THE UNIVERSITY OF AKRON
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Functions

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1. Introduction

In the world of Mathematics one of the most common creatures encountered is the *function*. It is important to understand the idea of a function if you want to gain a thorough understanding of *Calculus*.

Science concerns itself with the discovery of physical or scientific truth. In a portion of these investigations, researchers (or engineers) attempt to discern relationships between physical quantities of interest. There are many ways of interpreting the meaning of the word "relationships," but in *Calculus* we are most often concerned with *functional relationships*. Roughly speaking, a functional relationship between two variables is a relationship such that one of the two variables has the property that knowledge of it (or knowledge of its value) implies a knowledge of the value of the other variable.

For example, the physical quantity of *area*, A, of a circle is related to the *radius* of that circle, r. Indeed, it is internationally known that $A = \pi r^2$—an equation, I'm sure, you have had more than one occasion to examine in the past. The simple equation $A = \pi r^2$ sets forth the principle of a functional relationship: Given knowledge of the value of one variable (the *independent variable*), r, then we have total knowledge of the value of the other variable (the *dependent variable*), A. This causal (or deterministic) relationship one variable has with another variable is the essence of a functional relationship.

This only difference between the example of the previous paragraph and any other example of a function, either one taken from the applied fields or one that is of a more "purely abstract" nature, is the way in which the functional relationship is defined, and the complexity of that definition. There are many, many ways of defining (or describing) a functional relationship between one variable (or a set of variables) and another variable (or another set of variables). Some of these methods are rather "natural," and you will encounter them if you continue with reading through this tutorial on *Calculus I*; other methods are "unnatural"—you'll encounter them too.

Before we continue with this discussion, perhaps it is best to have a formalized definition of a function—in the next section.

2. The Concept of a Function

Let's begin by presenting a definition of a function. The definition is labeled "Junior Grade" because it is slightly less than rigorous; a rigorous definition, *for those who want to know more*, is given in the Appendix.

Definition 2.1. (Junior Grade) Let A be a set and B be a set. A function, f, from A into B is a rule that associates with each element in the set A a unique corresponding element in the set B. In this case, we write symbolically, $f\colon A \to B$, or $A \xrightarrow{f} B$.

This is a very general definition, and a number of remarks must be made concerning it. This definition will be taken apart in great detail in the next few paragraphs. Many of the other points discussed are expanded on and illustrated in more detail in subsequent sections. These remarks take the form of indented bulleted paragraphs listed below.

Throughout this discussion, f will refer to a *function* from a set A into a set B.

- The meaning of the phrase "a rule that associates" is not clear, imprecise, and is, therefore, non-rigorous. See the rigorous definition of function if desired.

- *Domain and Co-domain of a function.* The set A is called the *domain of the function* f, the set B is called the *co-domain of the function* f. Throughout these tutorials, the symbol $\mathrm{Dom}(f)$ will refer to the domain of the function of f.

In elementary calculus, typically A and B are subsets of the real line (in fact, we can always consider $B = \mathbb{R}$). In other fields of mathematics, however, the domain and co-domain of a function can be subsets of vectors, subsets of functions, and even subsets of quite abstract sets. The domain can consist of a finite set of elements, or consist of an infinite set of elements; similarly, the co-domain may be finite or infinite. The student should not maintain any preconceived notions concerning the nature of domains and co-domains. However, ...

> *Domain and Co-Domain in Calculus*:
> Throughout all of Calculus I and Calculus II, the domain and co-domain of all functions encountered are subsets of the real line, $\mathbb{R}$. (Perhaps there will be a *few* exceptions.) In Calculus III, the domain and co-domain may be subsets of vectors.

Exercise 2.1. Let $f\colon A \to B$, where $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$, discuss the parenthetical remark made above: " . . . in fact, we can always consider that $B = \mathbb{R}$."

■ In order to describe a function in more detail, it is customary to assign a generic symbol to the elements of the domain of a function and to the elements of the co-domain of a function. The "usual" symbolism is to denote a typical element of A, the domain, by the letter x, and a typical element of the set B, the co-domain, by y. These are the usual symbols, but mathematicians don't always stick to the usual—they can suddenly change these generic choices virtually without notice.

▪ *Function Notation.* With the aid of these symbolisms, the "action" of a function can be symbolically represented. From the definition above, the function f is a rule that associates with each element, x in A, a corresponding element y in B. If y is the element in B that corresponds to the element x in A, with respect to the function f, then we write $y = f(x)$ (spoken as y equals f of x), or sometimes we write $f\colon x \mapsto y$ or $x \overset{f}{\mapsto} y$ (but this is not seen too often at this level of play). To summarize, the proper notation for representing a function is

$$y = f(x) \quad \text{or} \quad f\colon x \mapsto y \quad \text{or} \quad x \overset{f}{\mapsto} y. \tag{1}$$

The student should strive the use the notation $y = f(x)$, and to understand the underlying meaning of the notation—that of a function. Throughout your studies, in addition to learning and mastering the different techniques of calculus, and understanding the various concepts of calculus, you should also seek most diligently to become *mathematically literate*: Learn to *write* and *speak* mathematics correctly.

> *Functional Notation*:
> Let f be a function. The symbolism
>
> $$y = f(x)$$
>
> means that, with respect to the function f, y is that element in the co-domain of f that corresponds to x in the domain, $\mathrm{Dom}(f)$ of f.

▪ *On the word "unique" in* Definition 2.1. The word "unique" here is important. It distinguishes the notion of *function* from more general mathematical concept of *relation*. For each element of A there must be associated one and only one element in the set B; two or more associations are not allowed—that would disqualify f as a function. Below are two contrasting trivial examples to illustrate the point.

Illustration 1. Let $A = \{ 0, 1 \}$ and $B = \mathbb{R}$. Define a f from A into B as follows:

$$f\colon 0 \mapsto 17$$
$$f\colon 1 \mapsto 17.$$

(Note: The notation defined in (1) is used to describe the *rule of association*.)

I have just defined a good and proper function. For each element in A there corresponds a *unique* (meaning one) element in B. Even though f associates the number 17 with both 0 and 1, this does not violate Definition 2.1. ■

Illustration 2. Let $A = \{ 0, 1 \}$ and $B = \mathbb{R}$. Define a f from A into B as follows:

$$f\colon 0 \mapsto 17$$
$$f\colon 0 \mapsto 20$$
$$f\colon 1 \mapsto 17.$$

(Note: The notation defined in (1) is used to describe the *rule of association*.)

Here, the rule of association described above associates with each element in A a corresponding element in the set B; however, f associates with the element $0 \in A$ *two* corresponding elements in the set B. This is in violation of the word "unique" in Definition 2.1. The construct f in this illustration is *not a function*. ■

EXERCISE 2.2. The mathematical concept of *relation* is typically introduced in a course on *PreCalculus*. Can you recall the meaning of the word *relation*? Write out a good definition of the notion of a mathematical relation. (Let me start you off. Let A be a set and B be a set. A relation R between sets A and B is ...)

■ *Independent Variable and Dependent Variable*. More terminology! Let $y = f(x)$, $x \in A$ and $y \in B$. The symbol x is called the *independent variable*, and y is the *dependent variable*; also, the variable y is said to be a *function* of the variable x.

The word *variable* refers to the fact that x can be any of the elements (numbers) in the set A. The way a functional relation is described, knowledge of the value of one variables gives us knowledge (through the rule of association) of the other variable. When a variable x is the *independent* variable, that means if we have knowledge of its value (or assign it a value), then, by the rule of association, we have knowledge of the other variable, y, in this case.

The statement that y is the *dependent variable* suggests that its value is determined by (or is dependent on) the value of another quantity— the *independent variable*, x.

■ *A Value of a Function.* Further, if y_0 is a particular element in the co-domain of f, then y_0 is called a *value* of f provided there is some element, $x_0 \in A$, such that $y_0 = f(x_0)$; that is, y_0 is a *value* of the function f if it corresponds, with respect to the rule of f, to some x_0 in the set $A = \text{Dom}(f)$.

Let me do away with the subscripts and rephrase the definition of the value of the function f.

> y is a value of the function f provided there is some x in $\text{Dom}(f)$ such that $y = f(x)$.

Turning things about a bit, if x_0 is a given element in the domain of the function f, and $y_0 = f(x_0)$, then we say that y_0 is the image of x_0 under f.

EXERCISE 2.3. Define a function f by $f(1) = 3$, $f(2) = 3$, $f(3) = 4$, $f(-1) = 7$.

a. Write the domain, $\text{Dom}(f)$, of f.
b. Is 3 a value of this function?
c. Is 1 a value of this function?
d. What is the image of 3 under f?
e. What is the value of $f(1)$?

■ *Co-Domain as a Statement of Type.* The co-domain B can be any set sufficiently large to contain all the values of the function f. Think of the co-domain of a function as a set that describes the *type* of values that a function, such as our f, can take on. Examples: for a function that takes on values in the real number system, the co-domain can be taken to be $\mathbb{R}$, the set of all real numbers; for a function that takes on complex numbers as values, the co-domain of such a function may be considered to be $\mathbb{C}$, the set of all complex numbers; for a function whose values are points in the plane, the co-domain may be considered to be $\mathbb{R}^2$, the plane.

- *Common Variable Specifications.* Throughout mathematics, you will read phrases such as, "Let f be a real-valued function of a real variable." What does this mean? The statement "real-valued" refers to the values of the function: It means that $\operatorname{coDom}(f) = \mathbb{R}$, i.e. the co-domain is taken to be the set of real numbers. (See above for an explanation.) The statement "of a real variable" means that $\operatorname{Dom}(f) \subseteq \mathbb{R}$. Thus, when

$$\operatorname{Dom}(f) \subseteq \mathbb{R} \text{ and } \operatorname{coDom}(f) = \mathbb{R},$$

the function f will be a "real-valued function of a real-variable."

EXERCISE 2.4. I declare f to be an integer-valued function of a integer variable. Describe what this means, and give an example of such a creature. In these notes, the set of all integers is denoted by $\mathbb{Z}$. (Note:

there are infinitely many such creatures roaming the mathematical planes ... just capture and put one on display one!)

Other common specifications that functions are often declared as are

a. Complex-valued function of a complex variable.
b. Real-valued function of a natural variable. (The natural numbers are the numbers 1, 2, 3, 4, ...) Such a function is called a *sequence*.
c. (Calculus III) Real-valued function of a vector-variable.
d. (Calculus III) Vector-valued function of a real-variable.

There are many, many other possibilities.

EXERCISE 2.5. Specify each of the functions described above in the numbered list, and give an example of each.

▪ *The Range of a Function.* As we have seen the role that the co-domain plays may be no more than a statement about the "type of value" a function takes on. If $B = \mathbb{R}$, then the function is real-valued, or if $B = \mathbb{C}$, then the function is complex valued. A concept related to the co-domain of a function is that of its *range*. As usual, $f\colon A \to B$, the *range* of f is that subset of the co-domain consisting of the set of values of the function f. This can be described more concisely as follows:

$$\begin{aligned}\operatorname{Rng}(f) &:= \{\, y \in B \mid y \text{ is a value of } f \,\} \\ &= \{\, y \in B \mid \text{there is some } x \in \operatorname{Dom}(f) \text{ such that } y = f(x) \,\},\end{aligned}$$

here, the notation $\operatorname{Rng}(f)$ denotes the range of f. (The latter rendering is due to the definition of a value of a function f.

Illustration 3. Define a function f from $\operatorname{Dom}(f) = \{\, 1, 2, 3 \,\}$ into $\mathbb{R}$ by $f(1) = 1$, $f(2) = 17$, and $f(3) = -100$. In this case, it is trivial to determine the range of the function f:

$$\operatorname{Rng}(f) = \{\, 1, 17, -100 \,\}.$$

For functions of this elementary type, it is very simple to calculate the range. The functions that we shall encounter in *Calculus* are a bit more complicated. The determination of their range is, as a result, considerably more involved. ■

■ *Defining the Rule of Association.* The last few paragraphs dealt with notation and terminology. Unless you have had some contact with functions in the past, these definitions may have little meaning to you. To rectify this potential problem, let us look at a simple example to illustrate these concepts. A function is a *rule of association between two sets*. At this level of presentation, the principle manner in which this rule is given is through an *algebraic equation*. Let that equation be $y = x^2$, and let us give our function a name, say f; thus, $f\colon x \mapsto y$, where $y = x^2$, is our function. This notation can be abbreviated more tersely as $f\colon x \mapsto x^2$, or as $f(x) = x^2$. This latter notation, $f(x) = x^2$, is by far the most common notation.

Particular correspondences are handled in the obvious way. To illustrate, let us continue with our old friend $f(x) = x^2$, $x \in \mathbb{R}$ (i.e., $\operatorname{Dom}(f) = \mathbb{R}$). Now we know that $4 \in \mathbb{R}$; to find the value in the range of the function that corresponds to the element 4 in the domain of the function, we apply the *rule*: $y = x^2$. Replacing x with 4 into the defining equation we get $y = 4^2 = 16$. Thus, 16 corresponds to 4 with respect to the function $f(x) = x^2$. (The number 16 is the *value* *of* f *at* 4 and 16 is the *image* of 4 under f) This may be written as either $f(4) = 16$, or $f\colon 4 \mapsto 16$.

Now we do not want to go through this analysis each time we wish to calculate the particular value, so usually we proceed as follows:

$$f(x) = x^2, \qquad x \in \mathbb{R}$$

put $x = 4$,

$$f(4) = 4^2,$$

thus,

$$f(4) = 16.$$

EXERCISE 2.6. Define a function by $f(x) = \dfrac{x}{x^2+1}$. Calculate $f(2)$.

▪ *Specifying the Domain and Co-Domain.* A function is more than a rule of association, such as $f(x) = x^2$. A function must have a specified *domain* and a *co-domain* (indeed, these were as much a part of the definition of function as the rule itself). The domain is defined by whomever is creating the function; in this case, I am creating this function, so I declare the domain to be the set of all real numbers, i.e. let $A = \mathbb{R}$, and let the co-domain be the set of all real numbers as well, i.e. let $B = \mathbb{R}$. Thus is the power of the mathematician; however, such power must be tempered by *reason*.

EXERCISE 2.7. Let $f(x) = x^2$. Is it possible to assign the function f an "unreasoned" domain (an unreasonable domain)? Is it possible to assign the function f an "unreasoned" co-domain (an unreasonable co-domain)?

Holding the rule of association *fixed* and changing the domain, in fact changes the *function*. To illustrate, define

$$f(x) = x^2, \text{ for } x \in \mathbb{R} = \operatorname{Dom}(f)$$

and,

$$g(x) = x^2, \text{ for } x \in [\,0, 1\,] = \operatorname{Dom}(g).$$

The function f and g have the same rule of association, but different domains. These functions are considered *different*; indeed, they cannot be the same function since $f(2)$ is a defined quantity, where as $g(2)$ is not. (See the definition of equality of functions below.)

▪ *More Discussion of Range.* The range of a function f obviously depends on the rule of association, but *it also depends on the domain of the function.* This is because a function consists of a rule of association *and* the specification of a *domain.* See discussion above. If the domain is changed, then the range *may be changed also.*

Illustration 4. The range for the function $f(x) = x^2$, $\text{Dom}(f) = \mathbb{R}$, is

$$\text{Rng}(f) = \{\, y \in \mathbb{R} \mid y \geq 0 \,\} = [\,0, \infty\,); \tag{2}$$

whereas the range of the function $g(x) = x^2$, where $\text{Dom}(g) = [\,-3, 2\,]$, is

$$\text{Rng}(g) = [\,0, 9\,].$$

▪

EXERCISE 2.8. Let $f(x) = x^2$, $x \in \mathbb{R}$. It was stated above that if we change the domain, the range *may be changed also*. In illustration 4, the range was changed when we changed the domain. *Your Assignment*: Change the domain of f in such a way that the range does *not change.*

EXERCISE 2.9. Suppose we have a function f whose domain consists of only a finite number of elements. Must the range consist of only a finite number of elements too?

- *The Argument of a Function.* Let $y = f(x)$. Let us agree that the quantity within the parentheses of f always be called the *argument* of the function f. The argument is the quantity upon which f acts to obtain the corresponding y-value. As a simple example, let $f(x) = x^2$, here the argument of f is x. But now, what is the argument of $f(x+1)$? Answer: $x+1$. What is the argument of $f(x^2)$? Answer: x^2.

EXERCISE 2.10. Let $f(x) = \sin(x)$. What are the arguments of f of each of the following: $\sin(x^2)$ and $\sin(\sqrt{x+1})$?

EXERCISE 2.11. What is the argument of the square-root function in the expression $\sqrt{x+1}$.

The next exercise contains within its solution some important teaching points concerning the notion of argument of a function. Read it and determine your responses before your peek at the answers.

EXERCISE 2.12. What are the arguments of each of the following:

a. $\sin(x^2)$.
b. $\cos(\sin(x))$.
c. $(x+1)^3$.

2.1. Constructing Functions

There are many, many ways of constructing functions; some of them require a knowledge of *Calculus*. In this section, we will take a survey of methods that are within your grasp at this time. Other methods will be brought out in the process of developing the *Calculus*.

• The Use of Algebraic Expressions

Let x denote a mathematical variable. An *algebraic expression in x* is any combination of sums, differences, products, quotients, and roots of x with itself and with other constants (whether numerical or symbolic).

I'm sure you have seen many examples; here is a few more.

1. $x^3 - 3x^2 + 12x + 10$.

2. $mx + b$.
3. $\dfrac{2y-1}{y^4-3}$.
4. $\dfrac{2.3x^2 - 27.3x}{\sqrt{x^2-3x+2}}$.

Some additional comments are in order. In item (2), we have two symbolic constants m and b. In item (3), we fooled you there, this is an algebraic expression in the variable y.

> **Rule**. As a general rule, any algebraic expression in a single variable determines a function.

The reason this rule is true is that when we give a value to the real variable, the algebraic expression evaluates to a *single* real number. This is the essence of a functional relation.

Illustration 5. Take $x^3 - 3x^2 + 12x + 10$, as defined above. To create a function we just say: Define a function f by

$$f(x) = x^3 - 3x^2 + 12x + 10,$$

and that's all there is to it! Well, not quite. A function also has a domain. We must consciously be aware of the domain of the function we are working with. The domain of f would be its natural domain of definition: $\operatorname{Dom}(f) = \mathbb{R}$.

Illustration 6. A second illustration will be sufficient. Consider the algebraic expression:

$$\frac{2.3x^2 - 27.3x}{\sqrt{x^2 - 3x + 2}}$$

Define a new function, g, by

$$g(x) = \frac{2.3x^2 - 27.3x}{\sqrt{x^2 - 3x + 2}}, \qquad x \in \operatorname{Dom}(g). \tag{3}$$

■ *Stand-alones.* A "stand-alone" algebraic expression is nothing more than an anonymous function. For example, the algebraic expression, x^2, is a function. (Just give it a name: $f(x) = x^2$.) The algebraic expression $\sqrt{3x^2 - 3x + 1}$ is also a function. The expression is the *rule of association*: Give x a value, evaluate expression, and *le voilá*, the value of the function is obtained.

EXERCISE 2.13. Calculate the domain of the function g defined in (3).

• Piecewise Definitions

Another method of creating functions is through a piecewise approach. In this method, you take algebraic expressions (or other functions that have already been defined) and 'piece' them together in such a way as to create a function. Here are a few examples.

Consider the function, $y = f(x)$ defined by

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

It was stated that this is a function, but is it? (Maybe I am playing with your head!) Is it true that for each value of x there is only one corresponding value $f(x)$ defined? Given any x, either $x < 0$, or $x \geq 0$ — that's obvious. Now in the first case, $x < 0$, the corresponding value of y is equal to the value of x; i.e. a *single* value y is defined. In the other case, $x \geq 0$, the corresponding value of y is the x^2, and there is only one number x^2, when x is a real number. So we can conclude that f is indeed a function.

EXERCISE 2.14. Consider the function defined above, i.e.:

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

You should be able to read this notation well enough to be able to compute $f(-2)$, $f(-1/2)$, $f(0)$, $f(1/2)$, $f(3)$. Give yourself 1 point for each correct answer!

In the above example we 'pieced' two functions together: the function $y = x$ and the function $y = x^2$. We can piece three functions together:

$$g(x) = \begin{cases} x^2 - 2x + 1 & x \le -1 \\ \sin x & -1 < x \le 3 \\ \dfrac{1}{3\sqrt{x}} & x > 3 \end{cases}.$$

It is possible to piece any number of functions together. You just have to piece them together so that you create a function.

EXERCISE 2.15. Define a 'function' by

$$f(x) = \begin{cases} x^2 & x \le 2 \\ x^3 & x \ge 1 \end{cases}.$$

Something is wrong here. As defined, is f a function?

EXERCISE 2.16. Based on your experience from the last exercise, would you say that

$$f(x) = \begin{cases} x^2 & x \le 1 \\ x^3 & x \ge 1 \end{cases}.$$

is a function or not? Note that the domains of the two pieces overlap!

Exercise 2.17. Is g a function? Where,

$$g(x) = \begin{cases} |x| & x \le -3 \\ 2x + 9 & -3 \le x \le 3 \\ x^3 - 10 & x \ge 3 \end{cases}$$

Exercise 2.18. Is h a function? Where,

$$h(x) = \begin{cases} |x| & x \le -3 \\ -x & -4 \le x \le 1 \\ x^2 & x > 1 \end{cases}$$

Exercise 2.19. Formulate some rules for constructing piecewise defined functions so that what you are defining is indeed a function.

An Applied Illustration. Piecewise defined functions do occur in the physical world. For example, imagine an electrical circuit. The switch in the circuit is "off" and so the current in the circuit is 0. At a certain time $t = 0$, you turn the circuit "on." You are interested in *current* as a function of *time*. This functional relationship might be described by a piecewise definition:

$$i(t) = \begin{cases} 0 & \text{if } t < 0 \\ \dfrac{t}{t+1} & \text{if } t \geq 0 \end{cases} \tag{4}$$

Here, $i(t) =$ current at a point in circuit at time t. As you can see, at time $t = 0$, there is a radical change in basic functional relationship between i and t.

Famous Piecewise Defined Functions.
There are several useful functions that are piecewise defined. Below is a listing and a brief discussion of each.

The Absolute Value Function. The absolute value function is defined by

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

The absolute function has a famous name, hence has a specialized notation for its name.

Notes: The *absolute value function* is a very useful critter. One use of it is that it reads back to you the distance a number is away from origin. The number 5 is $|5| = 5$ units away from the origin; the number -6.2 is $|-6.2| = 6.2$ units away from the origin.

■ Another related application of the *absolute value function* is that it is used to measure distance between two numbers on a number line. Let a and b be (real) numbers. Then,

$$|a - b| = \text{the distance between } a \text{ and } b. \tag{5}$$

Throughout *Calculus* you will see absolute values put to many uses.

■ An important fact you need to keep ever in mind when dealing with expressions involving an even root of a perfect square is

$$\sqrt{x^2} = |x| \quad \forall x \in \mathbb{R} \tag{6}$$

Commit this concept to memory. ■

Here's a "tricky" application to the above points.

Exercise 2.20. (The distance between horizontally oriented points) Let $P(a, b)$ and $Q(c, b)$ be two horizontally oriented points in the plane. (How do I know that?) Calculate the distance between P and Q.

Exercise 2.21. (The distance between vertically oriented points) Let $P(a, b)$ and $Q(a, c)$ be two vertically oriented points in the plane. Calculate the distance between P and Q.

The Heaviside Function. The Heaviside function is defined by

$$H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

This function is an “on-off” function, and is used as such.

Example 2.1. Consider the function defined in (4). Use the Heaviside function to describe this function.

Example 2.2. Define a function g to be

$$g(x) = H(x - 2) \qquad x \in \mathbb{R}.$$

Write g as a piecewise defined function.

Exercise 2.22. Define the function f by

$$f(x) = \begin{cases} 0 & x \le 2 \\ x^3 & x > 2 \end{cases}$$

Write the function f in terms of the Heaviside function.

Exercise 2.23. Consider the piecewise function

$$f(x) = \begin{cases} x^2 & x \le 2 \\ x^3 & x > 2 \end{cases}$$

Write the function f in terms of the Heaviside function.

Exercise 2.24. Consider the piecewise function

$$f(x) = \begin{cases} \sin(x) & x \le -1 \\ x\sin(x) & x > -1 \end{cases}$$

Write the function f in terms of the Heaviside function.

There is a pattern, do you see it?

• Descriptive or Conceptual Methods

The sciences are replete with examples of functions that are *described*. These functions may be only conceptual. It may not be possible to explicitly write down a calculation formula for them; however, these functions are among the most important in science. Quite often, scientists and mathematicians attempt to *approximate* or *model* these kinds of functions.

Functions that need to be modeled. Many functional relationships are defined by a descriptive definition. They are typically quite complex and need, therefore, to be modeled or approximated.

Illustration 7. For example, suppose we have a well-specified population of objects, and let t represent a time variable (measured in seconds, minutes, hours, days, years, etc.). Define a function, p, as a function of t, as follows:

$$p(t) = \text{size of the population at time } t.$$

I have *described* the relation of interest. Is this a function? One would say, "yes," because, at any given time t, one would think that there is exactly *one* population size at that time t.

Illustration 8. Another example might be from electrical circuits. At any given point in an electrical circuit, we can measure current. Let us choose and fix a point in the circuit at which to measure current, and let t represent time. Define

$$i(t) = \text{current at time } t.$$

Again, the definition of i is described, yet conceptually, it defines a function.

Functions that can be Calculated Exactly. Some functions are described (sometimes) geometrically, but are of such a nature that they can be written down in a analytical form.

The next example is of a function that is described, yet a formula can be obtained for calculating the values. You, no doubt, have seen these kinds of functions many times.

Example 2.3. Define $f(x) = x$ and $g(x) = x^2$ for $0 \leq x \leq 1$. Draw the graphs of these two functions. Observe that they enclose a region in the plane. For each x, $0 \leq x \leq 1$, draw a vertical line at x and extending upwards through the region. Define $L(x)$ to be the length of the line segment that (vertically) spans the region at x. *Calculate* $L(x)$.

Example 2.4. Define a function $f\colon x \mapsto y$. Let x be a real number, define $f(x)$, if it exists, to be the ordinate of intersection between the line $s = xt$ and the line $s = 2 - t$ in the ts-axis system. What is $\mathrm{Dom}(f)$?

Exercise 2.25. Define a function $f\colon x \mapsto y$. Let x be a real number, define $f(x)$, if it exists, be the t-intercept of the line $s = 2t - 5x$ in the ts-axes system. What is $\mathrm{Dom}(f)$?

Exercise 2.26. Define a function $g\colon x \mapsto y$. Let x be a real number, let $g(x)$, if it exists, be the y coordinate the point of intersection between the two lines $y = 3t + 4x$ and $y = -2t + 3x$ in the ty-axes.

EXERCISE 2.27. Define a function $h\colon x \mapsto y$. Let x be a real number. Consider $y = \sin t$ and the line $y = x(t-1)$. Define $f(x)$, if it exists, to be the y-coordinate of the point of intersection between these two curves. Is h a function? Explain! Draw a picture that illustrates the x and the corresponding y.

EXERCISE 2.28. Define a function $k\colon x \mapsto y$. Let x be a real number. Consider $y = \sin t$, $0 \le t \le \pi$, and the line $y = x(t-1)$. Define $k(x)$, if it exists, to be the y-coordinate of the point of intersection between these two curves. Is k a function? Explain! What is the domain of k? Draw a picture that illustrates the x and the corresponding y

2.2. Evaluation Issues

There are two ways to evaluate a function: numerically and symbolically. The former type you should be quite familiar with, and the latter being related to the notion of composition of functions.

• Numerical Evaluation

Let $y = f(x)$ be a function. The problem of calculating the value of a function with the symbol x is *replaced* by a numerical value is straight forward enough. Problems students encounter are largely algebraic: reading the formula and properly putting arithmetic operations together in their proper order.

Notice the use of the word "replace" in the previous paragraph. I chose this term deliberately in lieu of the more traditional term "substitute." When we do evaluations, we *replace* the symbolic variable, x, which acts as a place holder, with the value out of the domain at which we want to evaluate the function f. Think of evaluation as a process of replacement: To evaluate $f(x)$ at a particular value, we replace the symbol x with the particular value a to obtain $f(a)$.

Here are some simple examples for your consideration.

Example 2.5. Let $f(x) = \dfrac{x^3 - 1}{3x^2 + 1}$. Calculate $f(-2)$, $f(-1)$, $f(0)$, $f(1)$ and $f(2.12)$.

Take a close look at the function in the following exercise and carefully calculate the indicated values, please. You should be able to *clearly* see the order of computation.

EXERCISE 2.29. Let $f(x) = ((x+1)^3 - 2x)^2$. Calculate $f(-2)$, $f(0)$, and $f(1)$.

- **Symbolic Evalulation**

A function need not be evaluated at a numerical value; sometimes it is important to evaluate a function at a *symbolic value*.

- *The Independent Variable is a Dummy*!

Consider the following function:

$$f(x) = x^2 - 3x + 1.$$

The letter "x" is the independent variable. There is no particular significance to the actual alphabetic letter "x." Any other letter could have been used: t, s, u, v. Within the context of the definition of the function, the independent variable is sometimes referred to as a *dummy variable*.

Read on for an explanation.

A function is a rule of association. That being the case, in order to define a function we must be able to describe this rule of association — the calculation rule, if you will. When we are defining a function such as

$$f(x) = x^2 - 3x + 1, \tag{7}$$

the role that x plays is that of a *place holder*—the symbol x is standing in for the particular elements in the domain (which have not been chosen for calculation yet). The definition tells me, when I do choose an element from the domain of f, how to calculate the corresponding y-value: Simply replace x, the place holder, with the particular value of interest and evaluate.

All of the definitions below describe exactly the same function:

$$
\begin{aligned}
f(x) &= x^2 - 3x + 1 \\
f(t) &= t^2 - 3t + 1 \\
f(s) &= t^2 - 3s + 1 \\
f(A) &= A^2 - 3A + 1 \\
f(\rho) &= \rho^2 - 3\rho + 1 \\
f(\bar{x}) &= \bar{x}^2 - 3\bar{x} + 1.
\end{aligned}
$$

Do you get the idea? Note the use of the Greek letter *rho*, ρ, as a variable, and a compound symbol, x-bar ($\bar{x}$) as a variable.

To emphasize the place holder aspect of the independent variable, and the fact that the choice of the letter makes no difference, let's look at a strange idea. Consider the function

$$f(\ \ ) = (\ \ )^2 - 3(\ \ ) + 1. \tag{8}$$

Does this make sense? Or, is it rather silly? Does this describe the rule of association? Just place any (permissible) value inside the parentheses and *le voilà*! Obviously, this does away with the dependence on a particular letter to describe the rule of association, but it brings forth new monsters to frighten freshmen.

Exercise 2.30. Can you calculate $f(2)$ for the following function?

$$f(\quad) = \frac{(\quad)\sqrt{(\quad)^2 + (\quad) + 3}}{\cos^2((\quad)^2\pi) + (\quad)}$$

■ *What f found in its parentheses*: Another way of interpreting *rule of association* as described in (7) is as follows: the function f takes whatever it finds within its parentheses, squares it, subtracts 3 times what if found within its parentheses, and then adds 1. This point of view is useful in more complicated constructions because its interpretation of the rule of association does not depend on any symbol. For example,

$$\begin{aligned} f(x) &= x^2 - 3x + 1 \\ f(x^2) &= (x^2)^2 - 3(x^2) + 1 \\ &= x^4 - 3x^2 + 1. \end{aligned}$$

This is because the function f found x^2 within its parentheses (x^2 is then the argument of f). When f finds something within parentheses, is squares it $(x^2)^2$, subtracts 3 times what it finds in its parentheses $(x^2)^2 - 3(x^2)$, and finally, it adds 1 to obtain $(x^2)^2 - 3(x^2) + 1$. The "void argument" example in line (8) is a silly illustration of "what f finds in its parentheses."

Summary. I hope you will keep this "lesson" in mind when you study mathematics. The letter used in the definition of a function is a *dummy variable* — meaning that it is just a place holder to help us describe the rule of association.

2.3. What's in a Name

There are many ways of naming a function. In this section we enumerate a few of them. Giving a function a name, allows you to refer to it in an unambiguous way.

- **The "Standard" Way**

The method of naming functions you have encountered in most of your mathematics is using a single letter, such as f, g, or h.

For example,

$$f(x) = x^2, \qquad g(s) = \sqrt{s}, \qquad h(v) = \sin v.$$

This is the functional notation; however, there are other ways of naming functions.

As you will see in *Calculus I*, we will name functions using a combination of *letters* and *symbols*. Some examples of this are

$$f(x) = x^3, \qquad f'(x) = 3x^2, \qquad f''(x) = 6x.$$

Here, we have post-fixed a "prime symbol" to the single letter f; this creates new names f', and f''. Each function has a unique name, and so they can be referred to in an unambiguous manner; thus, I can say: "Consider the function f' ..."—and the classroom groans. (But at least there will be no confusion about what function they are groaning about.)

Again, consider the function $g(x) = x^3$ (groan?). Define an new function g^{-1} by

$$g^{-1}(x) = \sqrt[3]{x}.$$

We have, now, the functions g and g^{-1} defined — each with unique names.

Numerical operations can be used to name functions as well. If $h(x) = 2x^3$, then the function h^2 refers to the function $h^2(x) = 4x^6$.

More often than not, when we post-fix a symbol to the name of a function, f say, the new function is related to f in some mysterious way. In the case of the prime, f' refers to the *derivative* of f; f'' refers to the *second derivative* of f. In the example of g and g^{-1}, g^{-1} is the *inverse* of g. And, of course, h^2 is the *square* of h.

- **Functions Named by the Dependent Variable**

You have seen this kind of function before as well. For example,

$$y = x^2, \qquad y = x^3.$$

Here, I have defined two distinct functions, but if I wanted to refer to one of these two, I would have a bit of a problem. You might call these *anonymous functions* because they have no name—they do not have a built-in way of referencing themselves.

One method of overcoming this problem is to use different symbols for the dependent variable. To continue the discussion above, re-define the functions above as

$$u = x^2, \qquad v = x^3.$$

Now, I can refer to the function u or the function v. If I want to add these two functions together, I would write $u + v$. Notations such as $4u$, uv, $\sqrt{v}$, and so on, should have meaning to you.

Subscripting can also be employed to give anonymous functions distinctive labels:

$$y_1 = \sin(x) \qquad y_2 = \sin(x^2).$$

These two functions can now be referred to by their labels; for example, "Find where the functions y_1 and y_2 intersect."

You may be familiar with this technique in a more applied setting. The area of a circle is given by $A = \pi r^2$. The circumference of a circle is $C = 2\pi r$. The letters chosen help us to remember the physical significance of the symbols involved: $A =$ area, $C =$ circumference, and $r =$ radius.

Sometimes, rather than using single letters to name a function in this way, we use combinations of letters.

$$Area = \pi r^2$$

Now we can refer to this as the "Area" function. (This is actually just an anonymous with a unique name for the dependent variable.) Or, we might even write,

$$Area = \pi(radius)^2.$$

See the section entitled Descriptive Naming for more discussion of this technique.

Other times, combinations of letters and *symbols* are used. For example, define two functions,

$$y = x^2, \qquad y' = 2x.$$

Once again, each function has a unique name: the y-function and the y'-function. If I write $y + y'$ then that would refer to the function defined by the expression $x^2 + 2x$ (i.e. $y + y' = x^2 + 2x$). Thus, I can refer to the $y + y'$-function.

- *Evaluation.* Evaluation of functions so defined is a problem. If $y = x^2$, how do you describe notationally that we want the y-value corresponding to $x = 3$ without writing out a sentence as long as this one? The problem is that anonymous functions have no explicit argument. The solution is two-fold: force an explicit argument, or make up a new notation.

1. Force an explicit argument:

$$\begin{aligned} y &= x^2 & y(3) &= 9 \\ y' &= 2x & y'(3) &= 6 \\ v &= w^3 - w & v(2) &= 6. \end{aligned}$$

2. Use the Evaluation Notation:

$$\begin{aligned} y &= x^2 & y\big|_{x=3} &= 9 \\ y' &= 2x & y'\big|_{x=3} &= 6 \\ v &= w^3 - w & w\big|_{w=2} &= 6. \end{aligned}$$

This notation is used throughout *Calculus* as a notation of evaluation of anonymous functions.

EXERCISE 2.31. (Skill Level -1) Define the function $y = 2x^3 + 1$. Use the two notations defined above to calculate the value of y corresponding to $x = -1$.

- **Descriptive Naming**

In some areas, such as computer science, functions and their variables are named using complete words or phrases. The names of the functions are, if fact, the names of the subroutines that make the calculations.

For example, we could have functions such as

$$Area\_circle = \pi(radius)^2$$
$$Area\_circle(radius) = \pi(radius)^2$$
$$Area\_triangle(base, height) = \frac{1}{2}(base)(height).$$

The last example is a function of two variables. The domain of this function is

$$\mathrm{Dom}(Area\_triangle) = \{\,(\,b, h\,) \mid b > 0, h > 0\,\}.$$

EXERCISE 2.32. Let $Area\_circle = \pi(radius)^2$. If $radius = 2$, discuss how we notationally represent $Area\_circle$.

- **Famous Functions**

Some functions are so famous that they have their own names! The *sine function* is one of many examples of celebrity. Rather than giving the *sine function* a generic name like f, it has its own name: sin. Thus, $y = \sin(x)$ is internationally recognized as the *sine function*.

Some of the "famous" functions you will encounter in *Calculus I* are listed below.

Trigonometric Functions.

1. $y = \sin(x)$
2. $y = \cos(x)$
3. $y = \tan(x)$
4. $y = \cot(x)$
5. $y = \sec(x)$
6. $y = \csc(x)$

Inverse Trigonometric Functions.

1. $y = \sin^{-1}(x)$
2. $y = \cos^{-1}(x)$
3. $y = \tan^{-1}(x)$
4. $y = \cot^{-1}(x)$
5. $y = \sec^{-1}(x)$
6. $y = \csc^{-1}(x)$

Hyperbolic Functions.

1. $y = \sinh(x)$
2. $y = \cosh(x)$
3. $y = \tanh(x)$
4. $y = \coth(x)$
5. $y = \operatorname{sech}(x)$
6. $y = \operatorname{csch}(x)$

Inverse Hyperbolic Functions.

1. $y = \sinh^{-1}(x)$
2. $y = \cosh^{-1}(x)$
3. $y = \tanh^{-1}(x)$
4. $y = \coth^{-1}(x)$
5. $y = \operatorname{sech}^{-1}(x)$
6. $y = \operatorname{csch}^{-1}(x)$

Exponential Functions.

1. $y = \exp(x) = e^x$
2. $y = a^x$

Logarithmic Functions.

1. $y = \ln(x)$
2. $y = \log_a(x)$

In your calculus course you will study in detail each of these types of functions.

2.4. Models for Functions

In this section we present different ways of thinking about functions that may be of help to you.

• A Function as a Mapping

One traditional way of looking at a function is as a *mapping* or a *transformation*. Let $f\colon A \to B$ be a function, and let $x \in A$. As discussed above, $y = f(x)$ is the value of the function at x. We can also look upon f as a mapping or transformation: f maps x onto y, or, y is the *image* of x under f.

This interpretation is one of the origins of the notation introduced above:

$$x \overset{f}{\longmapsto} y.$$

Try to get the feeling for this interpretation. Imagine a bunch of arrows pointing from elements x in the set A to elements y in the set B. The arrows point from each x to the corresponding value of y, as the "arrow notation" above suggests. When we see x we immediately think of its corresponding value $f(x)$. The *Venn Diagram*, described next, is a more visual representation.

• Venn Diagram of a Function

Figure 1

In Figure 1, a pictorial representation of a function (mapping, transformation) is given. This graph represents f as it maps or transforms a typical element x from the domain set A into the co-domain set B. The *image* of x under this map, f, is denoted by y in the figure. Visualize a function as a bunch of "arrows" pointing from set A into set B. The tail of a typical arrow is at x, and the arrow "points" to the corresponding y-value.

This model is very useful in understanding functions and various operations performed on functions (such as composition of functions).

Figure 2

To further illustrate the point, Figure 2 depicts a relation that is *not* a function. A function is a rule that associates with each value x is a certain domain set, a corresponding *unique* y-value. A rule that associates with at least one x *more than one* corresponding y-value would not be a function—as illustrated in Figure 2. Observe that associated with x is *two* corresponding values—labeled y and z.

As a particular example of this, consider the equation: $x^2 + y^2 = 1$. For $x = 0$, there are two values of y that satisfy this equation: $y = 1$ and $y = -1$. This equation does not define, therefore, y as a function of x. (Visualize two arrow coming out of $x = 0$, one pointing to $y = 1$ and the other pointing to $y = -1$.

• A Function as a Black Box

This interpretation of function is often associated with the engineering world. A function is like a machine (a black box). We have a machine (a black box) that takes *input* into it, and, as a result, yields *output*. The black box is the function, the input are the values in the domain of the function, and the output of the box (function) are the values in the range of the function.

$$x \longrightarrow \boxed{\text{function}} \longrightarrow y.$$

Actually, this looks more like a *white* box to me `:={)`.

A black box you are familiar with is the *hand-held calculator*. This is usually, literally, a black box. You input x-values on the key pad, say $x = 12$. You then choose the black box to input this value of x into. Your calculator is actually made up of a large number of black boxes – called *function keys* (hey, function!). Choose the function key labeled

$$\boxed{x^2}$$

and press it – out comes the output. You will see (on your real or imagined display panel) the value 144.

This is a representation of the black box model.

$$x \longrightarrow \boxed{x^2} \longrightarrow x^2,$$

put $x = 12$,

$$12 \longrightarrow \boxed{x^2} \longrightarrow 144.$$

Input-output, input-output – and that's the way it works.

2.5. Calculating the Domain and Range

In this section we discuss techniques of calculating the domain and range of a function. These are important and fundamental techniques. Calculation of the domain is particularly useful.

• The Natural Domain of a Function

Suppose you want to write a computer program. As part of that program, you are to calculate values of the function

$$f(x) = \frac{\sqrt{x-1}}{x-2}. \tag{9}$$

In the course of this program, you ask the user to input a value of x, and your program will calculate the value of f for the user, and output the answer.

Naturally, you want to *protect your program against crashing*; therefore, you cannot allow the user to input a value of x outside the domain of the function. *Before you calculate* f, you must first check whether the x the user has input into your program is in the domain of the function.

Initially, you need to ask yourself two questions:

a. What type of values does the variable x take on: real-values, complex-values, or something more esoteric?
b. What type of values does $f(x)$ take on: real-values, complex-values, or something more esoteric?

For our example, perhaps based on the goals of our underlying computer program, we are determined to accept only *real-values* for x.

Even if x is a real-number, it is possible for $f(x)$ to be a complex-value. The thought of having complex-values for a function when we are still struggling with real-values is odious indeed; we'll decide that the values of $f(x)$ are real numbers. What we have just done is to decide on the *co-domain* of the function f. (See co-domain as a statement of type.)

We have just decided that f should be a *real-valued function* of a *real-variable*:

$$\mathrm{Dom}(f) \subseteq \mathbb{R} \qquad \mathrm{coDom}(f) = \mathbb{R}.$$

Now, let's turn to the problem of calculating the actual domain of our function, or, within the context of our computer program, what sort of values x can we allow the user to input?

Obviously, $x \neq 2$, for otherwise, we would have zero in the denominator of (9) — a no–no. The square root is the other expression in the definition of f in (9). We need $\sqrt{x-1}$ to be a real value. In order for $\sqrt{x-1}$ to be a real number, the radicand must be nonegative: $x - 1 \geq 0$, or $x \geq 1$.

Thus, if $x \geq 1$ the numerator is a real-value; if $x \neq 2$ the denominator is a nonzero real-number. If x is a real number that satisfies *both* of these conditions, then

$$\frac{\sqrt{x-1}}{x-2},$$

is a real-number.

Conclusion: The domain of f is all x that satisfy,

$$x \geq 1 \quad \text{and} \quad x \neq 2$$

or,

$$\text{Dom}(f) = \{\, x \in \mathbb{R} \mid x \geq 0 \text{ and } x \neq 2 \,\}.$$

With respect to our computer program our sequence of thoughts would be

a. input $\longrightarrow x$.
b. Determine whether x is a real number; if not ask user for a real number.
c. Is $x \neq 2$? If 'No' then ask user for a real number different from 2.
d. Is $x \geq 1$? If 'No' then ask user for a real number greater than or equal to 1.
e. Compute $f(x)$.

What we have just calculated is what is called the *natural domain* of the function f. Let's formalize this term into a quasi-formal definition.

Definition 2.2. Let f be a real-valued function of a real-variable. The *natural domain* of f is the set of all numbers $x \in \mathbb{R}$ for which $f(x)$ can be computed as a real number.

EXAMPLE 2.6. (Skill Level -1) Calculate the natural domain of the function $f(x) = x^2$.

EXAMPLE 2.7. (Skill Level 0) Calculate the natural domain of the function $f(x) = x^{-2}$.

EXAMPLE 2.8. (Skill Level 1) Calculate the natural domain of the function $f(x) = \sqrt{2x^2 - 1}$.

Many functions are either defined by algebraic expression or they involve arithmetic operations to some degree. The following are some fundamental guiding principles that you should be aware of and utilize when trying to discern the *natural domain* of a function.

- **Principles of Domain Analysis.**

1. When examining a root, the radicand of an *even root* must be *nonnegative* in order to evaluate to a real number. (This implies that you must be able to solve inequalities.)
2. The *denominator cannot be zero*. (This implies that you must be able to factor and/or solve equations.)
3. Determine the domain of each component (each factor, each term) of the expression.
4. Each component (each factor, each term) of the expression must evaluate to a real number. This means that we take the domain of each component and *intersect them*; a number lying the the domain of *each component* means each component evaluates to a real number, which, in turn, implies the whole expression evaluates to a real number.
5. In regards to the previous point, the word "intersects" is translated into english using the word "and." (See example below for an additional elaboration of this point.)

Reread the two previous examples in light of the above principles. Determine whether these little tidbits of knowledge were utilized. Or, read the next example for a slightly more complicated analysis . . . but not discouragingly so.

EXAMPLE 2.9. (Skill Level 1.1) Calculate the domain of the function

$$f(x) = \frac{x}{\sqrt{2x^2 - 1}}.$$

EXAMPLE 2.10. (Skill Level 2) Calculate the natural domain of

$$f(x) = \frac{\sqrt{4 - x^2}}{x - 1}.$$

EXAMPLE 2.11. (Skill Level 2) Calculate the natural domain of

$$f(x) = \frac{\sqrt{1 - x}}{\sqrt{1 + x}}.$$

Finding the natural domain of a function defined by an algebraic expression is simple enough. It's a matter of common sense, good algebraic methods, and the principles of domain analysis.

The next two problems are similar to the previous **Example**, but with a subtle change.

EXERCISE 2.33. Find the natural domain of the function

$$f(x) = \frac{\sqrt{x-1}}{\sqrt{x+1}}.$$

EXERCISE 2.34. Find the natural domain of the function

$$f(x) = \sqrt{\frac{x-1}{x+1}}.$$

The point of the last two examples was two-fold: (1) to supply the student with two exercises in domain analysis; (2) to illustrate a point that some studnets may be unaware:

$$\frac{\sqrt{x-1}}{\sqrt{x+1}} \neq \sqrt{\frac{x-1}{x+1}},$$

or the more general principle is

$$\frac{\sqrt{a}}{\sqrt{b}} \neq \sqrt{\frac{a}{b}}. \tag{10}$$

We do have equality when $a \geq 0$ and $b > 0$, but if both a and b are *negative*, then the right-hand side is a real number, whereas the left-hand side is not. Therefore, when utilizing a property such as (10), *be sure* you are dealing with nonnegative quantities.

Next example is of a lesser skill level but it is meant to illustrate domain analysis when *sums* of terms are involved—no problem though, just apply the principles.

As always try to solve this problem *before* peeking at its solution.

EXERCISE 2.35. Determine the natural domain of

$$f(x) = \sqrt[4]{2x-1} + \frac{1}{x-1}.$$

To further drive home the point, consider the following exercise.

Exercise 2.36. Determine the natural domain of the function

$$f(x) = \frac{\sqrt{1-x}}{\sqrt{1+x}} + \sqrt[4]{2x-1} + \frac{1}{x-1}.$$

Utilize the results of Example 2.11 and Exercise 2.35.

Additional practice can be found in the section entitled *The Algebra of Functions* below. In that section a more structured approach to analyzing the domains of functions — principles used in this section.

• Range Calculations

In this section we illustrate the logic associated with calculating the range of a function. Let's begin by reviewing the definition of range: Let f be a (real-valued) function (of a real-variable), then the range of f is the set of all *values* of f. Symbolically,

$$\operatorname{Rng}(f) = \{\, y \in \mathbb{R} \mid \text{there is some } x \in \operatorname{Dom}(f) \text{ such that } y = f(x)\,\}.$$

Therefore, to determine whether a given number, y, is in the range of f, we must try to find a number $x \in \operatorname{Dom}(f)$, such that $f(x) = y$. That is, we take the attitude that the symbol y is a given (known) number, and we try to find an $x \in \text{Dom}(f)$ such that $f(x) = y$, or, in otherwords, we solve the equation

$$f(x) = y \tag{11}$$

for x (again, treating y as a known quantity).

EXERCISE 2.37. What does it mean when there are *no solutions* to (11) for a given value of y?

EXERCISE 2.38. Quiz Question. If x is a solution to (11), does it follow that y must be in the range of f?

(a) True (b) False

EXERCISE 2.39. What does it mean when there are *multiple solutions* to (11) for a given value of y?

Range Testing a single point y.
Let f be a function having domain Dom(f). To determine whether a given number y belongs to the range of f we solve the equation

$$f(x) = y \qquad x \in \text{Dom}(f).$$

If there is a solution $x \in \text{Dom}(f)$, then $y \in \text{Rng}(f)$; otherwise, $y \notin \text{Rng}(f)$.

The above criteria can be used for testing a particular value y, but it can also be used to identify the entire domain of a function. The next example illustrates the first statement, most of the others illustrates the second.

EXAMPLE 2.12. Let $f(x) = 3 + \sqrt{x^2 - 4}$.

a. Is it true that $8 \in \text{Rng}(f)$?
b. Is it true that $2 \in \text{Rng}(f)$?

EXAMPLE 2.13. Let $f(x) = 3 + \sqrt{x^2 - 4}$. Find the range of f.

EXERCISE 2.40. Find the range of $f(x) = \dfrac{1}{3 + \sqrt{x^2 - 4}}$.

The calculation of the range may not be as important as the ability to calculate the natural domain of a function. Quite often, graphical methods can be used to discern the range of the function without all this algebraic hoopla anyway.

⇓ Click here to continue.

Appendix

Definition. Let $A \neq \emptyset$ and $B \neq \emptyset$, and $f \subseteq A \times B$. Then f is called a function from A into B provided

(1) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in f$.

(2) (a_1, b), $(a_2, b) \in f$ implies $a_1 = a_2$.

If $(a, b) \in f$, then we write $f(a) = b$.

Solutions to Exercises

2.1. The co-domain B can be any set sufficiently large to contain all the *values* of the function f. If the function generally takes on only real values, then we lose nothing by assuming that $B = \mathbb{R}$. More generally, think of the co-domain of a function as a set that describes the *type* of values that a function, such as our f, can take on. For a function that takes on complex numbers as values, the co-domain of such a function may be considered to be $\mathbb{C}$, the set of all complex numbers; for a function whose values are points in the plane, the co-domain may be considered to be $\mathbb{R}^2$. The possibilities are limitless.

Exercise 2.1. ■

2.2. ... is any subset of the *cartesian product* of A and B.

Definition R is a relation between A and B provided

$$R \subseteq A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

If $(a, b) \in R$ we sometimes write $a\, R\, b$ and say that the element a is R-related to b.

The symbol R is a generic one. For famous relations, R is replaced by specialized and more familiar symbols. For example, $\leq$ is a relation between $\mathbb{R}$ and $\mathbb{R}$. In this case, we generally don't write $(1, 2) \in \leq$; rather, we say $1 \leq 2$ (this corresponds to the general notation of $a\, R\, b$). (Now it makes sense!)

Other famous examples of relations on $\mathbb{R}$ are $<$, $=$, $>$, and $\geq$.

In its simplest form, a relation is just a subset of the Cartesian of two sets; in particular, any curve in the xy-plane defines a relation.

A function f from A into B can be used to define a relation. We'll call the relation $G(f)$.

$$G(f) = \{ (a, f(a)) \mid a \in A \} \subseteq A \times B.$$

▪ Can you guess why I used the symbol $G(f)$ to define the relation determined by a function?

Exercise 2.2. ▪

2.3. We present the answers in the form of multiple choice questions. If you have solved this exercise before jumping here, you expected score is 100%.

Definition of f: $f(1) = 3$, $f(2) = 3$, $f(3) = 4$, $f(-1) = 7$.

Part (a). Write the domain, $\mathrm{Dom}(f)$, of f.

(a) $\{3, 4, 7\}$ (b) $\{1, 2, 3, 4, 7, -1\}$
(c) $\{1, 2, 3, -1\}$ (d) none of these

Part (b). Is 3 a value of this function?

(a) Yes (b) No (c) Cannot tell

Part (c). Is 1 a value of this function?

(a) Yes (b) No (c) Cannot tell

Part (d). What is the image of 3 under f?

(a) 1 (b) 2 (c) 3 (d) 4

Part (e). What is the image of 3 under f?

(a) 1 (b) 2 (c) 3 (d) 4

These questions serve to illustrate the usage of the above defined terms. Your score indicates your understanding of these terms at the time you took the quiz. If you got less than 100%, review the definitions before continuing, please.

Exercise 2.3. ■

2.4. An integer-valued function of an integer variable is a function f, say, where $\operatorname{coDom}(f) = \mathbb{Z}$ (so f takes on integer values), and $\operatorname{Dom}(f) \subseteq \mathbb{Z}$ (so the independent variable x is an integer).

$$\operatorname{Dom}(f) \subseteq \mathbb{Z} \text{ and } \operatorname{coDom}(f) = \mathbb{Z}.$$

Trivial Example: Define a function f by

$$f(0) = 0.$$

That is, the domain of f is $\operatorname{Dom}(f) = \{\,0\,\}$ (small domain). The value of f at the only element in its domain is $f(0) = 0$, so f takes on integers as values.

Less Than Trivial Example: Define a function f by,

$$f(x) = 2^{|x|}, \qquad x \in \mathbb{Z}.$$

By the definition, $\operatorname{Dom}(f) = \mathbb{Z}$ so f has an integer variable, x. Are the values of f integers?

For $x \geq 0$,

$$f(x) = 2^{|x|} = 2^{x} \qquad x = 0, 1, 2, 3, \ldots .$$

This yields the integers: 1, 2, 4, 8, 16, 32, 64, etc. etc., and, of course, etc.

For $x < 0$,

$$f(x) = 2^{|x|} = 2^{-x} \qquad x = -1, -2, -3, \ldots .$$

This yields the integers: 2, 4, 8, 16, 32, 64, etc. etc., and, of course, etc.

Exercise 2.4. ■

2.5. This is for your pleasure only. I will not deny you.

Exercise 2.5. ■

2.6. $f(x) = \dfrac{2}{2^2+1} = \dfrac{2}{5}$. That was easy!

Exercise 2.6. ■

2.7. Yes. Yes. In the first case, the assignment of the domain must be consistent with the rule of association. (Here it is $y = x^2$.) The domain can be any set for which the symbol x^2 is a well-defined operation for every $x \in A$. Certainly, we can take A to be any subset of the real line, or any subset of the complex plane (in the complex plane, a multiplication is defined there). If A is Euclidean 3-space, $\mathbb{R}^3$, then there is no natural multiplication operation defined on this space and so the symbol x^2 is undefined there. (There are, however, multiplication-type operations, such as *dot-product* and *vector-product*, perhaps x^2 may refer to one of those.)

Now, regarding the co-domain question, given that the creator of the function has properly defined the domain of the function, the co-domain could then be any set that contains all the values of the function. If a set B is chosen that does not include all the values of the function, then B cannot be considered the co-domain of the function. For example, if $f(x) = x^2$, $x \in \mathbb{R}$ $(A = \mathbb{R})$, then we could not take the co-domain to be $B = (-\infty, 0)$ because this B in fact contains none of the values of f, nor could we take $B = [-1, 1]$, for this B

does not contain all the values of f. We can take the co-domain of this function to be $\mathbb{R}$, or a smaller set $[0, \infty)$; the latter set being the *range* of f.

Exercise 2.7. ■

2.8. There are infinitely many ways of responding to this question—I only have the time, patience, and money to give only *finitely many* alternatives; namely, one.

Define $h(x) = x^2$, for $x \in [\,0, \infty\,)$. We have the same rule of association, but a different domain; however, the range is

$$\operatorname{Rng}(h) = [\,0, \infty\,).$$

This is the same range as was given in (2).

Quiz. Read the alternatives, determine your response to each *before* testing your theories. Use your common sense, your knowledge of the squaring process, and your understanding of the symbolism and terminology of the question (let's hope). Graphical techniques may also be helpful.

Question: Which of the following sets, when used as the domain of the rule of association $x \mapsto x^2$ *does not* yield a range of $[\,0, \infty\,)$?

(a) $(\,-\infty, 0\,]$ (b) $(\,-3, 0\,] \cup [\,2, \infty\,)$

(c) $\{\, x \in \mathbb{R} \mid x \neq 3.3 \,\}$

(d) $(\, -\infty, -17) \cup (\, 17, \infty\,)$

End Quiz.

If you have errored in your considerations, think about why you errored. What lead you to believe a false conclusion? What is the correct reasoning?

Exercise 2.8. ■

2.9. Yes. Let the domain have n elements, n a natural number. Then we can represent the domain in the following manner:

$$\operatorname{Dom}(f) = \{\, x_1, x_2, x_3, \dots, x_n \,\}.$$

The range of f may then be represented by the set

$$\operatorname{Rng}(f) = \{\, f(x_1), f(x_2), f(x_3), \dots, f(x_n) \,\},$$

and this set has only finitely many elements; in fact, it has at most n elements (why?). Exercise 2.9. ■

2.10. This is easy, given the discussion on arguments.

If the underlying function is $f(x) = \sin(x)$, then the argument of f in $\sin(x^2)$ is x^2—since $f(x^2) = \sin(x^2)$. The argument of f in the expression $\sin(\sqrt{x+1})$ is $\sqrt{x+1}$—since $f(\sqrt{x+1}) = \sin(\sqrt{x+1})$.

Exercise 2.10. ■

2.11. It's $x + 1$!

Exercise 2.11. ■

2.12. We present that answers in the form of a Quiz.

Part (a). What is the argument of $\sin(x^2)$?

(a) x	(b) x^2
(c) $\sin(x^2)$	(d) insufficient information

Part (b). What is the argument of $\cos(\sin(x))$?

(a) x	(b) $\sin(x)$
(c) $\cos(\sin(x))$	(d) insufficient information

Part (c). What is the argument of $(x+1)^3$?

(a) x	(b) $x+1$
(c) x^3	(d) insufficient information

Exercise Notes: The notion of argument is made relative to a specific function. In the three given expressions,

$$\sin(x^2),\ \cos(\sin(x)),\ \text{and}\ (x+1)^3,$$

the underlying function was not specified or implied. Without knowledge of the underlying function, we cannot make a statement about its argument.

■ For take part (a) to illustrate the point: $\sin(x^2)$. If the underlying function had been stated to be $f(x) = \sin(x^2)$, then choice (a) would be correct. If the underlying function was $f(x) = \sin(x)$, then choice (b) would have been correct. And to be extremely tricky, if the underlying function had been specified as $f(x) = x$ then choice (c) would be correct. ■

Exercise 2.12. ■

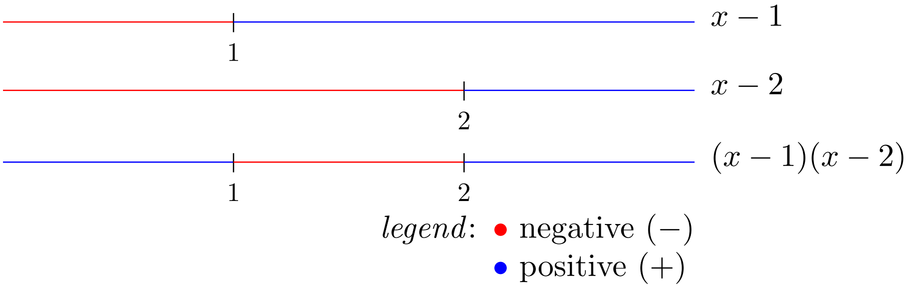
2.13. Symbolically, the domain of g is given by

$$\mathrm{Dom}(g) = \{\, x \in \mathbb{R} \mid x^2 - 3x + 2 > 0 \,\}.$$

This is because we cannot calculate the value of $g(x)$ if the radicand is negative. Also, if the radicand is 0, then we have 0 in the denominator. Therefore, the radicand must be *positive*.

Begin by factoring the radicand: $x^2 - 3x + 2 = (x - 1)(x - 2)$. Now use the *sign chart method* to solve this inequality.

The *Sign Chart* of $(x-1)(x-2)$



The solution to the inequality $x^2 - 3x + 2 > 0$ is $(-\infty, 1) \cup (2, \infty)$. Thus,

$$\mathrm{Dom}(g) = (-\infty, 1) \cup (2, \infty).$$

This problem was strictly an exercise in algebra. Exercise 2.13. ■

2.14. Let's present the answers if the form of a quiz.

Quiz $(1^{\text{pt}}_{\text{ea.}})$

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

1. The value of $f(-2)$ is

(a) -2 (b) 0 (c) 2 (d) 4

2. The value of $f(-1/2)$ is

(a) -1 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

3. The value of $f(0)$ is

(a) -1 (b) 0 (c) 1 (d) 4

4. The value of $f(1/2)$ is

(a) -1 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

5. The value of $f(3)$ is

(a) 0 (b) 1 (c) 3 (d) 9

Did you get 6 points? Based on the difficulty of the problem, I'm absolutely sure the answer is NO! Exercise 2.14. ■

2.15. No. The domains of the pieces overlap. Take $x = 2$, and calculate the corresponding y-value. First, it is true that $x \le 2$, since $x = 2$; thus, by the definition, $f(2) = 2^2 = 4$. Secondly, it is true that $x \ge 1$, since $x = 2$; thus, by the definition $f(2) = 2^3 = 8$. The above definition associates *two* values with $x = 2$. The conclusion is as advertised!

Note, we could have chosen any number $x \in [\,1, 2\,]$ and been able to draw the same conclusion. Any number but, $x = 1$. This value of x would not lead the conclusion that f is not a function. Do you understand why?

Exercise 2.15. ■

2.16. Yes, f is a function. Even though $x = 1$ falls into both cases ($x \leq 1$ and $x \geq 1$) the value of the functional pieces are the same at $x = 1$. That is, since $x \leq 1$, $f(1) = 1^2 = 1$; but since $x \geq 1$ too, we also have $f(1) = 1^3 = 1$. Thus, the rule of association only assigns one value corresponding to $x = 1$. Since $x = 1$ was the only problem child, f is a function. Exercise 2.16. ■

2.17. No. There is no problem at $x = -3$, but for the overlapping domains at $x = 3$, the pieces do not match $(2(3) + 9 \neq 3^3 - 10)$.

As an important variation on this problem. Consider once again

$$g(x) = \begin{cases} |x| & x \leq -3 \\ 2x + 9 & -3 \leq x \leq 3 \\ x^3 - c & x \geq 3 \end{cases}$$

Where c is a constant yet to be determined. *Question*: What value could you give c so that g, as defined above, is now a function? (Answer: 12.)

Exercise 2.17. ■

2.18. Yes. The definition of f is

$$h(x) = \begin{cases} |x| & x \leq -3 \\ -x & -4 \leq x \leq 1 \\ x^2 & x > 1 \end{cases}$$

The second and third domain specifications have no overlap — they are no problem. The first and second domain specifications do overlap. However, on the interval over which they intersect

$$(-\infty, -3\,] \cap [\,-4, 1\,] = [\,-4, -3\,]. \tag{A-1}$$

On the interval, $[\,-4, -3\,]$, the intersection of the two overlapping domains, the pieces is in question *agree*! Indeed, for any $x \in [\,-4, -3\,]$,

$$\begin{aligned} |x| &= -x && \text{since } x < 0 \\ -x &= -x && \text{self-obvious.} \end{aligned}$$

This means the two pieces agree on the domain of overlap. For each x in this overlap, no matter what definition you choose, $|x|$ or $-x$, the corresponding value of h is the same. Exercise 2.18. ■

2.19. Take any number of functions and piece them together by restricting their domains. These restricted domains must be either *disjoint*, or if they do overlap, the functions involved must agree.

In Exercise 2.15 and Exercise 2.17, the domains of the two pieces overlapped, but the functions did not agree on the overlap. In contrast, in Exercise 2.16 there was overlap of the domains, but the function pieces agreed on the overlap. See also the discussion at the end of the solution to Exercise 2.17.

Exercise 2.19. ■

2.20. The distance between two horizontally oriented points in the plane is the *absolute difference of their first coordinates*:

$$d(P, Q) = |a - c|.$$

The above rule is quite useful throughout *Calculus*. The above equation is directly deducible form (5)

The same result can be obtained from the *distance formula* for the plane:

$$\begin{aligned} d(P, Q) &= \sqrt{(a-c)^2 + (b-b)^2} \\ &= \sqrt{(a-c)^2} \\ &= |a - c| \qquad \triangleleft \ (6) \end{aligned}$$

Exercise 2.20. ■

2.21. Same reasoning as the previous exercise.

The distance between two vertically oriented points in the plane is the absolute difference in their second coordinates:

$$\mathrm{d}(\,P, Q\,) = |b - c|.$$

Exercise 2.21. ■

2.22. From our efforts of EXERCISE 2.2, we can easily see that

$$f(x) = x^3 H(x - 2).$$

Do we "see" it? Exercise 2.22. ■

2.23. By first considering $x \leq 2$ then $x > 2$, verify that

$$f(x) = x^2 + (x^3 - x^2)H(x - 2).$$

(When you do the above analysis, keep x are a symbolic variable.)

Exercise Notes: Piecewise defined functions are used often in engineering and there are special techniques for handling them. One technique, called *Laplace Transforms*, requires that you represent all piecewise defined functions in terms of the HEAVISIDE function. This exercise attempts to prepare the way for you. ∎

Exercise 2.23. ∎

2.24. I leave the analysis to you:

$$f(x) = \sin(x) + (x - 1)\sin(x)H(x + 1).$$

Exercise 2.24. ■

2.25. One of the initial major inhibiting factors in successfully solving this problem is *understanding the question*! This can be done with a careful reading of the problem, and some graph sketching.

Draw a Cartesian axis system. Label the horizontal axis the t-axis, and label the vertical axis the s-axis. Let x be any real number. (What this means is that you are to consider the symbol x as a given or know quantity. You can even think of x as being different values if you wish; think of x as being -2, -1, 0, 1, 2, 3, for example.) Now draw the line $s = 2t - 5x$ on the ts-axis you have already drawn. All these lines have slope $m = 2$, but different intercepts. (Here, you can use your particular values of x, if you wish to get a "feel" for what's going on.) Now, $f(x)$ is defined to be the t-intercept of the line. (The t-intercept of the line will depend on x since the line $s = 2t - 5x$ depends on x.)

Think about these points and try again to solve the problem. The solution is on the next page; **don't look** until you have made another attempt at solving the problem, given the above discussion.

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$

⋮ ⋮ ⋮ ⋮ ⋮ ⋮

Continued on the next page

⋮ ⋮ ⋮ ⋮ ⋮ ⋮

⋮ ⋮ ⋮ ⋮ ⋮ ⋮

Exercise 2.25 (cont.) Let x be a given real number, and consider the straight line $s = 2t - 5x$. By definition, $f(x)$ is the t-intercept of this line. The t-intercept can be calculated in the "usual way" by putting $2t - 5x = 0$ and solving for t (the t-intercept is occurs when $s = 0$). Solving for t we get $t = 5x/2$. Thus, $f(x)$ being the t-intercept, we must have

$$f(x) = \frac{5}{2}x.$$

Obviously, we are able to calculate the t-intercept no matter what the value of x, so $\mathrm{Dom}(f) = \mathbb{R}$. Exercise 2.25. ■

2.26. Let x be a real number, then $g(x)$ is the y-coordinate (in the ty-axes system) of the two lines: $y = 3t + 4x$ and $y = -2t + 3x$. This y-coordinate can be calculate by setting up the equation:

$$3t + 4x = -2t + 3x$$

$$5t = -x$$

Therefore,

$$t = -x/5$$

This (i.e. $t = -x/5$) is the t-coordinate of the point of intersection. The y-coordinate can be obtained by substituting this into either of the two functions: $y = 3(-x/5) + 4x = (17x)/5$. But this is the definition of $g(x)$. Thus,

$$g(x) = \frac{17}{5}x.$$

The domain $\mathrm{Dom}(g) = \mathbb{R}$, since there was only one y-coordinate of the point of intersection, no matter what the value of x. Exercise 2.26. ■

2.27. *h* is *not* a function. For if we take x to be a little larger than zero, then the line $y = x(t - 1)$ would have a slightly positive slope. Looking at the two graphs, $y = \sin(t)$ and $y = x(t - 1)$, we see that $y = x(t-1)$ intersect the sine graph at least three times. Thus, for that x (slightly positive) there is (at least) corresponding y-values defined — all in violation of the property of a function. Exercise 2.27. ■

2.28. In this case, in contrast to Exercise 2.27, *k is a function*! We have restricted the domain of $y = \sin(t)$ so as to eliminate the problem we ran into in the analysis of h in Exercise 2.27.

Draw the graph of $y = \sin(t)$ and $y = x(t-1)$ in the ty-axis. Do you see that the line intersects the sine curve at only one place between $0 \leq t \leq \pi$; hence, there is only one y-coordinate of intersection? There is one interesting anomaly: $x = 0$. In this case, the corresponding line intersects the sine curve at *two* points; but we are saved, these two points have the same y-coordinate! When $x = 0$, we have $y = 0$ — only one y-coordinate of intersection. Therefore,

$$\mathrm{Dom}(k) = \mathbb{R}.$$

Speaking of the graph of k, can you (yes, you) make a rough sketch of the graph of k *without* using your graphing calculator to help you? It is obvious that k has an horizontal asymptote.

Below are some questions to stimulate your thinking. Think about the problem *before daring* to see the answer.

1. The function $y = k(x)$ has an horizontal asymptote at ...

(a) $y = -1$ (b) $y = 0$ (c) $y = \sin(1)$ (d) $y = 1$

2. For the horizontal asymptote for $x > 0$, does the graph ...

(a) approach the horizontal asymptote from below, or
(b) cross over the horizontal asymptote, then approach it from above?

3. For the horizontal asymptote for $x < 0$, does the graph ...

(a) approach the horizontal asymptote from below, or
(b) cross over the horizontal asymptote, then approach it from above?

4. As observed above, $k(0) = 0$. This is really the only value of k that we know. The graph starts at the origin and moves upwards towards the asymptotes. What does the graph look like at the origin?

(a) A nice smooth curve
(b) A sharp corner
(c) A vertical asymptote
(d) Insufficient data

5. What is the highest altitude the graph of k attains?

(a) 0 (b) $\dfrac{1}{2}$ (c) $\dfrac{\sqrt{2}}{2}$ (d) 1

6. What is the x-coordinate of the highest point on the graph of k?

(a) $x = \dfrac{\pi}{4}$ (b) $x = \dfrac{\pi}{2}$ (c) $x = \dfrac{1}{\frac{\pi}{2} - 1}$ (d) $x = \dfrac{1}{\sin(\frac{\pi}{2} - 1)}$

Hopefully, given this information, you can now make a good rough sketch of the graph of k.

Exercise 2.28. ■

2.29. Let's have the details of the first calculation.

Calculation of $f(-2)$:

$$\begin{aligned} f(x) &= ((x+1)^3 + 2x)^2 \\ f(-2) &= ((-2+1)^3 - 2(-2))^2 \\ &= ((-1)^3 + 4)^2 \\ &= (-1+4)^2 \\ &= 3^2 \\ &= \boxed{9.} \end{aligned}$$

Care must be taken when dealing with negative numbers. A plentiful use of signs is often desirable.

Calculate the other two values, if you haven't done so already. Be methodical. The object is *not* to make errors.

Quiz. Passing Score: 2 out of 2 (100%). **1.** Which of the following is $f(0)$?

(a) 0 (b) 1 (c) 2 (d) 9

2. Which of the following is $f(1)$?

(a) 10 (b) 36 (c) 64 (d) 100

End Quiz

Exercise 2.29. ■

2.30. $f(2) = 2!$ That's strange.

Exercise 2.30. ■

2.31. The function is $y = 2x^3 + 1$.

1. Force an explicit argument. $y(-1) = 2(-1)^3 + 1 = -1$.

2. Use the Evaluation Notation:

$$y\big|_{x=-1} = (2x^3 + 1)\big|_{x=-1} = 2(-1)^3 + 1 = -1.$$

Exercise Notes: One of the advantages of the Evaluation Notation is that it can be used to work through a complex calculation. In **2.** above, we replaced y with $2x^3 + 1$ in the Evaluation Notation. This enabled me to continue the calculation so that the reader (that's you) could follow.

Exercise 2.31. ■

2.32. Remember there are two methods for handling anonymous functions.

1. Force an explicit argument:

$$Area\_circle(2) = \pi(2)^2 = 4\pi.$$

2. Use the Evaluation Notation:

$$Area\_circle\big|_{radius=2} = 4\pi.$$

I hope you answered this correctly, á priori. Exercise 2.32. ■

2.33. Here is a brief outline.

The Numerator. The numerator, given by $\sqrt{x-1}$, requires by (1) that $x - 1 \geq 0$, or $x \geq 1$.

The Denominator. The denominator, $\sqrt{x+1}$, requires by (1) that $x + 1 > 0$, or $x > -1$.

Final Analysis. The natural domain of this function would be all x that satisfy $x \geq 1$ and $x > -1$.

$$\mathrm{Dom}(f) = [\,1, \infty\,)$$

Exercise 2.33. ■

2.34. Keeping in mind the principles of domain analysis, when looking at the expression

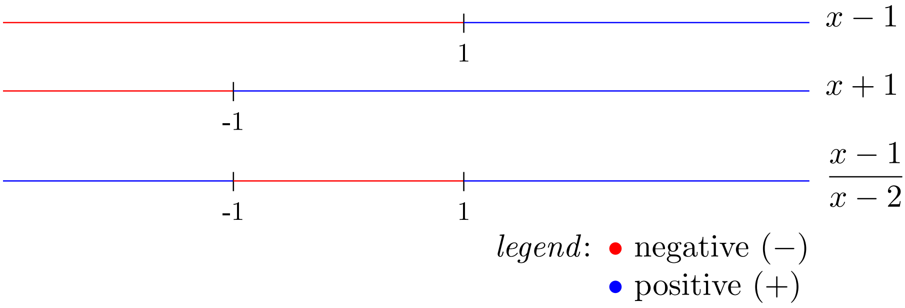
$$\sqrt{\frac{x-1}{x+1}}$$

we would require

$$\frac{x-1}{x+1} \geq 0 \text{ and } x \neq -1.$$

We now need to solve the first inequality. An easy way of doing so is to use the *sign chart method.*

The *Sign Chart* of $\dfrac{x-1}{x+1}$



Therefore, we see that the solution to the inequality is

$$x < -1 \text{ or } x \geq 1$$

In formal terms, then, the domain of the function, f, is

$$\begin{aligned} \mathrm{Dom}(f) &= \{\, x \in \mathbb{R} \mid x < -1 \text{ or } x \geq 1 \,\} && \triangleleft \text{ Set Notation} \\ &= (\, -\infty, -1 \,) \cup [\, 1, \infty \,) \end{aligned}$$

Note: The word "or" is translated as "set union (∪)."

Exercise 2.34. ■

2.35. We proceed along standard lines of inquiry. The function is

$$f(x) = \sqrt[4]{2x - 1} + \frac{1}{x - 1}.$$

Analysis of the first term. The first term involves an *even root*; by (1), we require

$$2x - 1 \geq 0 \text{ or } \boxed{x \geq \frac{1}{2}}$$

Analysis of the first term. The second term involves a ratio. The numerator is the constant 1 and puts no constraint on the domain. The denominator is $x - 1$. By (2), we require

$$\boxed{x \neq 1}.$$

Final Analysis. Therefore, we require

$$x \geq \frac{1}{2} \text{ and } x \neq 1.$$

Now, using good and standard notation we have that,

$$\begin{aligned} \mathrm{Dom}(f) &= \{\, x \in \mathbb{R} \mid x \geq \frac{1}{2} \text{ and } x \neq 1 \,\} && \triangleleft \text{ Set Notation} \\ &= [\, \frac{1}{2}, 1 \,) \cup (\, 1, \infty \,) && \triangleleft \text{ Interval Notation} \end{aligned}$$

Exercise 2.35. ■

2.36. In this problem, you should have realized that you can take a shortcut. The function is

$$f(x) = \frac{\sqrt{1-x}}{\sqrt{1+x}} + \sqrt[4]{2x-1} + \frac{1}{x-1}$$

$$= \underbrace{\left[\frac{\sqrt{1-x}}{\sqrt{1+x}}\right]}_{\text{Example 2.11}} + \underbrace{\left[\sqrt[4]{2x-1} + \frac{1}{x-1}\right]}_{\text{Exercise 2.35}}$$

The First Bracketed Term has domain $(-1, 1]$.

The Second Bracketed Term has domain $[\frac{1}{2}, 1) \cup (1, \infty)$.

Final Analysis. The domain of our given function is then all $x \in \mathbb{R}$ such that

$$x \in (-1, 1] \text{ and } x \in [\frac{1}{2}, 1) \cup (1, \infty)$$

Now we are confronted with the problem of getting the visualization of this set.

$$\boxed{\mathrm{Dom}(f) = [\,\frac{1}{2}, 1\,).}$$

Exercise 2.36. ■

2.37. In other words, there is no x such that $f(x) = y$. This means that $y \notin \mathrm{Rng}(f)$.

Exercise 2.37. ■

2.38. The answer is **False** in general. When we solve (11) and obtain a solution x, then logically it follows that either that x belongs to the domain of f ($x \in \mathrm{Dom}(f)$), *or* that x does not belong to the domain of f ($x \notin \mathrm{Dom}(f)$).

It is the first case that allows us to deduce that $y \in \mathrm{Rng}(f)$, but the second case causes us to say that the answer to the question of **False**.

Exercise 2.38. ■

2.39. It means one of three things: (1) y is in the range of f; and (2) y is not in the range of f.

1. If any of the multiple solutions belong to the domain of f, then $y \in \mathrm{Rng}(f)$.

2. If none of the multiple solutions belongs to the domain of f, then $y \notin \mathrm{Rng}(f)$.

Exercise 2.39. ■

2.40. The natural domain of this function is

$$\operatorname{Dom}(f) = (-\infty, 2) \cup (2, \infty)$$

A number $y \in \operatorname{Rng}(f)$ if and only if there is some $x \in \operatorname{Dom}(f)$ such that

$$\frac{1}{3 + \sqrt{x^2 - 4}} = y. \tag{A-2}$$

Obviously,

$$y > 0. \tag{A-3}$$

because the left-hand side of (A-2) is never zero and is always non-negative.

Now invert (A-2)

$$3 + \sqrt{x^2 - 4} = \frac{1}{y}$$

or,

$$\sqrt{x^2 - 4} = \frac{1}{y} - 3$$

Again, since the square roon is always nonnegative, we have,

$$0 \le \sqrt{x^2 - 4} = \frac{1}{y} - 3$$

or,

$$\frac{1}{y} - 3 \ge 0$$

or,

$$y < \frac{1}{3} \tag{A-4}$$

Putting (A-3) and (A-4) together we get

$$0 < y \le \frac{1}{3} \tag{A-5}$$

Do these specifications define the range of f? We must ask ourselves the question: "Given y, $0 \le y \le 3$, does there exist an $x \in \text{Dom}(f)$ such that $y = f(x)$?"

To answer this question, we must continue with the calculations and solve for x.

$$
\begin{aligned}
f(x) = y &\iff \frac{1}{3+\sqrt{x^2-4}} = y \\
&\iff 3+\sqrt{x^2-4} = \frac{1}{y} \\
&\iff \sqrt{x^2-4} = \frac{1}{y} - 3 \\
&\iff x^2 - 4 = (\frac{1}{y} - 3)^2 \\
&\iff x^2 = 4 + (\frac{1}{y} - 3)^2 \\
&\iff x = \pm\sqrt{4 + (\frac{1}{y} - 3)^2}
\end{aligned}
$$

Do these two x's belong to the domain of f, i.e. is it true that $|x| \geq 2$? **Yes!** Indeed,

$$\begin{aligned} |x| &= \sqrt{4 + (\frac{1}{y} - 3)^2} \\ &\geq \sqrt{4} \\ &= 2. \end{aligned}$$

Where, again, we have used the property that $a \leq b$ implies $\sqrt{a} \leq \sqrt{b}$.

Summary: $\mathrm{Rng}(f) = (\,0, \frac{1}{3}\,]$, from (A-5).

Exercise 2.40. ■

Solutions to Examples

2.1. Here is the function reproduced for you convenience.

$$i(t) = \begin{cases} 0 & \text{if } t < 0 \\ \dfrac{t}{t+1} & \text{if } t \geq 0 \end{cases}$$

I claim that, if fact,

$$i(t) = \frac{t}{t+1} H(t). \tag{S-1}$$

To prove this equality, we must show that for every value of t, the left-hand side of (S-1) equals the right-hand side. There are two cases: $t < 0$ and $t \geq 0$.

Case $t < 0$: By definition of $i(t)$, $i(t) = 0$. By the definition of the Heaviside function, $H(t) = 0$ for $t \leq 0$. This makes the right-hand side of (S-1) equal to zero too. Thus, for this case, the equality of (S-1) is verified — they are both equal to zero.

Case $t > 0$: By definition of $i(t)$,

$$i(t) = \frac{t}{t+1} \qquad t > 0.$$

Now, for $t > 0$, the Heaviside function takes on a value of $H(t) = 1$, this means that for $t > 0$

$$\frac{t}{t+1} H(t) = \frac{t}{t+1} = i(t) \qquad t > 0.$$

Case $t = 0$: When $t = 0$, by the definition of $i(t)$:

$$i(0) = \frac{0}{0+1} = 0$$

For $t = 0$, $H(0) = 0$, hence, the right-hand side of (S-1) is zero as will. Thus, the two sides of (S-1) agree for this case too.

Thus we have shown that

$$\boxed{i(t) = \frac{1}{t+1} H(t).}$$

Example Notes: The Heaviside function can be used to represent piecewise defined functions as single expressions like the one above. This is useful in engineering mathematics when trying to calculate something called the *Laplace Transform*. ■

Example 2.1. ■

2.2. According to the definition of the Heaviside function, H, H has a value of 0 if its argument is less than or equal to zero; therefore,

$$\begin{aligned} H(x-2) &= 0 \quad \text{whenever } x-2 \leq 0 \\ &= 0 \quad \text{whenever } x \leq 2. \end{aligned}$$

Similarly, H takes on a value of 1 whenever its argument is greater than zero; thus,

$$\begin{aligned} H(x-2) &= 1 \quad \text{whenever } x-2 > 0 \\ &= 1 \quad \text{whenever } x > 2. \end{aligned}$$

From this reasoning we conclude,

$$g(x) = H(x-2) = \begin{cases} 0 & x \leq 2 \\ 1 & x > 2. \end{cases}$$

Exercise Notes: The introduction of the (dummy) function name of g was not needed. Oftentimes we just speak of the function $H(x-2)$.

- The function $H(x-2)$ is a horizontal shifting of the basic $H(x)$ function. Obviously, realization of this point is important. With this observation, you should have knowledge of the functions $H(x-5)$, $H(x+2)$, and $H(2x-1)$. ■

■ Using paper and pencil only to write down your final answer, what is the piecewise definition of $H(2x-1)$?

Example 2.2. ■

2.3. A quick sketch of the graphs of these two functions shows that

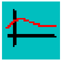


Figure S-1

$$f(x) = x \qquad g(x) = x^2 \qquad 0 \le x \le 1$$

For each x, $0 \le x \le 1$, the vertical line enters the region by crossing the graph of $g(x) = x^2$, it passes through the region, and exits when it crosses the graph of $f(x) = x$. The endpoints of the line segment is then $P(\,x, x^2\,)$ and $Q(\,x, x\,)$. The length of the line segment that goes from P to Q is given by

$$L(x) = x - x^2.$$

This defines a function: $L(x) = x - x^2$ having domain $0 \le x \le 1$.

You will encounter this kind of problem later when we use this function to calculate the *area* of the region. Example 2.3. ■

2.4. The introduction of the ts-axis system is necessary for clarity.

For any given $x \in \mathbb{R}$, the intersection between the lines $s = xt$ and $s = 2 - t$ is obtained by solving these two equations for s. (Why s, because s is the ordinate axis in the ts-axis system. Solve for t in one equation and substitute it into the other equation:

$$s = 2 - t \implies t = 2 - s$$

$$s = xt \text{ and } t = 2 - s \implies s = x(2 - s)$$

Finally take the equation $s = x(2 - s)$, and solve for s.

$$\begin{aligned} s = x(2 - s) &\iff s = 2x - sx \\ &\iff s + sx = 2x \\ &\iff s(1 + x) = 2x \\ &\implies s = \frac{2x}{1 + x} \qquad x \neq -1 \end{aligned}$$

Thus, for $x \neq -1$,

$$s = \frac{2x}{1 + x}.$$

What about the case $x = -1$? Go back to the original equations and consider this case:

$$s = xt \text{ and } s = 2 - t$$

put $x = -1$

$$s = -t\text{s} = \text{2-t}$$

These two lines each have slope $m = -1$; hence they are *parallel*. They are either the same line (not!) or they have *no points in common* (Yes!).

The solution to the problem is then,

$$\boxed{f(x) = \frac{2x}{1+x} \qquad x \neq -1,}$$

the latter specification indicates the natural domain of the function f.

Example 2.4. ■

2.5. The key to successful calculation the is principle of *replacement*: The symbol x is a place holder; simply replace x everywhere with a particular numerical value in question.

Calculation of $f(-2)$:

$$\begin{aligned} f(x) &= \frac{x^3 - 1}{3x^2 + 1} \\ f(-2) &= \frac{(-2)^3 - 1}{3(-2)^2 + 1} = \frac{-8 - 1}{3(4) + 1} \\ &= \boxed{-\frac{9}{13}.} \end{aligned}$$

Notice the clever use of parentheses. This is standard algebraic technique. The parentheses were inserted to avoid problems with the negative sign.

Calculation of $f(-1)$:

$$f(x) = \frac{x^3 - 1}{3x^2 + 1}$$

$$f(-1) = \frac{(-1)^3 - 1}{3(-1)^2 + 1} = \frac{-1 - 1}{3(1) + 1}$$

$$= -\frac{2}{4} = \boxed{-\frac{1}{2}.}$$

In addition to the use of parentheses to protect oneself against unwanted sign errors, there is another point to be made: *All fractions should be reduced.*

Calculation of $f(0)$:

$$f(x) = \frac{x^3 - 1}{3x^2 + 1}$$

$$f(0) = \boxed{-1.}$$

In this case the calculation was quite simple. A mental calculation was all that was needed. Try to do simple calculations and algebraic manipulations in your head—your head may need the exercise!

Calculation of $f(1)$:

$$f(x) = \frac{x^3 - 1}{3x^2 + 1}$$

$$f(1) = \boxed{-\frac{1}{4}.}$$

Again a mental calculation. There is no need to use a calculator when dealing with simple calculations.

Calculation of $f(2.12)$:

$$\begin{aligned} f(x) &= \frac{x^3 - 1}{3x^2 + 1} \\ f(2.12) &= \frac{(2.12)^3 - 1}{3(2.12)^2 + 1} \\ &= \frac{8.528128}{14.4832} \\ &\approx 0.5888 \end{aligned}$$

Example 2.5. ■

2.6. The natural domain of the function $f(x) = x^2$ is, by the description of Definition 2.2, the set of all numbers $x \in \mathbb{R}$ for which $f(x) = x^2$ can be computed as a real number. Obviously, if x is *any* real number, then x^2 is a defined quanity, and the result of the calculation is a real number; therefore, *any* real number x is in the natural domain of f. Thus,

$$\text{Dom}(f) = \mathbb{R}.$$

Example 2.6. ■

2.7. The natural domain of the function $f(x) = x^{-2}$ is, by the description of Definition 2.2, the set of all numbers $x \in \mathbb{R}$ for which

$$f(x) = \frac{1}{x^2}$$

can be computed as a real number.

Obviously, we run into problems when $x = 0$. In this case, we would be dividing by 0 — a no-no.

$$\begin{aligned} \mathrm{Dom}(f) &= \{\, x \in \mathbb{R} \mid x \neq 0 \,\} && \triangleleft \text{ Set Notation} \\ &= (\, -\infty, 0 \,) \cup (\, 0, \infty \,) && \triangleleft \text{ Interval Notation.} \end{aligned}$$

Example 2.7. ■

2.8. We need to find all x for which $f(x) = \sqrt{2x^2 - 1}$ can be computed as a real number. First we must recall some . . .

Fundamental Knowledge. $\sqrt{z}$ is a real number provided $z \geq 0$.

Therefore, $\sqrt{2x^2 - 1}$ is a real number provided

$$2x^2 - 1 \geq 0.$$

We now invoke our *algebraic module of knowledge* and solve this inequality. Indeed,

$$\begin{aligned} 2x^2 - 1 \geq 0 &\iff 2x^2 \geq 1 \\ &\iff x^2 \geq \frac{1}{2} \\ &\iff |x| \geq \frac{1}{\sqrt{2}} \end{aligned} \qquad \text{(S-2)}$$

where the symbol $\iff$ means "if and only if" or "is equivalent to."

Push Address on Stack (Computer Science Jargan)

Algebraic Note: To arrive at (S-2), I took the square root of both sides of the inequality in the previous line. In doing so, I utilized two fundamental facts that you need to be aware of

Fundamental Fact. (Square roots preserves inequalities) For $a \geq 0$ and $b \geq 0$,

$$a \leq b \iff \sqrt{a} \leq \sqrt{b}.$$

A similar statement is true for all the root functions.

Fundamental Fact. (The square root of a perfect square) Let $a \in \mathbb{R}$, then

$$\sqrt{a^2} = |a|.$$

This equation is *very important.*

Pop Address Off Stack (Computer Science Jargan)

The natural domain of f then is, from (S-2)

$$\begin{aligned} \mathrm{Dom}(f) &= \{\, x \in \mathbb{R} \mid |x| \geq \frac{1}{\sqrt{2}} \,\} && \triangleleft \text{ Set Notation} \\ &= (\, -\infty, -\frac{1}{\sqrt{2}} \,] \cup [\, \frac{1}{\sqrt{2}}, \infty \,) && \triangleleft \text{ Interval Notation} \end{aligned}$$

Example 2.8. ■

2.9. The function is

$$f(x) = \frac{x}{\sqrt{2x^2 - 1}}.$$

For us to calculate the real numbers for the values of f, the numerator and the denominator must be real numbers. There is no problem with the numerator: For *any* $x \in \mathbb{R}$, the numerator, x, is real. The denominator is a slight problem. We must require two things of the denominator: It be a real number *and* it be nozero.

Analysis of Denominator. We require that:

It be real: We saw from Example 2.8, that

$$\sqrt{2x^2 - 1} \text{ is a real number } \iff |x| \geq \frac{1}{\sqrt{2}}.$$

It be nonzero:

$$\sqrt{2x^2 - 1} \neq 0 \iff |x| \neq \frac{1}{\sqrt{2}} \iff x \neq \pm\frac{1}{\sqrt{2}}.$$

Correlating these two pieces of information, we get

$$f(x) \text{is a real number} \iff |x| > \frac{1}{\sqrt{2}}.$$

Thus, using proper notation,

$$\begin{aligned} \operatorname{Dom}(f) &= \{\, x \in \mathbb{R} \mid |x| > \frac{1}{\sqrt{2}} \,\} && \triangleleft \text{ Set Notation} \\ &= (\, -\infty, -\frac{1}{\sqrt{2}} \,) \cup (\, \frac{1}{\sqrt{2}}, \infty \,) && \triangleleft \text{ Interval Notation} \end{aligned}$$

Example 2.9. ■

2.10. The function is

$$f(x) = \frac{\sqrt{4 - x^2}}{x - 1}.$$

There are two "constraints" on the natural domain of f: the numerator and the denominator.

Analysis of the Numerator. As we have seen already, the numerator is a real number if and only if

$$\begin{aligned} 4 - x^2 \geq 0 &\iff x^2 \leq 4 \\ &\iff |x| \leq 2 \end{aligned} \tag{S-3}$$

For an explantion of (S-3), see the aside in the solution to Example 2.8.

Analysis of the Denominator. The height of triviality! We require only that

$$x \neq 1 \tag{S-4}$$

Final Analysis. We require

$$|x| \leq 2 and \ x \neq 1.$$

bf Note: The word "and" above represents the set operation of "intersection." This speaks to my point mentioned earlier. The numerator is a real number provided $|x| \leq 2$, the denominator is a real (nonzero) number provided $x \neq 1$; therefore, for the whole expression to be a real number, both numerator *and* denominator must be real numbers: $|x| \leq 2$ *and* $x \neq 1$.

If x is so constrained, then $f(x)$ will surely by a computable real number. Therefore,

$$\begin{aligned} \text{Dom}(f) &= \{ x \in \mathbb{R} \mid |x| \leq 2 \text{ and } x \neq 1 \} \\ &= \{ x \in \mathbb{R} \mid -2 \leq x \leq 2 \text{ and } x \neq 1 \} \\ &= [-2, 1) \cup (1, 2] \end{aligned}$$

Example 2.10. ■

2.11. The function is

$$f(x) = \frac{\sqrt{1-x}}{\sqrt{1+x}}.$$

Analysis of the Numerator. We require

$$1 - x \geq 0 \text{ or } x \leq 1 \tag{S-5}$$

Analysis of the Denominator. We require

$$1 + x > 0 \text{ or } x > -1 \tag{S-6}$$

Note that x cannot be equal to -1 for otherwise we would have 0 is the denominator — a no-no!

Final Analysis. Overall, we require, from (S-5) and (S-6),

$$x \leq 1 \text{ and } x > -1$$

or,

$$-1 < x \leq 1.$$

Now, using good and proper notation, we present:

$$\begin{aligned}\operatorname{Dom}(f) &= \{\, x \in \mathbb{R} \mid -1 < x \leq 1 \,\} \\ &= (\,-1, 1\,].\end{aligned}$$

Example 2.11. ■

2.12. The first question is . . . what is the domain of f? Without an explicit specification, it is to be understood that the domain of f is its natural domain. By the methods already developed (see The Natural Domain of a Function), it is easy to see that

$$f(x) = 3 + \sqrt{x^2 - 4}$$
$$\operatorname{Dom}(f) = (\,-\infty, -2\,] \cup [\,2, \infty\,) \tag{S-7}$$

(Verify?)

Solution to (a): Is it true that $8 \in \operatorname{Rng}(f)$?

Following the advice outlined above, we setup the equation,

$$f(x) = 8$$

or,

$$3 + \sqrt{x^2 - 4} = 8,$$

and solve for x. Solving,

$$\begin{aligned} 3+\sqrt{x^2-4}=8 &\iff \sqrt{x^2-4}=5 \\ &\iff x^2-4=25 \\ &\iff x^2=29 \\ &\iff x=\pm\sqrt{29}. \end{aligned}$$

Note that we have multiple solutions (see EXERCISE 2.39). For the case $x=\sqrt{29}$, a consultation with your calculator (or using your knowledge of numbers — no calculator necessary) it is easy to see that both $x=\sqrt{29}\geq 2$ and $x=-\sqrt{29}\leq -2$ this means, by (S-7) that $29\in \operatorname{Dom}(f)$ and the $y=8\in \operatorname{Rng}(f)$.

Conclusion: $8\in \operatorname{Rng}(f)$ and, in fact, $f(\sqrt{29})=8$.

Solution to (b): Is it true that $2\in \operatorname{Rng}(f)$?

As before we setup the equation,

$$f(x) = 2$$

or,

$$3 + \sqrt{x^2 - 4} = 2,$$

and solve,

$$3 + \sqrt{x^2 - 4} = 2 \iff \sqrt{x^2 - 4} = -1. \tag{S-8}$$

Wait! Time Out! The equation $\sqrt{x^2 - 4} = -1$ has no solutions x that are real numbers because, by definition, the square root of any nonnegative number is a nonnegative number. (See Exercise 2.37)

Conclusion: $3 \notin \mathrm{Rng}(f)$. Example 2.12. ■

2.13. Based on our experience of testing the same function for particular points in EXAMPLE 2.12, we should be able to determine the entire range of f.

Let y be *any real number*. When is it true that there is an $x \in \mathrm{Dom}(f)$ such that $f(x) = y$? Recall,

$$\mathrm{Dom}(f) = (-\infty, -2] \cup [2, \infty).$$

Setup the standard equation:

$$f(x) = y$$

or,

$$3 + \sqrt{x^2 - 4} = y.$$

Under what conditions on y is it possible to solve this equation for x such that $|x| \geq 2$? Let's investigate.

$$3 + \sqrt{x^2 - 4} = y \iff \sqrt{x^2 - 4} = y - 3.$$

Now because $\sqrt{z} \geq 0$, for any $z \geq 0$, it is certainly the case that we must have

$$0 \leq \sqrt{x^2 - 4} = y - 3$$

or,

$$y - 3 \geq 0$$

or,

$$y \geq 3 \tag{S-9}$$

It is clearly the case that $y \geq 3$. But we have not entirely finished the analysis. We must ask ourselves the question: "Is it true that for any $y \geq 3$, there is some $x \in \mathrm{Dom}(f)$ such that $f(x) = y$?"

Indeed, let $y \geq 3$ be given, then

$$\begin{aligned} 3 + \sqrt{x^2 - 4} = y &\iff \sqrt{x^2 - 4} = y - 3 \\ &\iff x^2 - 4 = (y - 3)^2 \\ &\iff x^2 = 4 + (y - 3)^2 \\ &\iff x = \pm\sqrt{4 + (y - 3)^2} \end{aligned}$$

Concentrate on the positive solution for now.

$$x = \sqrt{4 + (y-3)^2}$$

We now ask the question, "Does this x belong to the domain of f, i.e., is it true that $x \geq 2$?" The answer is **Yes**. Why? We know that,

$$4 \leq 4 + (y-3)^2$$

since $(y-3)^2 \geq 0$. Therefore we deduce,

$$2 = \sqrt{4} \leq \sqrt{4 + (y-3)^2}. \tag{S-10}$$

Here, we have used the property of square roots:

Basic Fact. For $a \geq 0$ and $b \geq 0$, $a \leq b \iff \sqrt{a} \leq \sqrt{b}$.

But, (S-10) implies that $x = \sqrt{4 + (y-3)^2} \in \mathrm{Dom}(f)$.

Summary: We have argued that for any $y \geq 3$, (S-9), there is an $x \in \text{Dom}(f)$, (S-10), such that $f(x) = y$. This means that

$$\begin{aligned} \text{Rng}(f) &= \{\, y \in \mathbb{R} \mid y \geq 3 \,\} && \triangleleft \text{ Set Notation} \\ &= [\, 3, \infty \,) && \triangleleft \text{ Interval Notation} \end{aligned}$$

Example Notes: The negative solution, $x = -\sqrt{4 + (y-3)^2}$, now need not be considered since we already have $y \in \text{Rng}(f)$! Example 2.13. ■

Important Points

Because the relation

$$G(f) = \{ (a, f(a)) \mid a \in A \} \subseteq A \times B.$$

is nothing more than the *graph of* f! G is for *graph* the parentheses '()' are pronounced 'of' and f for f. Obvious! Important Point ■

This is a horizontal shift to the right 1/2 a unit.

$$H(2x-1) = \begin{cases} 0 & x \le \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases}$$

Important Point ■