx &= 2\int (u^2-1)^2\,du \\
&= 2\int u^4 - 2u^2 + 1\,du \\
&= 2\left(\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right) + C \\
&= 2u\left(\frac{1}{5}u^4 - \frac{2}{3}u^2 + 1\right) + C
\end{aligned}$$

Resubstitute: Recall that $u = \sqrt{x+1}$.

$$\begin{aligned}\int \frac{x^2}{\sqrt{x+1}}\,dx &= 2u\left(\frac{1}{5}u^4 - \frac{2}{3}u^2 + 1\right) + C \\ &= 2\sqrt{x+1}\left(\frac{1}{5}(x+1)^2 - \frac{2}{3}(x+1) + 1\right) + C \\ &= \boxed{\frac{2}{15}\sqrt{x+1}(3x^2 - 4x + 8) + C.}\end{aligned}$$

In the last line, I felt that I had to be true to my algebraic heritage — I simplified to a final magnificent answer. (Yes, I checked to see whether $3x^2 - 4x + 8$ can be factored some more. Did you, can you?)

Example 6.4. ■

6.5. We use the Butterfly pattern of thinking.

Begin at the top of the left-hand column of the list: the Power Rule. Does the power rule solve this problem?

If we designate $u = \csc(1/x)$, then we are trying to make our given integral

$$\int \frac{\csc^2(\frac{1}{x})}{x^2}\, dx \tag{S-20}$$

look like $\int u^2\, du$. Does it? If we put $u = \csc(1/x)$, then we calculate $du = (1/x^2)\csc(1/x)\cot(1/x)\, dx$ (You had better check this, in case I made a mistake, thanks.) We don't have the du; therefore, the *Power Rule* does not apply.

Moving on down that list, skipping over the integral formulas that don't apply: we don't have any sines, no cosines, no secant squares, (I'm going down the list). Let's see where was I, no cosecant squared, no … Wait! I do have cosecant squared in my integral. STOP.

Let's investigate whether Trig. (4):

$$\int \csc^2(u)\,du = -\cot(u) + C.$$

Looking at this formula, we see that u is the argument of the cosecant squared function. Compare this formula with our given integral (S-20), we see that we must let u be

$$u = \frac{1}{x} \text{ and so } du = -\frac{1}{x^2}.$$

Rearrange our given integral (S-20) to make it look more like the formula integral:

$$\int \csc^2(\tfrac{1}{x})\,\frac{1}{x^2}\,dx.$$

Is it true that everything after the $\csc^2(u)$ is directly proportional to the calculated value of du? *Ans*: Yes. Therefore, the formula solve the problem.

Evaluation:

$$\int \csc^2(\tfrac{1}{x})\,\frac{1}{x^2}\,dx = -\int \underbrace{\csc^2(\tfrac{1}{x})}_{\csc^2(u)}\,\underbrace{\frac{-1}{x^2}\,dx}_{du} \qquad \triangleleft \text{ insert fudge}$$

$$= -(-\cot(\tfrac{1}{x})) + C \qquad \triangleleft \text{ Trig. (4)}$$

$$= \boxed{\cot(\tfrac{1}{x}) + C.}$$

Example Notes: Don't forget the *Power Rule*. It can potentially solve any integral no matter what kinds of functions are involved.

▪ The dialog that I carried on above represents the simple minded approach of the Butterfly Method. Go down the list, stop at a formula that has some of the attributes of your own integral problem — in this case, it was the cosecant squared. Use formula checking to check it out.

▪ Do not get bothered by the ugliness of the integrals. Just pick out the most important components of your integral: the expression $\csc^2$(of something) in the numerator and the x^2 in the denominator.

■ Despite its initial ugliness, this problem was almost an exact integral formula integral. We just had to see that it was. ■

Example 6.5. ■

6.6. The problem is

$$\int \frac{x \sin(x^2)}{\sqrt{\cos(x^2)}}\, dx,$$

a mean-looking dude, if I may say so. But, let's not panic. Proceed along our proven standard technique of analysis: the Butterfly Method.

Go the first formula on the left-hand column of the list of formulas and techniques. This is the much often repeated and used Power Rule. Does the power rule solve this problem?

Formula Checking for Power Rule: The only thing that is being raised to a power, other than power 1, is the cosine function in the denominator. Rewrite the integral to make it look more *Power Rule-ish*:

$$\int \frac{x \sin(x^2)}{\sqrt{\cos(x^2)}}\, dx = \int (\cos(x^2))^{-1/2}\, x \sin(x^2)\, dx$$

Now let us meditate upon the possibilities. If we let

$$u = \cos(x^2), \text{ and } du = -\sin(x^2)\, 2x\, dx = -2x \sin(x^2)\, dx,$$

then is it true that everything that follows our power function is directly proportional to the calculated value of du? *Ans*: YES. Therefore, the *Power Rule* solves this problem.

Evaluation:

$$\begin{aligned}
&\int (\cos(x^2))^{-1/2}\, x\sin(x^2)\, dx \\
&\quad = -\frac{1}{2}\int \underbrace{(\cos(x^2))^{-1/2}}_{u^r}\,\underbrace{(-2x)\sin(x^2)\,dx}_{du} && \triangleleft \text{ fudge factor} \\
&\quad = -\frac{1}{2}\,\frac{(\cos(x^2))^{1/2}}{1/2} + C && \triangleleft \text{ Power Rule} \\
&\quad = \boxed{-\sqrt{\cos(x^2)} + C.}
\end{aligned}$$

Example 6.6. ■

Important Points

Generally, if a function has one antiderivative, then it has infinity many. (Some functions have no antiderivative—can you give an example of one such creature?)

In the case of $f(x) = 2x$, each of the functions are antiderivatives of f:

$$F_1(x) = x^2 + 1 \qquad F_4(x) = x^2 + 100$$
$$F_2(x) = x^2 + 2 \qquad F_5(x) = x^2 - 234.12$$
$$F_3(x) = x^2 - \frac{1}{2} \qquad F_6(x) = x^2 - \pi$$

More generally, a function of the form $F(x) = x^2 + C$, where C is any constant, is an antiderivative of $f(x) = 2x$—because, in all cases, $F'(x) = f(x)$, for all $x \in \mathbb{R}$.

Given the observation that any function of the form $F(x) = x^2 + C$ is an antiderivative of $f(x) = 2x$, what is a *natural* question to ask yourself in this regard? Important Point ■

The answer is ‘Yes.’ The definition requires that

$$F'(x) = f(x) \quad \text{for all } x,$$

well, let’s check it out.

The definition of f is $f(s) = 4s^3$ and so $f(x) = 4x^3$.

The definition of F is $F(t) = t^4$ and so, by the rules of differentiation, $F'(t) = 4t^3$. Thus, $F'(x) = 4x^3$.

Therefore,

$$F'(x) = 4x^3 = f(x) \quad \text{for all } x,$$

as required by the definition. Important Point ■

Important Points (continued)

This problem was given to me by a colleague. When he/she gave it to me, he/she left explicit instructions that I was to integrate with respect to the variable z. Therefore,

$$\int x^2 = x^2 z + C.$$

If you missed this problem, it was probably because you weren't around when my colleague communicated to me what the variable of integration was to be ... sorry! $\mathfrak{DPS}$ Important Point ■

As you work your way through calculus, what is the one big thing that prevents your success? What one thing do your always struggle with? What one thing requires most of your time and concentration—perhaps ultimately taking away from your study of the calculus itself? The answer, most probably, ALGEBRA!

Imagine how things would be if you where an algebraic whiz. You could concentrate more on the ideas and techniques of calculus. Since algebra is no problem, you could do more problems—that would help your problem solving abilities, give more practice to the different techniques, and increase the speed at which you solve problems (that's always good).

Algebra is a foundation block of higher mathematics; it is the language of mathematics. If you don't know the language, you can't operate effectively in a mathematics environment.

So it goes with, in this instance, the integral formulas. If you don't know the formulas, you can solve problems. (If you know the formulas, and don't know algebra, you still can't solve the problems!)

Finally, having a solid knowledge increases the rate at which you can learn new ideas! *The more you know, the faster you can learn*. Knowledge builds on itself.

My advice to you is KNOW THE FORMULAS! Important Point ■

Here are a few more details from a slightly different point of view.

The integrand is $f(x) = 0$. Define the function $F(x) = 0$ as well. We know from differential calculus that

$$F'(x) = 0 = f(x)$$

therefore, from the definition of antiderivative, F is an antiderivative of f. Hence, we are entitled to say

$$\int 0\,dx = \int f(x)\,dx = F(x) + C = C.$$

The equality of the extreme left side with the extreme right side is the substance of (1).

Important Point ■

Choose an irrational number $x_0 \in (0, 1)$. Then $F'(x_0) = f(x_0) = 1$.

For any $h \neq 0$, the MEAN VALUE THEOREM states that there is a number c_h between x_0 and $x_0 + h$ such that

$$\frac{F(x_0 + h) - F(x_0)}{h} = F'(c_h) = f(c_h). \tag{I-1}$$

(I have used the notation c_h because the value of 'c' as given to us by the MEAN VALUE THEOREM depends the value of the endpoint $x_0 + h$, which depends, in turn, on the value of h. So 'c' will depend on the value of h.)

Since the limit of the left-hand side of (I-1), as $h \to 0$, is $F'(x_0) = 1$, we deduce for h small enough, say $-\delta < h < \delta$, for some positive number δ, that

$$\frac{F(x_0 + h) - F(x_0)}{h} = f(c_h) = 1. \tag{I-2}$$

Remember that f only takes on values of 0 and 1. If $f(c_h)$ in equation (I-2) is equal to 0 for values of h 'arbitrarily' close to 0, that would imply $F'(x_0)$ does not exist. (Why?)

Given the validity of (I-2), we now see that

$$F(x_0 + h) = F(x_0) + h \qquad -\delta < h < \delta. \tag{I-3}$$

Differentiate both sides of (I-3) with respect to h to obtain,

$$F'(x_0 + h) = 1 \qquad -\delta < h < \delta.$$

But $F'(x_0 + h) = f(x_0 + h)$, and so,

$$f(x_0 + h) = 1 \qquad -\delta < h < \delta. \tag{I-4}$$

To obtain our contradiction, we simply choose a value for h_0, such that $-\delta < h_0 < \delta$ and $x_0 + h_0$ is a rational number. (Is it *always* possible to do that?) Thus,

$$f(x_0 + h_0) = 1 \quad \text{from (I-4)}$$

but,

$$f(x_0 + h_0) = 0 \quad \text{since } x_0 + h_0 \text{ is rational}$$

This is a *contradiction*. Therefore, there is no function F such that $F'(x_0) = f(x_0)$ for *any* irrational number $x_0 \in (0, 1)$.

A similar argument can be made in the case that x_0 is a rational number. I now claim that f is a function having all the stated properties.

Important Point ■