2.6. Recognizing Functions

In this section we examine the following question:

"Is the variable s a function of the variable t?"

The goal of this section is develop a "feel" for functional relationships, and furthermore, do it in such a way that your understanding of a functional relationship does not depend on the particular letters (variables) used. (This is why I have used different letters to denote the variables – to avoid biasing your thinking towards our special variables x and y.)

• Interpreting the Terminology

In the quotation,

"Is the variable s a function of the variable t?"

the letter s is usually y and the letter t is usually x and so the above question becomes: "Is y a function of x?". However, it may be that letter s refers to the letter x and the letter t means letter y, the question now becomes: "Is x a function of y?" The letters s and t could represent any pair variables of interest.

Suppose I make the assertion: "s is a function of t." What does this statement mean, what are its implications?

1. The variable t is to be considered the *independent variable.*
2. The variable s is to be considered the *dependent variable.*
3. The acceptable values of the variable t vary over a set of numbers that is referred to as the domain of the function.
4. The values of the function are symbolically represented by the letter s. The values of the function come from the range of the function.
5. There is some rule of association – a rule that associates with each value of t in the domain of the function, a corresponding value of s in the range. The rule of association my be given *explicitly* or *implicitly.*
6. Notationally, "s is a function of t" means $s = f(t)$.

Quiz.

1. Consider w as a function of z. Then, corresponding to each value of z is only one value of w.

(a) True (b) False

2. Consider h as a function of q. Then h is the independent variable.

(a) True (b) False

3. Consider x a function of y. Then x may be considered a member of the domain of the function.

(a) True (b) False

4. Consider z as a function of x. Then, corresponding to each value of z is only one value of z.

(a) True (b) False

5. Consider w as a function of s. Then w may be considered a member of the range of the function.

(a) True (b) False

6. Consider t as a function of w. Then symbolically, this means that $t = f(w)$.

(a) True (b) False

Passing Score: 6 out of 6.

End Quiz.

Let's now practice recognizing functions through a series of examples and exercises.

Example 2.14. Let x and y be real-variables. Suppose it is known that y is related to x by the equation $2x^2 - 3y = 1$.

a. Is y a function of x?
b. Is x a function of y?

Exercise 2.41. Suppose x and y are related by the equation $2x - 5y^3 = 1$.

a. Is y a function of x?
b. Is x a function of y?

Exercise 2.42. Let s and t be related to each other by way of the equation $s - 4t + t^2 = 1$.

a. Is s a function of t?
b. Is t a function of s?

Exercise 2.43. Consider the equation $x^2 + y + 2 = 1$. I wouldn't think of asking you whether y is a function of x or whether x is a function of y — they are not. Let m be any number. Consider the straight line given by $y = mx$ and visualize the intersection of the line $y = mx$ with the circle $x^2 + y^2 = 1$.

a. Is x a function of m, where x is the x-coordinate of the point(s) of intersection between $y = mx$ and $x^2 + y^2 = 1$.
b. Let x be the variable described in part (a). Is m a function of x?

• The Vertical Line Test

Suppose you have a curve $\mathcal{C}$ drawn in the xy-plane. How can we tell whether this curve $\mathcal{C}$ represents y as a function of x? There is a simple graphical test.

> *Vertical Line Test*:
> A curve $\mathcal{C}$ in the xy-plane defines y as a function of x if it is true that every vertical line intersects the curve at *no more than* one point.

Important. The x-axis is assumed to be the *horizontal axis*, and so the meaning of *vertical* is perpendicular to the x-axis.

EXERCISE 2.44. Taking the definition of function into consideration, the orientation of the axes (x-axis is horizontal), and the geometry of the graph of a curve, justify in your own mind the *Vertical Line Test*.

EXERCISE 2.45. Assume the usual orientation of the xy-axis system (i.e. the x-axis is horizontal). Suppose we have a curve $\mathcal{C}$ in the xy-axis plane. Under what conditions, similar to the *Vertical Line Test*, under which we can assert that the curve defines x as a function of y?

The Vertical and the results of Exercise 2.45 can be consolidated into a single statement which is stated independently of orientation of the axis system.

> *The Function Line Test*:
> A curve $\mathcal{C}$ in the xy-plane defines y as a function of x if it is true that every line perpendicular to the x-axis intersects the curve at *no more than* one point.

Where, in this test, we do not assume that the x-axis is necessarily the horizontal axis.

The above concepts are independent of the letters used to describe them. Here are a couple of questions using other letters.

Exercise 2.46. Let $\mathcal{C}$ be a curve in the st-plane. Under what conditions, similar to the *Function Line Test*, under which we can assert that

a. the curve defines s as a function of t;

b. the curve defines t as a function of s.

Exercise 2.47. Quiz.

3. Graphing: First Principles

This section still under construction. It is my intention to have a section here discussing the fundamental principles and techniques of graphing a function.

4. Methods of Combining Functions

Functions can be combined in a variety of ways to create new functions. In this section, we discuss ways in which we can use *arithmetic operations* for this purpose.

4.1. The Algebra of Functions

- **Equality of Functions**

Let f and g be functions. We say that $f = g$ provided:

1. $\mathrm{Dom}(f) = \mathrm{Dom}(g)$;
2. $f(x) = g(x)$, for all $x \in \mathrm{Dom}(f)$.

More informally, two functions are the same if they have the same *domain of definition* (condition 1), and *pointwise* they have the *same values* (condition 2).

The first example illustrates the equality of two functions. It is a two-step method: (1) Check whether the domains are equal; (2) Check whether the functions, pointwise, have the same values.

Example 4.1. Consider the following two functions:

$$f(x) = x, \qquad g(x) = \frac{x^3 + x}{x^2 + 1}.$$

Is it true that $f = g$?

The next example is almost the same as the previous one, but with two subtle changes. The signs in the numerator and denominator of the function g have been changed to negative signs. As in the previous example, the numerator and denominator have a common factor, when you cancel the common factor you get $g(x) = x$, this is the same definition of f. So the two functions are equal, right?

Example 4.2. Are the following two functions equal?

$$f(x) = x, \qquad g(x) = \frac{x^3 - x}{x^2 - 1}.$$

Example 4.3. Consider the following two functions:

$$f(x) = x + 2 \qquad x \in \mathbb{R},$$
$$g(x) = \begin{cases} \dfrac{x^2 + x - 2}{x - 1} & \text{for } x \neq 1 \\ 3 & \text{for } x = 1 \end{cases}$$

Is it true that $f = g$?

Exercise 4.1. Determine whether the two functions f and g are equal.

$$f(x) = \frac{1}{\sqrt{x+1} - \sqrt{x}} \qquad g(x) = \sqrt{x+1} + \sqrt{x}.$$

Exercise 4.2. Determine whether the two functions f and g are equal.

$$f(x) = \frac{x}{\sqrt{x^2+x} - x} \qquad g(x) = \sqrt{x^2+x} + x.$$

- **Scalar Multiplication**

Let f be a real-valued function of a real variable, and let $k \in \mathbb{R}$ be a constant. Define a new function, denoted kf, to be a function whose domain is

$$\mathrm{Dom}(kf) = \mathrm{Dom}(f),$$

such that,

$$(kf)(x) := kf(x), \qquad x \in \mathrm{Dom}(kf).$$

The function kf is called a *scalar multiple* of f (the constant k, in this context, is referred to as a *scalar*).

Below is a sequence of simple examples.

$$\begin{cases} f(x) = x^2, & x \in \mathbb{R} \\ 5f(x) = 5x^2, & x \in \mathbb{R} \end{cases}$$

$$\begin{cases} g(x) = \sqrt{x^2 - 4}, & x \in [\,4, \infty\,) \\ -.23g(x) = -.23\sqrt{x^2 - 4}, & x \in [\,4, \infty\,) \end{cases}$$

$$\begin{cases} h(x) = \sin x, & x \in \mathbb{R} \\ \frac{1}{2}h(x) = \frac{1}{2}\sin x, & x \in \mathbb{R} \end{cases}$$

The above discussion covered the *construction* of a scalar multiple of a function. A related topic is the *recognition* of scalar multiples of functions. For example, consider the function $G(x) = 2\sin x$, you should recognize (or realize) that his function G is, in fact, a scalar multiple of a *more elementary function*, namely the sin function. Thus, $G = 2\sin$. (Looks strange doesn't it.)

This ability to recognize scalar multiples is fundamentally a very important skill.

- **The Addition/Subtraction of Functions**

Let f and g be real-valued functions of a real variable. Define $f + g$ to be a function whose domain is

$$\mathrm{Dom}(f+g) = \mathrm{Dom}(f) \cap \mathrm{Dom}(g),$$

such that

$$(f+g)(x) := f(x) + g(x), \qquad x \in \mathrm{Dom}(f+g).$$

The function is called the *sum* of f and g. Subtraction of two functions is defined similarly.

Exercise 4.3. Study the definition of the sum of two functions. Discuss why the domain of $f + g$ is defined as it is.

Let's look at some illuminating examples.

Example 4.4. Let $f(x) = \sqrt{x}$ and $g(x) = \sin x$, then the sum and difference of f and g are

$$(f + g)(x) = f(x) + g(x) = \sqrt{x} + \sin x$$
$$(f - g)(x) = f(x) - g(x) = \sqrt{x} - \sin x.$$

What are the domains of these functions?

Exercise 4.4. Let $f(x) = \sqrt{1 - x}$ and $g(x) = \sqrt{x - 2}$, and let $F = f + g$. Is there something strange about this definition?

The sum and difference of functions is extended to sums and differences of many functions.

Exercise 4.5. Let f, g, and h be functions. Define $F = f + g - h$. What is the domain of F, and how do calculate the values of F?

You can combine scalar multiplication and addition/subtraction of functions.

EXAMPLE 4.5. Let $f(x) = \sqrt{x}$, $g(x) = \sin x$, and $h(x) = \sqrt{4-x}$, and define $F = 2f - 3g + 4h$. Discuss the domain of F and obtain a calculation formula for F.

Recognition: Frankly, a quite a bit more important ability is that of recognizing sum and differences of functions. For example consider the function:

$$y = 6x^2 \sin(x) - 4x^3(x+1)^{1/2}.$$

Ideally, when you look at this function, you would, perhaps, in the natural course of things, make a number of observations: (1) the function is the *difference* of two other functions $y = 6x^2 \sin(x)$ and $4x^3(x+1)^{1/2}$; (2) these two functions are scalar multiples of the functions $y = x^2 \sin(x)$ and $y = x^3(x+1)^{1/2}$. Here, we mentally *decompose* the original function $y = 6x^2 \sin(x) - 4x^3(x+1)^{1/2}$ into smaller functional "pieces."

When you look at certain problems in *Calculus*, for example, this ability to visually decompose a function this way is fundamental to correctly analyzing the problem and successfully solving the problem. (Of course, the functions $y = x^2 \sin(x)$ and $y = x^3(x+1)^{1/2}$ can, themselves, be broken down into smaller functional "pieces;" namely, into x^2, $\sin(x)$, x^3, $(x+1)^{1/2}$ — as we shall see in subsequent paragraphs below.)

Exercise 4.6. Break the function

$$f(x) = 5x^3 \sin(x) \cos(x) - 3\sqrt{x} \tan(x)$$

down into more elementary functions.

• The Multiplication of Functions

Let f and g be real-valued functions of a real variable. Define fg to be a function whose domain is

$$\operatorname{Dom}(fg) = \operatorname{Dom}(f) \cap \operatorname{Dom}(g)$$

such that,

$$(fg)(x) := f(x)g(x), \qquad x \in \operatorname{Dom}(fg).$$

The function is called the *product* of f and g.

Examples in this section are much the same as in the previous section on addition and subtraction of functions. Here we have an abbreviated discussion.

One of the common examples of function multiplication is power functions. For example, Consider the functions$F(x) = x^3$. Now it may be convenient to think of F as a "stand-alone" function. Sometimes it is useful to realize that F is a product of functions; which functions? Well, define a function $f(x) = x$, then $F(x) = f(x)f(x)f(x)$, for all $x \in \text{Dom}(F) = \mathbb{R}$. This can be written as $F(x) = [f(x)]^3$. Or, if we want to utilize the concept of equality of functions, we can say: $F = f^3$. (The f-cubed function, is the function whose value at an $x \in \text{Dom}(f)$ is given as $[f(x)]^3$.

EXERCISE 4.7. In light of the previous discussion, what can be said about the function $F(x) = (2x^3 - 1)^5$ and the function $G(x) = \sin^2(x)$?

Now let's get back to a more direct illustration of the product of functions definition.

EXAMPLE 4.6. Consider the functions: $f(x) = 4$, $g(x) = \sqrt{x^2 - 1}$, and $h(x) = \sin(x^2)$. Define a new function by $F = fgh$. Find the domain of definition of F, and write the calculating formula for F.

Recognition: As in the previous section, it is important to recognize products of functions.

EXAMPLE 4.7. Consider the function $F(x) = 6x^3 \sin(x) \cos^2(x)$, define the most basic of functions such that F is the product of them.

EXERCISE 4.8. Consider the function $F(x) = x^4 \sin^2(x) - \cos^2(x)$. If you wanted to write F in terms of sums, differences, and products, of "elementary functions," what is the *minimum* number of distinct functions needed to do this?

• The Quotient of Functions

Let f and g be real-valued functions of a real variable. Define f/g to be a function whose domain is

$$\text{Dom}(f/g) = \text{Dom}(f) \cap \{\, x \in \text{Dom}(g) \mid g(x) \neq 0 \}$$

such that,

$$\frac{f}{g}(x) := \frac{f(x)}{g(x)}, \qquad x \in \text{Dom}(f/g).$$

The function is called the *quotient* of f and g.

Notice the domain of f/g is a bit more involved than the previous definitions. Obviously, we cannot divide by 0, so we must ensure that the x we use to evaluate the expression $f(x)/g(x)$, cannot yield 0 in the denominator; i.e. we require $g(x) \neq 0$.

A skill level 0 example would be the following.

EXAMPLE 4.8. Let $f(x) = x^3$, $g(x) = (x^2 - 1)$, and define $F = f/g$. Discuss the natural domain of definition of F, and write a calculating formula for F.

EXAMPLE 4.9. Decompose the function $F(x) = \dfrac{\sqrt{x}\sin(x)}{x^2 - 3}$ into a products and quotients of "elementary functions." Do a domain analysis on same.

• Comparison of Functions

Comparing functions is quite important in mathematics. It is very important to understand what is meant by it.

EXERCISE 4.9. Let f and g be two given functions. Now that you have seen a large number of definitions, can you give a good definition of the following phrase:

$$\text{“}f < g \text{ over the set } A\text{.”}$$

5. The Composition of Functions

Composition of two functions is a very fundamental and important concept. If you think of a function as a calculation of some type, then, roughly speaking, the composition of two functions, is the process of calculating the value of one function, then based on that result, calculate the value of the second function. The technical definition of composition follows.

Let f and g be functions such that $\text{Rng}(g) \subseteq \text{Dom}(f)$. Define $f \circ g$ to be a function whose domain is

$$\text{Dom}(f \circ g) = \text{Dom}(g),$$

such that,

$$(f \circ g)(x) := f(g(x)), \qquad x \in \text{Dom}(f \circ g).$$

The function $f \circ g$ is called the *composition* of f with g.

Compatible for Composition: Given a pair of functions, f and g, if $\text{Rng}(g) \subseteq \text{Dom}(f)$, we us agree to say that f is *compatible* with g for *composition*.

Example 5.1. State the criterion under which the functions h is compatible for composition with the function f. (That is, we wish to compose $h \circ f$.)

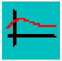
Figure 1

Figure 1 shows the **Venn Diagram** of the composition of two functions. When thinking of composition, it is important to visualize this picture. It often helps you to reason your way through a difficult composition.

In terms of a black box interpretation we have the following diagram

$$x \longrightarrow \boxed{g} \longrightarrow g(x) \longrightarrow \boxed{f} \longrightarrow f(g(x))$$

Example 5.2. Consider the two functions $f(x) = \sin(x)$ and $g(x) = x^2$. (a) Is f compatible with g for composition? (b) Calculate the domain of the composed function. (b) Compose f with g.

Multiple Compositions. Very often, functions of interest are, in fact, the composition of several simple functions. The extension from two function to three is obvious — at least after a few examples.

EXAMPLE 5.3. Let $f(x) = x^5$, $g(x) = \sin(x)$, and $h(x) = \sqrt{x}$. Calculate the function $(f \circ g \circ h)(x) = f(g(h(x)))$.

EXERCISE 5.1. Consider the functions

$$f(x) = \tan(x) \qquad g(x) = \frac{x}{x^2+1} \qquad h(x) = x^2.$$

Calculate the composition $(f \circ g \circ h)(x) = f(g(h(x)))$. Perform the calculation two ways: In-to-Out, and Out-to-In.

The concept of composition is independent of the letters used to define the functions and the variables.

EXERCISE 5.2. Consider the functions

$$W(x) = \cos(x) \qquad M(u) = u^2.$$

Calculate $W \circ M$ and $M \circ W$.

Composition of Anonymous Functions. Very often, the functions are given in anonymous form, say $y = sin(x)$ and $y = x^2$. Now we have a bit of a syntactical problem: How to explain to the user (you) what we want to do. We could say, that we want to compose the first function with the second function — until we reverse the order of the functions, in which case we change our minds and now want to compose the second with the first.

A popular convention is to relabel the variables so as to suggest our intentions. Instead of writing $y = \sin(x)$ and $y = x^2$, write instead

$$y = \sin(u) \text{ and } u = x^2.$$

Does this now suggest our intention? The composition of the these two functions is then

$$y = \sin(x^2),$$

where now composition is reduced to a process of *substitution of variables*; to tell you the truth, that's all composition is anyway.

Exercise 5.3. Consider the pair of functions $w = s^2 + 2$ and $s = t^2 - 1$, compose these two functions together in the obvious order to obtain w as a function of t.

A function may be the composition of three functions, four functions, five functions, or any number of functions. If we label the variables properly, multiple compositions is *une piecè du gateaux*.

EXAMPLE 5.4. Suppose $y = u^4$, $u = v^2 + 1$, and $v = \sin(x)$. Perform the implied composition.

Actually, this relabeling of variables is sometimes used even when the functions have names. In the abstract, we may describe two functions by saying: Define,

$$y = f(u) \text{ and } u = g(x).$$

That is, f defines y as a function of u, and g defines u as a function of x. Again, the choice of the variables suggests our intention to compose f with g to get

$$y = f(g(x)),$$

the new function defines y as a function of x.

Composition and your Calculator. Composition of functions is an operation you perform almost every time you use your hand-held calculator. On your calculator, there is a series of buttons called *function* *keys*. When you press two consecutive function keys on your calculator, you are composing functions together.

For example, suppose you wanted to calculate the expression $\sin(x^2)$, for some particular number x. How would you do it? You would perform a series of calculation steps. Step (1): Use the keypad to enter x into the display. Step (2): Press the function key labeled x^2. Step (3): Now press the function key labeled $\sin(x)$. You have just composed functions! Here is a diagram of the sequence of operations:

$$x \longrightarrow \boxed{x^2} \longrightarrow x^2 \longrightarrow \boxed{\sin(x)} \longrightarrow \sin(x^2).$$

The "squaring box" squares whatever is input into its box, the "sine box" take the sine of whatever is input into its box.

Now that I have amazed you with this observation, let me pop you balloon. You don't actually *have* to press two function keys to compose functions together. Consider the functions $g(x) = x^2 + 2x + 3$ and $f(x) = x^2$. The composition of these two functions is

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \triangleleft \text{ by defn of composition} \\
&= f(x^2 + 2x + 3) && \triangleleft \text{ by defn of } g \\
&= (x^2 + 2x + 3)^2. && \triangleleft \text{ by defn of } f
\end{aligned}
$$

To make this calculation on the calculator, we start with the initial input value of x, some particular value. We then build up the expression $x^2 + 2x + 3$ through a series of keypresses on our keypad, utilizing the multiplication and addition buttons, as well as perhaps the x^2 function key. Once we build up the value of $x^2 + 2x + 3$, then we would press (possibly for a second time) the x^2 function key. This key would then take the value in the display, which is $x^2 + 2x + 3$ and square it. The process is diagramed as

$$x \longrightarrow x^2 + 2x + 3 \longrightarrow \boxed{x^2} \longrightarrow (x^2 + 2x + 3)^2.$$

As you can see, this process is indeed a composition: the output of a function $(x^2 + 2x + 3)$ is input back into another function x^2.

Patterns Observed. There is a pattern to composition that is important that you be able to be aware of. To see the pattern let me present a whole list of compositions, $f \circ g$, where, in each example, the functions f is $f(x) = \sin(x)$, but the function g is different.

$g(x)$	$f(g(x))$
x^2	$\sin(x^2)$
x^3	$\sin(x^3)$
$x^2 + 1$	$\sin(x^2 + 1)$
$\sqrt{x}$	$\sin(\sqrt{x})$
$\cos x$	$\sin(\cos x)$
$\cos^2 x$	$\sin(\cos^2 x)$

Let's do the same thing but with a different function f, say, $f(x) = x^3$. Repeating the above table with the new function f, we get:

$g(x)$	$f(g(x))$
x	x^3
x^2	$(x^2)^3$
x^3	$(x^3)^3$
x^2+1	$(x^2+1)^3$
$\sqrt{x}$	$(\sqrt{x})^3$
$\cos x$	$(\cos x)^3$
$\cos^2 x$	$(\cos^2 x)^3$

Of course, some of the entries in the second column can be simplified. I left them that way so you could see the results of composition.

One more table. This one a little more abstract. Let the function $f(x) = (3x+1)^4$. Compose f with some functions of the form $g(x) = x + h$, where h algebraic quantity (unspecified); or $g(x) = ax$, where, again a is an algebraic quantity — plus variations on these two.

$g(x)$	$f(g(x))$
x	$(3x+1)^4$
$x+h$	$(3(x+h)+1)^4$
$x-h$	$(3(x-h)+1)^4$
ax	$(3(ax)+1)^4$
$-ax$	$(3(-ax)+1)^4$
$ax+h$	$(3(ax+h)+1)^4$
$ax-h$	$(3(ax-h)+1)^4$
$\frac{x-a}{h}$	$(3(\frac{x-a}{h})+1)^4$

Have you gotten the "feel" for composition? As a rough rule, if you want to compose a function $f(x) = \sin(x)$ with another function $g(x) = x^2 + 1$, that is, if you want to compute the function $f \circ g$, you take the "outer function," f in this case, and replace its "argument," that's x, with the value, $g(x)$ of g, that's x^2. Thus,

$$f \circ g(x) = f(g(x)) = \sin(g(x)) = \sin(x^2 + 1).$$

Uncomposing functions. In *calculus*, in order to perform certain calculations on functions, it is important to analyze the function under consideration. Part of that analysis is, perhaps, to realize that the function you are studying can be thought of as the composition of two other (simpler) functions. The importance of this ability to spot composed functions *cannot be overemphasized*!

With the experience of the tables above, you should be able to solve the following exercise without looking at the answer first.

EXAMPLE 5.5. Consider the function $F(x) = (3x^3 - 2x + 1)^6$. Write F as the composition of two other functions — let's call these two functions f and g (That's original!).

EXAMPLE 5.6. Consider the function $F(x) = \sin(1 + x^3)$. Write f as the composition of two other functions f and g; i.e. write $F(x) = f(g(x))$.

The question a student might ask, *if I permitted it*, is how can this decomposition be discerned? I might answer in any of three ways. (1)

Look at a large number of worked out examples, similar to the last example, until you finally get a "feel" for the process, or see the pattern of composition. Or, (2) Imagine how you would calculate the function on your calculator — the consecutive pressing of function buttons. Reread the discussion above. Or, (3) uncompose by substitution.

Uncompose by Substitution. The formal technique of substitution is a way of uncomposing a function. Let me illustrate with a hideous example:

$$y = \left(\frac{x^2 \sin(x^3) + 2x = 1}{\cos(x) - \sqrt{x}} \right)^{23/3}.$$

Isn't that ugly now? That base function offends me! Let me *mask it over*, or *substitute it away*. Let u be defined by

$$u = \frac{x^2 \sin(x^3) + 2x = 1}{\cos(x) - \sqrt{x}}. \tag{1}$$

Now, my original function is not so bad; it becomes

$$y = u^{23/3}, \tag{2}$$

where, u is the quantity defined in (1). What I have just done is uncompose the hideous function into a non hideous part, $u^{23/3}$, and a semi-hideous part, (1).

What equations (2) and (1) represent is the implicit composition of functions: Define

$$y = f(u) = u^{23/3}$$
$$u = g(x) = \frac{x^2 \sin(x^3) + 2x = 1}{\cos(x) - \sqrt{x}}$$

This substitution method is a very important tool used in differentiation of complicated functions and in the integration of functions.

Exercise 5.4. Make an appropriate substitution of variables to help you uncompose the function $y = \tan(x^3 - 2x + 2)$.

6. Shifting and Rescaling

Shifting and rescaling are a terms applied to particular algebraic and composite functions – that's clear I'm sure. Let me divide the discussion into three topics: horizontal shifting, vertical shifting, and rescaling.

- **Horizontal Shifting**

Let $y = f(x)$ be a function of a real variable, and let $c > 0$ be a fixed constant. The graph of f is a certain curve in the xy-plane. Sometimes we want to move the graph *horizontally* a distance of 'c.' We may want to shift horizontally to the right or to the left.

Shift Horizontally to the Right. Define a new function g (whose graph is going to be the graph of f shifted over c units to the right) by

$$g(x) := f(x - c), \qquad x \in \mathrm{Dom}(f).$$

Shift Horizontally to the Left. Define a new function g (whose graph is going to be the graph of f shifted over c units to the left) by

$$g(x) := f(x + c), \qquad x \in \text{Dom}(f).$$

- **Vertical Shifting**

Let $y = f(x)$ be a real-valued function, and let $c > 0$ be a fixed constant. The graph of f is a certain curve in the xy-plane. Sometimes we want to move the graph *vertically* a distance of 'c.' We may want to shift vertically upward or downward.

Shift Vertically Up. Define a new function g (whose graph is going to be the graph of f shifted up c units) by

$$g(x) := f(x) + c, \qquad x \in \text{Dom}(f).$$

Shift Vertically Down. Define a new function g (whose graph is going to be the graph of f shifted down c units) by

$$g(x) := f(x) - c, \qquad x \in \text{Dom}(f).$$

- **Rescaling**

Rescaling is a term taken from the applications this technique has in many of the applied sciences. In the sciences, the variables of interested are observable, perhaps measurable quantities. Quite often the variables are measured in a certain *scale of measurement*: inches, meters, pounds, liters, etc. Sometimes, it is desirable to change a relationship from one scale of measurement to another: this is the origin of the term.

Let $y = f(x)$ be a function of a real variable, and let c be a fixed constant (positive or negative). Define a new function g by

$$g(x) := f(cx), \qquad x \in \mathrm{Dom}(f).$$

The constant c is sometimes called the *scale factor*.

For example, suppose a car travels at a constant speed of $v = 55\,\mathrm{mi/hr}$. As we know, the distance traveled is given by $d = vt = 55t$, where it is understood that t is measured in hours. Let's put this in functional notation: $d = f(t)$, where $f(t) = 55t$. Suppose now we want to measure time in minutes. We still want to know the distance traveled, but the input value, t, will be measured in seconds. Let u denote time as measured in seconds; we know that $u = 60t$, or $t = u/60$. (i.e. when $t = 1$, we want $u = 60$.) The distance function, now is $d = f(t) = f(u/60)$, or, more formally, define

$$g(u) = f(u/60) = 55\frac{u}{60}$$

or,

$$g(u) = \frac{11}{12}u$$

Let's make a calculation. After 6 minutes, how far has the car gone?

$$d = g(6) = \frac{11}{12}(6) = \frac{11}{2}.$$

That is, the car has gone 5.5 miles.

7. Classification of Functions

A brief discussion some of the very common types of functions seen in pure and applied mathematics. At this introductory level, we will survey only the follow types at this time.

1. Polynomials.
2. Rational Functions.
3. Algebraic Functions.

- **Polynomial Functions**

Let be define a polynomial function in a series of definitions.

Polynomial of degree 0. A polynomial of degree 0 is any function of the form:

$$y = a_0,$$

where a_0 is any constant. For example, $y = 2$ is considered a polynomial of degree 0.

Polynomial of degree 1. A polynomial of degree 1 in x is any function of the form:

$$y = a_0 + a_1 x,$$

where a_0 and a_1 are constants. A polynomial of degree 1 is also called a *linear function*. For example, $y = 3+2x$, or more commonly written, $y = 2x + 3$ – this is a polynomial of degree 1 in x. A polynomial of degree 1 in t might be $y = 7t - 3$.

Polynomial of degree 2. A polynomial of degree 2 in x is any function of the form:

$$y = a_0 + a_1 x + a_2 x^2,$$

where a_0, a_1 and a_2 are constants — called the *coefficients of the polynomial*. Such a polynomial is called a *quadratic*, meaning that it is of degree 2. The graph, as you know is typically a *parabola*. A simple example would be $y = 1 + 2x + 2x^2$; this a polynomial of degree 2 in x. Whereas, $y = z^2 - 3z + 5$ is a polynomial of degree 2 in z.

Polynomial of degree 3. A polynomial of degree 3 in x is any function of the form:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

where a_0, a_1, a_2 and a_3 are constants (the coefficients of the polynomial). Examples abound: $y = 1 - 2x + 4x^2 - 8x^3$; $y = 1.23 - 3.42z + 3.141z^2 + 4.4z^3$.

Polynomial of degree 4. A polynomial of degree 4 in x is any function of the form:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4,$$

where a_0, a_1, a_2, a_3, and a_4 are constants (the coefficients of the polynomial). Some examples would be $y = x^4$ (the coefficients $a_0 = a_1 = a_2 = a_3 = 0$; $y = 2t - 7t^4$, and so on.

The reason that I introduced polynomials in this rather monotonous way is for you see them: see what they look like, see what they look like relative to each other. As you can see, a polynomial of degree 2, is a polynomial of degree 1 *plus* one additional term of higher power (the x^2 term). Similarly, a polynomial of degree 3 is a polynomial of degree 2 *plus* one additional term of higher power (the x^3 term).

Here is a chart to drive home this point.

degree 0	2
degree 1	$2 + 5x$
degree 2	$2 + 5x - 12x^2$
degree 3	$2 + 5x - 12x^2 - 7x^3$
degree 4	$2 + 5x - 12x^2 - 7x^3 + x^4$
degree 5	$2 + 5x - 12x^2 - 7x^3 + x^4 - 2x^5$

Let's tackle the general definition of a polynomial.

Definition 7.1. Let n be a positive integer ($n = 1, 2, 3, 4, \ldots$). A *polynomial of degree £n£* in x is any function of the form:

$$\boxed{y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n.}$$

Where, as before, the symbols $a_0, a_1, a_2, a_3, \ldots, a_n$ are constants. These constants are called the *coefficients* of the polynomial.

Domain Analysis: The *natural domain* of any polynomial is $\mathbb{R}$, the set of all real numbers.

More terminology: a_0 is referred to as the *constant term*; a_1 is the coefficient of x (or is the coefficient of the degree 1 term); a_2 is the coefficient of x^2; a_3 is the coefficient of x^3; and of course etc., etc., so on and so forth.

More^2 terminology: If all the coefficients of a polynomial are integer, we say that it is polynomial a *polynomial with integer coefficients*. For example, $y = 3 - 2x^5 + 5x^{23}$ is an polynomial with integer coefficients in x of degree 23. If all the coefficients of a polynomial are rational numbers, we say that it is a *polynomial with rational coefficients*. For example, $y = 3 - \frac{1}{2}x^3$ is a polynomial with rational coefficients. What do you think a polynomial with *real coefficients* is?

EXERCISE 7.1. Describe the function $y = -3 + \dfrac{1}{2}t^5 - 5t^2 + \dfrac{100}{3}t^5$ using the terminology of this section.

EXERCISE 7.2. Argue that any polynomial with rational coefficients can be written as a rational scalar multiple of the polynomial with integer coefficients.

- **Rational Functions**

A *rational function* is any function that can be written as the quotient of two polynomials. More technically,

Definition 7.2. Let $N(x)$ and $D(x)$ be polynomials define a new function $r(x)$ by

$$r(x) = \frac{N(x)}{D(x)}.$$

Domain Analysis: The natural domain of a rational function given by

$$\operatorname{Dom}(r) = \{ x \in \mathbb{R} \mid D(x) \neq 0 \}.$$

This analysis follows from the section on quotients of two functions.

Illustration 1. Here are some quick examples of rational functions:

$$y = \frac{2x^3 - 5x^2}{x^2 - 3x + 2} \qquad y = \frac{x}{x^2 + 1} \qquad y = 2x + \frac{x}{x + 1}$$

The latter example is considered a rational function of x since it can be written as a quotient of two polynomial (get common denominator).

- **Algebraic Functions**

Let's begin by setting the terminology.

Definition 7.3. An*algebraic function* in x is any combination of sums, differences, products, quotients, and roots of x with itself and with other constants (whether numerical or symbolic).

Domain Analysis: The domain of an algebraic function is naturally limited by the presence of *even root functions* and, of the function consists of a ratio of two expressions, the presence of *zeros* in the denominator.

Example 7.1. The following are algebraic functions. Do a domain analysis of each.

$$f(x) = \frac{x^{1/3} - 5x^2}{\sqrt{x^2 - 3x + 2}} \qquad g(x) = \frac{x}{x^{5/4} - 1} \qquad h(x) = \frac{x}{x + 1}$$

The last example is actually a rational function; however, it is true that a rational function is also a algebraic function.

Solutions to Exercises

2.41. *Solution of* (a). Is y a function of x? **Yes**.

Indeed, take the given equation $2x - 5y^3 = 1$ and solve, if possible, for y. We get

$$\begin{aligned} 2x - 5y^3 = 1 &\iff 5y^3 = 2x - 1 \\ &\iff y^3 = \frac{2x-1}{5} \\ &\iff y = \sqrt[3]{\frac{2x-1}{5}} \end{aligned}$$

Thus, for each value of x there is only *one* corresponding value of y. We could name this function as $y = f(x) = \sqrt[3]{\frac{2x-1}{5}}$. It is easy to see that the natural domain is $\operatorname{Dom}(f) = \mathbb{R}$.

Important Fact. (The Existence and Uniqueness of Odd Roots) The pivotal fact used in this calculation is that *any* real number has a *unique* real cube root. More generally, if $n \in \mathbb{N}$ is an *odd natural number* and $z \in \mathbb{R}$ is *any* real number, then $\sqrt[n]{z}$ exists as a real number and is unique. What this means in terms of solving equations is

$$w^n = z \iff w = \sqrt[n]{z} = z^{1/n} \qquad n \text{ odd.}$$

Solution to (b): Is x a function of y? **Yes.**

Take the equation $2x - 5y^3 = 1$ and try to write x in terms of y:

$$\begin{aligned} 2x - 5y^3 = 1 &\iff 2x = 1 + 5y^3 \\ &\iff x = \frac{1 + 5y^3}{2}. \end{aligned}$$

These calculations justifies the conclusion. Each value of y determines only one corresponding value of x. Let's name this function

$$x = g(y) = x = \frac{1 + 5y^3}{2}.$$

It is easy to see that $\text{Dom}(g) = \mathbb{R}$ is the natural domain of g.

Exercise 2.41. ■

2.42. (a) Yes. (b) No.

Solution to (a): Is s a function of t?

Try to solve for s in terms of t. (Why?)

$$s - 4t + t^2 = 1 \iff \boxed{s = 1 + 4t + t^2.}$$

That was simple. Thus, each value of t yields only one value of s; s is indeed a function of t. We can use the notation

$$\boxed{s = f(t) = 1 + 4t + t^2} \qquad \boxed{\mathrm{Dom}(f) = \mathbb{R}.}$$

Solution to (b): Is t a function of s?

Try to solve for t in terms of s. (Why?)

$$s - 4t + t^2 = 1 \iff t^2 - 4t + (s-1) = 0 \tag{A-1}$$

$$\iff t = \frac{4 \pm \sqrt{16 - 4(s-1)}}{2} \tag{A-2}$$

This shows that for each value of s (for which the radicand is nonnegative) there is *two* value of t. Thus, t is not a function of s.

Example Notes: In line (A-2) we used the infamous *quadratic formula*. The solution to the equation

$$ax^2 + bx + c = 0$$

is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is why I set up the equation in line (A-1). Exercise 2.42. ■

2.43. (a) No. (b) No.

Solution to (a): Is x a function of m?

I need to find the x-coordinate(s) of the point(s) of intersection between the line $y = mx$ and the circle $x^2 + y^2 = 1$.

At any point of intersection (x, y), the two variables satisfy both equations simultaneously: If (x, y) is a point of intersection then

$$y = mx \text{ and } x^2 + y^2 = 1.$$

Thus,

$$x^2 + (mx)^2 = 1.$$

Now, solving this last equation for x gives the x-coordinate(s) of the point(s) of intersection.

$$\begin{aligned} x^2 + (mx)^2 = 1 &\iff x^2 + m^2x^2 = 1 \\ &\iff x^2(1 + m^2) = 1 \\ &\iff x^2 = \frac{1}{1 + m^2} \\ &\iff \boxed{x = \pm\sqrt{1}1 + m^2.} \end{aligned} \tag{A-3}$$

Equation (A-3) indicates to me that x is not a function of m: Each value of m yields *two* values of x.

Solution to (b): Is m a function of x?

Given that x is the x-coordinate of the point(s) of intersection between the circle $x^2 + y^2 = 1$ and any line of the form $y = mx$, we must determine the value of m.

Just as in part (a) we have $x^2 + (mx)^2 = 1$, but now we want to solve for m in terms of x (Why?)

$$\begin{aligned} x^2 + (mx)^2 = 1 &\iff x^2 + m^2x^2 = 1 \\ &\iff m^2x^2 = 1 - x^2 \\ &\iff m^2 = \frac{1 - x^2}{x^2} = 1 - \frac{1}{x^2} \\ &\iff m = \pm\sqrt{1 - \frac{1}{x^2}} \qquad \text{(A-4)} \end{aligned}$$

This shows that form each x, $x \neq 0$, $-1 \leq x \leq 1$, there corresponds *two* values of m. This means that m is *not* a function of x.

Example Notes: I assume you have drawn a picture of the described situation. The two algebraic solutions were more difficult than merely looking at the picture of the situation and reaching the proper conclusions based on your understanding of the concepts. Equally well, it is obvious, geometrically, why $x \neq 0$ in (A-4): When x *is* zero, there is no corresponding m at all; for in that case the corresponding line is vertical, the slope of a vertical line is undefined. Exercise 2.43. ■

2.44. I cannot justify it in your *own* mind, only in mine.

Exercise 2.44. ■

2.45. Perhaps one could call this the horizontal line test?

> A curve $\mathcal{C}$ in the xy-plane defines x as a function of y if it is true that every horizontal line intersects the curve at *no more than* one point.

If this be true, then for each y there corresponds at most one x — this is descriptive of the concept of x is a function of y. (Note: if for a given y the horizontal line at altitude y does not intersect the curve, this means that y does not belong to the domain of the function.)

Exercise 2.45. ■

2.46. Simply restate the Function Line Test using these different letters:

Solution to (a): $\mathcal{C}$ defines s as a function of t provided every line perpendicular to the t-axis intersects the curve at no more than one point.

Solution to (b): $\mathcal{C}$ defines t as a function of s provided every line perpendicular to the s-axis intersects the curve at no more than one point.

Exercise 2.46. ■

2.47. Let $\mathcal{C}$ be a curve in the pq-plane.

Under what conditions can we assert that p is a function of q? Review the definition of function in your head, and answer this question without error. Passing Score: 1 out of 1.

(a) Every line perpendicular to the p-axis intersects the curve at no more than one point.

(b) Every line perpendicular to the q-axis intersects the curve at no more than one point.

Exercise 2.47. ■

4.1. Yes.

$$f(x) = \frac{1}{\sqrt{x+1} - \sqrt{x}} \qquad g(x) = \sqrt{x+1} + \sqrt{x}.$$

Domains Equal? We must have $x+1 \geq 0$ and $x \geq 0$ for the radicals in the two functions to be real numbers. The next question, is whether the denominator of f can ever be zero; indeed,

$$\begin{aligned} \sqrt{x+1} - \sqrt{x} = 0 &\implies \sqrt{x+1} = \sqrt{x} \\ &\implies x + 1 = x \\ &\implies 1 = 0 \end{aligned}$$

We have argued that *if* $\sqrt{x+1} - \sqrt{x} = 0$, *then* $1 = 0$; therefore, we conclude $\sqrt{x+1} - \sqrt{x} \neq 0$ for any $x \in \mathbb{R}$.

The domains of these two functions requires $x + 1 \geq 0$ and $x \geq 0$; therefore,

$$\boxed{\operatorname{Dom}(f) = \operatorname{Dom}(g) = [\,0, \infty\,).}$$

Pointwise Equal? Suppose $x \in \mathrm{Dom}(f) = \mathrm{Dom}(g)$, then

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{x+1} - \sqrt{x}} \\ &= \frac{1}{\sqrt{x+1} - \sqrt{x}} \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}} \\ &= \frac{\sqrt{x+1} + \sqrt{x}}{(x+1) - x} \\ &= \sqrt{x+1} + \sqrt{x} \\ &= g(x) \end{aligned}$$

Thus,

$$\boxed{f(x) = g(x) \qquad \text{for all} x \in \mathrm{Dom}(f) = \mathrm{Dom}(g).}$$

Conclusion: Yes indeed, $f = g$.

Example Notes: You did remember the trick of multiplying by the conjugate, didn't you? It is also important to note that the quantity

$\sqrt{x+1}+\sqrt{x}$ is never equal to zero; therefore, multiplying the numerator and denominator by this quantity is equivalent to multiplying by one, no matter what the value of x.

Exercise 4.1. ■

4.2. No. Recall,

$$f(x) = \frac{x}{\sqrt{x^2 + x} - x} \qquad g(x) = \sqrt{x^2 + x} + x.$$

Domains Equal? I think we can take a useful shortcut. Note that $0 \notin \mathrm{Dom}(f)$ since $x = 0$ makes the denominator equal to zero (and the numerator too). But $0 \in \mathrm{Dom}(g)$ since we can calculate $\sqrt{x^2 + x} + x$ for the case of $x = 0$; indeed, $g(0) = 0$. Thus, we have argued that $0 \notin \mathrm{Dom}(f)$ and $0 \in \mathrm{Dom}(g)$. Therefore,

$$\mathrm{Dom}(f) \neq \mathrm{Dom}(g),$$

and so,

$$f \neq g.$$

All done!

Exercise Notes: However, we can say $\mathrm{Dom}(f) \subseteq \mathrm{Dom}(g)$ and that for all $x \in \mathrm{Dom}(f)$ we have $f(x) = g(x)$. In this case, we say that g is an *extension* of f.

Exercise 4.2. ■

4.3. Look at the calculating formula. The concept is to add the values of f and g together. So given an x, we need to have defined the value $f(x)$. This implies $x \in \mathrm{Dom}(f)$. Similarly, we need to add $f(x)$ to $g(x)$ — $g(x)$ needs to be a defined quantity; therefore, $x \in \mathrm{Dom}(g)$ as well. We have argued that in order to carry out the concept of summing two functions together, we must choose an x in both $\mathrm{Dom}(f)$ *and* in $\mathrm{Dom}(g)$. Thus, $x \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$. 'Nuff said. Exercise 4.3. ■

4.4. Yes. The domain of F is void (empty).

$$\mathrm{Dom}(f) = (\,-\infty, 1\,], \text{ and}, \mathrm{Dom}(g) = [\,2, \infty\,)$$

Thus,

$$\begin{aligned} \mathrm{Dom}(F) &= \mathrm{Dom}(f) \cap \mathrm{Dom}(g) \\ &= (\,-\infty, 1\,] \cap [\,2, \infty\,) \\ &= \emptyset \end{aligned}$$

The "function" F has no domain, hence, F remains *undefined.*

Exercise 4.4. ■

4.5. This is a simple generalization of the sum/difference of two functions.

$$\mathrm{Dom}(F) = \mathrm{Dom}(f) \cap \mathrm{Dom}(g) \cap \mathrm{Dom}(h).$$

How do you calculate the values of F?

$$F(x) = f(x) + g(x) - h(x), \qquad x \in \mathrm{Dom}(F).$$

Exercise 4.5. ■

4.6. At the first level f can be broken down into two pieces:

$$5x^3 \sin(x) \cos(x), \text{ and } 3\sqrt{x} \tan(x).$$

At a second level:

$$5, \quad x^3, \quad \sin(x), \quad \cos(x), \quad 3, \quad \sqrt{x}, \quad \tan(x).$$

Each (much simpler) functions are the ones used to "build" the functions f – through a combination of scalar multiplication, addition, subtraction and multiplication of functions. Exercise 4.6. ■

4.7. Define $f(x) = 2x^3 - 1$, and $g(x) = \sin(x)$, then $F = f^5$, and $G = g^2$.

Exercise 4.7. ■

4.8. Three! But is it the three you had in mind? Define $f(x) = x$, $g(x) = \sin(x)$, and $h(x) = 1$. Then,

$$F = f^4 g^2 + g^2 - h$$

or,

$$F(x) = x^3[\sin(x)]^2 + [\sin(x)]^2 - 1, \qquad x \in \mathrm{Dom}(F).$$

What is the natural domain of definition of F? Is there another way of writing F?

Exercise 4.8. ■

4.9. We say that $f < g$ over the set A provided

$$f(x) < g(x) \qquad \forall x \in A.$$

Geometrically, this means that the graph of f is always *below* the graph of g when plotting these functions over the set A.

To be true to our mathematical roots we should realize what about the set A?

(a) $A \subseteq \mathrm{Dom}(f)$ (b) $A \subseteq \mathrm{Dom}(g)$
(c) $A \subseteq \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ (d) n.o.t.

Part of successfully answering this question is being able to read and *understand* the proposed questions. Then after some thought, respond correctly with error. How did you do? Exercise 4.9. ■

5.1. I'll leave the details to you. But I will tell you,

$$(f \circ g \circ h)(x) = \tan((\frac{1}{x^2+1})^2).$$

Verify!

Exercise 5.1. ■

5.2. A challenge to our perspicacity, without doubt.

Calculation of $W \circ M$. By convention, we take as the independent variable symbol, the symbol used by the inner-most function for its independent variable. Thus

$$\begin{aligned}(W \circ M)(u) &= W(M(u)) \\ &= W(u^2) \\ &= \cos(u^2)\end{aligned}$$

But, of course, the independent variable is a dummy, so it really didn't matter what letter we used. Thus,

$$(W \circ M)(u) = \cos(u^2) \quad (W \circ M)(x) = \cos(x^2) \quad (W \circ M)(t) = \cos(t^2)$$

all define exactly the same function. The use of u was just a little more convenient to use than the other variables.

Calculation of $M \circ W$. By convention, we take as the independent variable symbol, the symbol used by the inner-most function for its independent variable. Thus

$$\begin{aligned}(M \circ W)(x) &= M(W(x)) \\ &= W(\cos(x)) \\ &= (\cos(x))^2 \\ &= \cos^2(x)\end{aligned}$$

Ditto the comments made above. Exercise 5.2. ■

5.3. The height of triviality: $w = (t^2 - 1)^2 + 2$. I hope you got it.

Exercise 5.3. ■

5.4. Let $u = x^3 - 2x + 2$, then

$$y = \tan(u), \text{ where } u = x^3 - 2x + 2.$$

Exercise 5.4. ■

7.1. This is a polynomial in t of degree 5 with rational coefficients.

Exercise 7.1. ■

7.2. First "proof by example." This is not a proof, but it is frequently used to get insight into how to formally prove an assertion. Take as an example:

$$y = \frac{1}{2} - \frac{2}{3}x + \frac{5}{6}x^3.$$

Get a common denominator *for the coefficients* — that would be 6. Thus,

$$y = \frac{3}{6} - \frac{4}{6}x + \frac{5}{6}x^3.$$

Now, factor out 1/6.

$$y = \frac{1}{6}\left(3 - 4x + 5x^3\right).$$

As you can see, the original polynomial is written as a rational scalar multiple (that is, a scalar multiple that is a rational number) times a polynomial with integer coefficients.

The general proof is an abstract manifestation of this example.

Exercise 7.2. ■

Solutions to Examples

2.14. *Solution of* (a).
If we are posing the question: "Is y a function of x?", then we take the attitude that x is the independent variable and y is the dependent variable. The statement that "x and y are real variables" asserts that the domain of the x variable (since it is the independent variable) is a subset of $\mathbb{R}$, and that the co-domain of the y variable (the dependent variable) is a subset of $\mathbb{R}$. (Which we can assume to be equal to $\mathbb{R}$, see the Exercise above.)

We ask the question now, does the equation $2x^2 - 3y = 1$ establish a relationship between x and y such that if we assign x a value, then the equation determines a corresponding *unique* value for y?

Typically at this elementary level, we need only try to express y, the dependent variable, in terms of x. Indeed, by elementary algebra we have

$$y = \frac{1}{3}(1 - 2x^2).$$

Therefore, it is clear that if we do assign x a value, and replace x in this equation by this assigned value, then the equation evaluates to what we interpret as the corresponding value of y. For example, if we put $x = 2$ and evaluate the right-hand side of the above equation we obtain the corresponding value of $y = -7/3$.

The answer to the question, then, is "yes," y is a function of x. Let's give this function a name, say f. Thus,

$$f(x) = \frac{1}{3}(1 - 2x^2),$$

and so,

$$f(2) = -\frac{7}{3},$$

as above.

But the analysis is really not complete until we determine at least the domain of this function f. Since x is supposed to be a *real variable*, we know $\text{Dom}(f) \subseteq \mathbb{R}$. Without any further information about the domain (possibly put on the function by physical constraints – certainly, not the issue in this simple academic problem), we take $\text{Dom}(f)$ to be natural domain: the largest subset of $\mathbb{R}$ (since x is a real variable), for which the correspondence can be made. This would be

$$\text{Dom}(f) = \mathbb{R}.$$

This is because, for any $x \in \mathbb{R}$, we can evaluate the left-hand side of the equation $y = \frac{1}{3}(1 - 2x^2)$ to obtain a *real value* for y.

Solution of (b).
Now let's study the next question: "Is x a function of y?" Again, x and y are related by the equation $2x^2 - 3y = 1$. Given a value of y (in the domain – yet to be determined), does this y *uniquely* determine the value of x? If we take our defining equation and try to solve for x in terms of y we get

$$x = \pm\sqrt{\frac{1}{2}(1 - 3y)}.$$

The '$\pm$' is a signal to us that this is not a functional relation.

To illustrate, give y a value, say $y = 2$, and formally, substitute this value into the above expression to obtain: $x = \pm\sqrt{-5/2}$. Ops! The right-hand side does not evaluate to a real number (as required by the condition that x be a real variable). This does not mean that x is or is not a function of y, it simply means that we have chosen a value of y that is not in the proper domain.

Now look at the case $y = -1$; substituting this value into the equation we get $x = \pm\sqrt{2}$. Thus, corresponding to a value of $y = -1$ there are *two* values of x – $x = \sqrt{2}$ and $x = -\sqrt{2}$. This *denies* the concept (definition) of a function. Therefore "x is not a function of y."

Example 2.14. ■

4.1. *Solution*:

Domains the Same? The domains of these two functions were not given; therefore, we take their *natural* domains of definition. In this case, both functions have domain of $\mathbb{R}$. Thus,

$$\mathrm{Dom}(f) = \mathrm{Dom}(g) = \mathbb{R}.$$

Functions Pointwise Equal? Now we must argue that these functions, pointwise, have the same values. This can be seen by noticing:

$$\begin{aligned} g(x) = \frac{x^3 + x}{x^2 + 1} &= \frac{x(x^2 + 1)}{x^2 + 1} \\ &= x \\ &= f(x). \end{aligned}$$

You did notice the common factor, didn't you? Thus

$$f(x) = g(x), \qquad x \in \mathbb{R}.$$

Example 4.1. ■

4.2. *Solution*: With the experience of the previous example, we first need to determine the respective domains of these two functions.

$$\operatorname{Dom}(f) = \mathbb{R}$$
$$\operatorname{Dom}(g) = \{ x \in \mathbb{R} \mid x \neq \pm 1 \}$$

Thus $\operatorname{Dom}(f) \neq \operatorname{Dom}(g)$ and therefore $f \neq g$.

Example Notes: For $x \neq \pm 1$,

$$g(x) = \frac{x^3 - x}{x^2 - 1} = \frac{x(x^2 - 1)}{x^2 - 1} = x = f(x).$$

Thus, the two functions f and g have the same rule of association but they have slightly different domains. As a result, they are considered different functions.

▪ This is an important point: To be equal, a two functions *must* have the exactly the same domain, and exactly the same rule of association.

▪ In this example, the functions are considered different, but they are *related*. Because $\operatorname{Dom}(g) \subseteq \operatorname{Dom}(f)$ and $f(x) = g(x)$ for all $x \in$

Dom(g), we say that f *extends* the definition of g to a larger domain (namely, Dom(f)).

Example 4.2. ■

4.3. *Solution*: Yes. Equal domains: $\text{Dom}(f) = \text{Dom}(g) = \mathbb{R}$.

Now for $x \neq 1$,

$$
\begin{aligned}
g(x) &= \frac{x^2 + x - 2}{x - 1} \\
&= \frac{(x+2)(x-1)}{x-1} \\
&= x + 1 \\
&= f(x).
\end{aligned}
$$

Thus, $g(x) = f(x)$ for all $x \neq 1$. What about the case $x = 1$? Let's see: $f(1) = 3$ and $g(1) = 3$, so, $f(1) = g(1)$. Finally, we can make the declaration:

$$\text{Dom}(f) = \text{Dom}(g) = \mathbb{R} \qquad \triangleleft \ (1)$$

$$f(x) = g(x), \ x \in \mathbb{R}. \qquad \triangleleft \ (2)$$

This means, by definition, that $f = g$. Example 4.3. ■

4.4. *Solution*: The domains of $f+g$ and $f-g$ are specified in the definition:

$$\operatorname{Dom}(f \pm g) = \operatorname{Dom}(f) \cap \operatorname{Dom}(g).$$

Now the domains of f and g are unspecified; therefore, we take their *natural* domains:

$$\operatorname{Dom}(f) = [\,0, \infty\,), \text{ and, } \operatorname{Dom}(g) = \mathbb{R},$$

Obviously, $\operatorname{Dom}(f+g) = \operatorname{Dom}(f-g)$ and

$$\operatorname{Dom}(f+g) = [\,0, \infty\,) \cap \mathbb{R} = [\,0, \infty\,).$$

Thus,

$$\boxed{\operatorname{Dom}(f+g) = \operatorname{Dom}(f-g) = [\,0, \infty\,)}.$$

That seemed straight forward. Example 4.4. ■

4.5. *Solution*: The domain analysis:

$$\text{Dom}(f) = [\,0, \infty\,), \quad \text{Dom}(g) = \mathbb{R}, \quad \text{Dom}(h) = (\,-\infty, 4\,].$$

Thus,

$$\begin{aligned} \text{Dom}(F) &= \text{Dom}(2f) \cap \text{Dom}(3g) \cap \text{Dom}(4h) \\ &= \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h) \\ &= [\,0, \infty\,) \cap \mathbb{R} \cap (\,-\infty, 4\,] \\ &= [\,0, 4\,]. \end{aligned}$$

Some Harping: *Flame On!* Study the above domain analysis. Notice how the notation for the domains of scalar multiplication, sum and difference of functions is utilized as a tool for working through a "complicated" problem. You as a student should strive to master the concepts and the mechanics; however, a more subtle aspect that is oft not emphasized, is that of *mathematical literacy*. Try to master the *notation* introduced in these tutorials and develop a *style* for using it. Notation can be a very important tool for manipulating ideas and working through problems. *Flame off!*

The calculating formula is given by,

$$F(x) = 2\sqrt{x} - 3\sin x + 4\sqrt{4 - x}, \qquad x \in [\,0, 4\,].$$

Example 4.5. ■

4.6. Let's do a domain analysis first. Observe:

$$\text{Dom}(f) = \mathbb{R}, \quad \text{Dom}(g) = (\,-\infty, 1\,] \cup [\,1, \infty\,), \quad \text{Dom}(h) = \mathbb{R}.$$

The domain of g is obtained by solving the inequality: $x^2 - 1 \geq 0$ (verify!). Thus,

$$\begin{aligned} \text{Dom}(F) &= \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h) \\ &= \mathbb{R} \cap ((\,-\infty, 1\,] \cup [\,1, \infty\,)) \cap \mathbb{R} \\ &= (\,-\infty, 1\,] \cup [\,1, \infty\,). \end{aligned}$$

Example 4.6. ■

4.7. One of the first things to question me about is the meaning of the phrase "the most basic of functions." Admittedly, this has perhaps no definite meaning to you, but does it have some intuitive meaning? Maybe that the meaning that I had in mind.

Define three functions: $f(x) = x$, $g(x) = \sin(x)$, and $h(x) = \cos(x)$. Then

$$F = 6f^3gh^2.$$

Is that clear? Example 4.7. ■

4.8. First note:

$$\mathrm{Dom}(f) = \mathrm{Dom}(g) = \mathbb{R}.$$

Now since g is to appear in the denominator, we need to determine when $g(x) \neq 0$. This is more easily accomplished by finding for what values of x is $g(x) = 0$. To this end,

$$\begin{aligned} g(x) &= 0 \\ x^2 - 1 &= 0 \\ (x-1)(x+1) &= 0 \end{aligned}$$

therefore,

$$x = 1, \text{or } x = -1$$

We conclude from this that

$$\{\, x \in \mathrm{Dom}(g) \mid g(x) \neq 0 \,\} = \{\, x \in \mathbb{R} \mid x \neq \pm 1 \,\}.$$

The domain of $F = f/g$ is then,

$$\begin{aligned} \mathrm{Dom}(F) = \mathrm{Dom}(f/g) &= \mathrm{Dom}(f) \cap \{\, x \in \mathrm{Dom}(g) \mid g(x) \neq 0 \} \\ &= \mathbb{R} \cap \{\, x \in \mathbb{R} \mid x \neq \pm 1 \,\} \\ &= \{\, x \in \mathbb{R} \mid x \neq \pm 1 \,\}. \end{aligned}$$

Thus,

$$\boxed{\mathrm{Dom}(F) = \{\, x \in \mathbb{R} \mid x \neq \pm 1 \,\}}.$$

Or, more informally, the domain of $F = f/g$ is all $x \in \mathbb{R}$ different from ± 1.

The calculating formula is the height of triviality to compute — yet, I will do it:

$$F(x) = \frac{x^3}{x^2 - 1}, \qquad x \neq \pm 1.$$

Note the domain specification. Example 4.8. ■

4.9. Let $f(x) = \sqrt{x}$, $g(x) = \sin(x)$, and $h(x) = x^2 - 3$, then $F = (fg)/h$.

What about the domain analysis? Can you determine it by inspection? What is it? Clearly it has to be

$$\begin{aligned} \mathrm{Dom}(F) &= \{\, x \mid x \geq 0 \text{ and } x \neq 1 \,\} \\ &= [\, 0, 1\,) \cup (\, 1, \infty\,) \end{aligned}$$

Verify this please using the meticulous methods demonstrated in the earlier examples/exercises. Example 4.9. ■

5.1. Taking the definition are doing symbolic replacement we get the criterion: for h to be compatible with f for composition we require $\mathrm{Rng}(f) \subseteq \mathrm{Dom}(h)$.

One of the fundamental skills in the mathematical sciences is *replacement*; that is the ability to take a sentence that uses certain symbolics in it and replacing those symbolics with another set of symbolics. Here are some examples:

Terminology	**Notation**	**Compatibility**	**Calculation**
compose f with g	$f \circ g$	$\mathrm{Rng}(g) \subseteq \mathrm{Dom}(f)$	$f(g(x))$
compose g with f	$g \circ f$	$\mathrm{Rng}(f) \subseteq \mathrm{Dom}(g)$	$g(f(x))$
compose h with w	$h \circ w$	$\mathrm{Rng}(w) \subseteq \mathrm{Dom}(h)$	$h(w(x))$
compose w with h	$w \circ h$	$\mathrm{Rng}(h) \subseteq \mathrm{Dom}(w)$	$w(h(x))$
compose A with B	$A \circ B$	$\mathrm{Rng}(B) \subseteq \mathrm{Dom}(A)$	$A(B(x))$

Example 5.1. ■

5.2. *Solution of* (a): In order to argue that f is compatible with g for composition we must show $\operatorname{Rng}(g) \subseteq \operatorname{Dom}(f)$. Now $\operatorname{Dom}(f) = \mathbb{R}$, and the $\operatorname{Rng}(g) = [\,0, \infty\,)$. Thus,

$$\operatorname{Rng}(g) = [\,0, \infty\,) \subseteq \mathbb{R} = \operatorname{Dom}(f).$$

Solution of (b): $\operatorname{Dom}(f \circ g) = \operatorname{Dom}(g) = [\,0, \infty\,)$. That was easy! (Of course, I am controlling the questions that I ask myself.)

Solution of (c): The composition of f with g is

$$(f \circ g)(x) = f(g(x)) = f(x^2) = \sin(x^2).$$

That's that. Example 5.2. ■

5.3. For your reference, the three functions are

$$f(x) = x^5 \qquad g(x) = \sin(x) \qquad h(x) = \sqrt{x}.$$

We can work from the outside function inward, or work from the inner-most function outward. In either case, we simply exhibit the courage to follow the formula: (In-to-Out)

$$\begin{aligned}
(f \circ g \circ h)(x) &= f(g(h(x))) && \triangleleft \text{ defn of composition} \\
&= f(g(\sqrt{x})) && \triangleleft \text{ inner-most function } h \\
&= f(\sin(\sqrt{x})) && \triangleleft \text{ next func. out, } g \\
&= (\sin(\sqrt{x}))^5 && \triangleleft \text{ outer-most func., } f
\end{aligned}$$

Or, we can do: (Out-to-In)

$$\begin{aligned}
(f \circ g \circ h)(x) &= f(g(h(x))) && \triangleleft \text{ defn of composition} \\
&= (g(h(x)))^5 && \triangleleft \text{ outer-most func., } f \\
&= (\sin(h(x)))^5 && \triangleleft \text{ next in, } g \\
&= (\sin(\sqrt{x}))^5 && \triangleleft \text{ inner-most, } h
\end{aligned}$$

Any way you cut it, it's the same.

Example 5.3. ■

5.4. Let's do it in two stages. Recall,

$$y = u^4 \qquad u = v^2 + 1 \qquad v = \sin(x).$$

Stage 1: $u = v^2 + 1$ and $v = \sin(x)$ and so $u = (\sin(x))^2 + 1$, or $u = \sin^2(x) + 1$.

Stage 2: $y = u^4$ and $u = \sin^2(x) + 1$ and so

$$y = (\sin^2(x) + 1)^4$$

Example 5.4. ■

5.5. Take $f(x) = x^6$ and $g(x) = 3x^3 - 2x + 1$, now calculate $f \circ g$.

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \triangleleft \text{ by definition} \\
&= (g(x))^6 && \triangleleft \text{ by defn of } f \\
&= (3x^3 - 2x + 1)^6 && \triangleleft \text{ by defn of } g \\
&= F(x). && \triangleleft \text{ by defn of } F
\end{aligned}$$

Example 5.5. ■

5.6. Take $f(x) = \sin(x)$ and $g(x) = 1 + x^3$. Now calculate $f \circ g$:

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \triangleleft \text{ by definition} \\
&= \sin(g(x)) && \triangleleft \text{ by defn of } f \\
&= \sin(1 + x^3) && \triangleleft \text{ by defn of } g \\
&= F(x). && \triangleleft \text{ by defn of } F
\end{aligned}$$

That's nice, a "proof" by definition!

Example 5.6. ■

7.1. We state the domains only, leaving the details to the interested reader — that's you!

$$\mathrm{Dom}(f) = \{\, x \in \mathbb{R} \mid x \neq 1 \text{ and } x \neq 2 \,\}$$
$$\mathrm{Dom}(g) = [\,0, 1\,) \cup (\,1, \infty\,)$$
$$\mathrm{Dom}(h) = \{\, x \in \mathbb{R} \mid x \neq 1 \,\}$$

Example 7.1. ■

Important Points

The correct response is $A \subseteq \text{Dom}(f) \cap \text{Dom}(g)$. This "set-theoretic relation" can translated as "A is a subset of $\text{Dom}(f)$ *and* A is a subset of $\text{Dom}(g)$." That is, the set over which you are comparing the two functions must, of course, lie in the domains of the functions involved. Obviously! Important Point ■