8 Applications

This chapter presents some applications of manifold theory and tensor analysis to physics and engineering. Our selection is a of limited scope and depth, with the intention of providing an introduction to the techniques. There are many other applications of the ideas of this book as well. We list below a few *selected* references for further reading in the same spirit.

1. Arnol'd [1982], Abraham and Marsden [1978], Chernoff and Marsden [1974], Weinstein [1977], Marsden [1981], Marsden [1992], and Marsden and Ratiu [1999] for Hamiltonian mechanics.
2. Marsden and Hughes [1983] for elasticity theory.
3. Flanders [1963], von Westenholz [1981], and Bloch, Ballieul, Crouch and Marsden [2001] for applications to control theory.
4. Hermann [1980], Knowles [1981], and Schutz [1980] for diverse applications.
5. Bleecker [1981] for Yang–Mills theory.
6. Misner, Thorne, and Wheeler [1973] and Hawking and Ellis [1973] for general relativity.

8.1 Hamiltonian Mechanics

Newton's Second Law. Our starting point is ***Newton's second law*** in $\mathbb{R}^3$, which states that a particle which has mass $m > 0$, and is moving in a given potential field $V(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^3$, moves along a curve $\mathbf{x}(t)$ satisfying the equation of motion $m\ddot{\mathbf{x}} = -\operatorname{grad} V(\mathbf{x})$. If we introduce the momentum $\mathbf{p} = m\dot{\mathbf{x}}$ and the energy

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m}\|\mathbf{p}\|^2 + V(\mathbf{x}),$$

then the equation $\dot{\mathbf{x}} = \mathbf{p}/m$ and Newton's law become ***Hamilton's equations***:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \quad i = 1, 2, 3.$$

To study this system of first-order equations for given H, we introduce the matrix

$$\mathbb{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

where I is the 3×3 identity; note that the equations become

$$\dot{\xi} = \mathbb{J} \operatorname{grad} H(\xi)$$

where $\xi = (\mathbf{x}, \mathbf{p})$. In complex notation, setting $z = \mathbf{x} + i\mathbf{p}$, they may be written as

$$\dot{z} = -2i \frac{\partial H}{\partial \bar{z}}.$$

Suppose we make a change of coordinates, $w = f(\xi)$, where $f : \mathbb{R}^6 \to \mathbb{R}^6$ is smooth. If $\xi(t)$ satisfies Hamilton's equations, the equations satisfied by $w(t)$ are

$$\dot{w} = A\dot{\xi} = A\mathbb{J} \operatorname{grad}_\xi H(\xi) = A\mathbb{J}A^* \operatorname{grad}_w H(\xi(w)),$$

where $A^i_j = (\partial w^i / \partial \xi^j)$ is the Jacobian matrix of f, A^* is the transpose of A and $\xi(w)$ denotes the inverse function of f. The equations for w will be Hamiltonian with energy $K(w) = H(\xi(w))$ if $A\mathbb{J}A^* = \mathbb{J}$. A transformation satisfying this condition is called ***canonical*** or ***symplectic.*** One of the things we do in this chapter is to give a coordinate free treatment of this and related concepts.

The space $\mathbb{R}^3 \times \mathbb{R}^3$ of the ξ's is called the ***phase space***. For a system of N particles one uses $\mathbb{R}^{3N} \times \mathbb{R}^{3N}$. However, many fundamental physical systems have a phase space that is a manifold rather than Euclidean space, so doing mechanics solely in the context of Euclidean space is too constraining. For example, the phase space for the motion of a rigid body about a fixed point is the tangent bundle of the group SO(3) of 3×3 orthogonal matrices with determinant $+1$. This manifold is diffeomorphic to $\mathbb{RP}^3$ and is topologically nontrivial. To generalize the notion of a Hamiltonian system to the context of manifolds, we first need to geometrize the symplectic matrix $\mathbb{J}$. In infinite dimensions a few technical points need attention before proceeding.

Weak and Strong Metrics and Symplectic Forms. Let $\mathbf{E}$ be a Banach space and $B : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$ a continuous bilinear mapping. Then B induces a continuous map $B^\flat : \mathbf{E} \to \mathbf{E}^*$, $e \mapsto B^\flat(e)$ defined by $B^\flat(e) \cdot f = B(e, f)$. We call B ***weakly nondegenerate*** if $B^\flat$ is injective, that is, $B(e, f) = 0$ for all $f \in \mathbf{E}$ implies $e = 0$. We call B ***nondegenerate*** or ***strongly nondegenerate*** if $B^\flat$ is an isomorphism. By the open mapping theorem, it follows that *B is nondegenerate iff B is weakly nondegenerate and $B^\flat$ is onto.*

If $\mathbf{E}$ is finite dimensional there is no difference between strong and weak nondegeneracy. However, if infinite dimensions the distinction is important to bear in mind, and the issue does come up in basic examples, as we shall see in Supplement 8.1A.

Let M be a Banach manifold. By a ***weak Riemannian structure*** we mean a smooth assignment $g : x \mapsto \langle \, , \rangle_x = g(x)$ of a weakly nondegenerate inner product (not necessarily complete) to each tangent space T_xM. Here smooth means that in a local chart $U \subset \mathbf{E}$, the mapping $g : x \mapsto \langle \, , \rangle_x \in L^2(\mathbf{E}, \mathbf{E}; \mathbb{R})$ is smooth, where $L^2(\mathbf{E}, \mathbf{E}; \mathbb{R})$ denotes the Banach space of bilinear maps of $\mathbf{E} \times \mathbf{E}$ to $\mathbb{R}$. Equivalently, smooth means g is smooth as a section of the vector bundle $L^2(TM, TM; \mathbb{R})$ whose fiber at $x \in M$ is $L^2(T_xM, T_xM; \mathbb{R})$. By a ***Riemannian manifold*** we mean a weak Riemannian manifold in which $\langle \, , \rangle_x$ is nondegenerate. Equivalently, the topology of $\langle \, , \rangle_x$ is complete on T_xM, so that the model space $\mathbf{E}$ must be isomorphic to a Hilbert space.

For example the L^2 inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx \quad \text{on } \mathbf{E} = C^0([0,1], \mathbb{R})$$

is a weak Riemannian metric on $\mathbf{E}$ but is not a Riemannian metric.

8.1.1 Definition. *Let P be a manifold modeled on a Banach space $\mathbf{E}$. By a **symplectic form** we mean a two-form ω on P such that*

(i) *ω is closed, that is, $\mathbf{d}\omega = 0$;*

(ii) *for each $z \in P$, $\omega_z : T_zP \times T_zP \to \mathbb{R}$ is weakly nondegenerate.*

If ω_z in (ii) *is nondegenerate, we speak of a **strong symplectic form.** If* (ii) *is dropped we refer to ω as a **presymplectic form.*** (*For the moment the reader may wish to assume P is finite dimensional, in which case the weak–strong distinction vanishes.*)

The Darboux Theorem. Our proof of this basic theorem follows Moser [1965] and Weinstein [1969].

8.1.2 Theorem (The Darboux Theorem)**.** *Let ω be a strong symplectic form on the Banach manifold P. Then for each $x \in P$ there is a local coordinate chart about x in which ω is constant.*

Proof. The proof proceeds by the Lie transform method Theorem 5.4.7. We can assume $P = \mathbf{E}$ and $x = 0 \in \mathbf{E}$. Let ω_1 be the constant form equaling $\omega_0 = \omega(0)$. Let $\Omega = \omega_1 - \omega$ and $\omega_t = \omega + t\Omega$, for $0 \le t \le 1$. For each t, $\omega_t(0) = \omega(0)$ is nondegenerate. Hence by openness of the set of linear isomorphisms of $\mathbf{E}$ to $\mathbf{E}^*$, there is a neighborhood of 0 on which ω_t is nondegenerate for all $0 \le t \le 1$. We can assume that this neighborhood is a ball. Thus by the Poincaré lemma, $\Omega = \mathbf{d}\alpha$ for some one-form α. We can suppose $\alpha(0) = 0$. Define a smooth vector field X_t by

$$\mathbf{i}_{X_t}\omega_t = -\alpha,$$

which is possible since ω_t is strongly non-degenerate. Since $X_t(0) = 0$, by Corollary 4.1.25, there is a sufficiently small ball on which the integral curves of X_t will be defined for time at least one. Let F_t be the flow of X_t starting at $F_0 =$ identity. By the Lie derivative formula for time-dependent vector fields (Theorem 5.4.4) we have

$$\begin{aligned}\frac{d}{dt}(F_t^*\omega_t) &= F_t^*(\pounds_{X_t} - \omega_t) + F_t^*\frac{d}{dt}\omega_t \\ &= F_t^*\mathbf{d}\mathbf{i}_{X_t}\omega_t + F_t^*\Omega = F_t^*(\mathbf{d}(-\alpha) + \Omega) = 0.\end{aligned}$$

Therefore, $F_t^*\omega_1 = F_0^*\omega_0 = \omega$, so F_1 provides the chart transforming ω to the constant form ω_1. ∎

We note without proof that such a result is not true for Riemannian structures unless they are flat. Also, the analogue of Darboux theorem is known to be not valid for weak symplectic forms. (For the example, see Abraham and Marsden [1978], Exercise 3.2-8 and for conditions under which it is valid, see Marsden [1981] and Bambusi [1999].)

8.1.3 Corollary. *If P is finite dimensional and ω is a symplectic form, then*

(i) *P is even dimensional, say* $\dim P = 2n$;

(ii) *locally about each point there are coordinates $x^1, \ldots, x^n, y^1, \ldots, y^n$ such that*

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

*Such coordinates are called **canonical**.*

Proof. By elementary linear algebra, any skew symmetric bilinear form that is nondegenerate has the canonical form

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

where I is the $n \times n$ identity. (This is proved by the same method as Proposition 6.2.9.) This is the matrix version of (ii) pointwise on P. The result now follows from Darboux theorem. ∎

As a bilinear form, ω is given in canonical coordinates by

$$\omega((x_1, y_1), (x_2, y_2)) = \langle y_2, x_1 \rangle - \langle y_1, x_2 \rangle.$$

In complex notation with $z = x + iy$ it reads

$$\omega(z_1, z_2) = -\operatorname{Im}\langle z_1, z_2 \rangle.$$

This form for canonical coordinates extends to infinite dimensions (see Cook [1966], Chernoff and Marsden [1974], and Abraham and Marsden [1978, Section 3.1] for details).

Canonical Symplectic Forms. Of course in practice, symplectic forms do not come out of the blue, but must be constructed. The following constructions are basic results in this direction.

8.1.4 Definition. *Let Q be a manifold modeled on a Banach space $\mathbf{E}$. Let T^*Q be its cotangent bundle, and $\pi : T^*Q \to Q$ the projection. Define the **canonical one-form** θ on T^*Q by*

$$\theta(\alpha)w = \alpha \cdot T\pi(w),$$

*where $\alpha \in T_q^*Q$ and $w \in T_\alpha(T^*Q)$. The **canonical two-form** is defined by $\omega = -\mathbf{d}\theta$.*

In a chart $U \subset \mathbf{E}$, the formula for θ becomes

$$\theta(x, \alpha) \cdot (e, \beta) = \alpha(e),$$

where $(x, \alpha) \in U \times \mathbf{E}^*$ and $(e, \beta) \in \mathbf{E} \times \mathbf{E}^*$. If Q is finite dimensional, this formula may be written

$$\theta = p_i dq^i,$$

where $q^1, \ldots, q^n, p_1, \ldots, p_n$ are coordinates for T^*Q and the summation convention is enforced. Using the local formula for $\mathbf{d}$ from formula (6) in the table of identities in §6.4,

$$\omega(x, \alpha)((e_1, \alpha_1), (e_2, \alpha_2)) = \alpha_2(e_1) - \alpha_1(e_2),$$

or, in the finite-dimensional case,

$$\omega = dq^i \wedge dp_i.$$

In the infinite-dimensional case one can check that ω is weakly nondegenerate and is strongly nondegenerate iff $\mathbf{E}$ is reflexive.

If $\langle\, , \rangle_x$ is a weak Riemannian (or pseudo-Riemannian) metric on Q, the smooth vector bundle map

$$\varphi = g^\flat : TQ \to T^*Q$$

defined by $\varphi(v_x) \cdot w_x = \langle v_x, w_x \rangle_x, x \in Q$, is injective on fibers. If $\langle\, , \rangle$ is a strong Riemannian metric, then φ is a vector bundle isomorphism of TQ onto T^*Q. In any case, set $\Omega = \varphi^*\omega$ where ω is the canonical two–form on T^*Q. Clearly Ω is exact since $\Omega = -\mathbf{d}\Theta$ where $\Theta = \varphi^*\theta$.

In the finite-dimensional case, the formulas for Θ and Ω become

$$\Theta = g_{ij}\dot{q}^j dq^i,$$

and

$$\Omega = g_{ij} dq^i \wedge d\dot{q}^j + \frac{\partial g_{ij}}{\partial q^k} \dot{q}^j dq^i \wedge dq^k,$$

where $q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n$ are coordinates for TQ. This follows by substituting $p_i = g_{ij}\dot{q}^j$ into $\omega = dq^i \wedge dp_i$.

In the infinite-dimensional case, if $\langle\,,\rangle$ is a weak metric, then ω is a weak symplectic form locally given by

$$\Theta(w,e)(e_1,e_2) = -\langle e, e_1\rangle_x,$$

and

$$\begin{aligned}\Omega(x,e)((e_1,e_2),(e_3,e_4)) &= \mathbf{D}_x\langle e,e_1\rangle_x e_3 - \mathbf{D}_x\langle e,e_3\rangle_x e_1\\ &\quad + \langle e_4,e_1\rangle_x - \langle e_2,e_3\rangle_x,\end{aligned}$$

where $\mathbf{D}_x$ denotes the derivative with respect to x. One can also check that if $\langle\,,\rangle_x$ is a strong metric and Q is modeled on a reflexive space, then Ω is a strong symplectic form.

Symplectic Maps. Naturally, since we have the notion of a symplectic manifold, we should consider the mappings that preserve this structure.

8.1.5 Definition. *Let (P,ω) be a symplectic manifold. A (smooth) map $f : P \to P$ is called* ***canonical*** *or* ***symplectic*** *when $f^*\omega = \omega$.*

It follows that $f^*(\omega\wedge\cdots\wedge\omega) = \omega\wedge\cdots\wedge\omega$ (k times). If P is $2n$-dimensional, then $\mu = \omega\wedge\cdots\wedge\omega$ (n times) is nowhere vanishing, so is a volume form; for instance by a computation one finds μ to be a multiple of the standard Euclidean volume in canonical coordinates. In particular, note that symplectic manifolds are orientable. We call μ the ***phase volume*** or the ***Liouville form***. Thus a symplectic map preserves the phase volume, and so is necessarily a local diffeomorphism. A map $f : P_1 \to P_2$ between symplectic manifolds (P_1,ω_1) and (P_2,ω_2) is called ***symplectic*** if $f^*\omega_2 = \omega_1$. As above, if P_1 and P_2 have the same dimension, then f is a local diffeomorphism and preserves the phase volume.

Cotangent Lifts. We now discuss symplectic maps induced by maps on the base space of a cotangent bundle.

8.1.6 Proposition. *Let $f : Q_1 \to Q_2$ be a diffeomorphism; define the* ***cotangent lift*** *of f by*

$$T^*f : T^*Q_2 \to T^*Q_1; \quad T^*f(\alpha_q)\cdot v = \alpha_q \cdot Tf(v),$$

where $q \in Q_2, \alpha_q \in T_q^*Q_2$ and $v \in T_{f^{-1}(q)}Q_1$; that is, T^*f is the ***pointwise adjoint*** *of Tf. Then T^*f is symplectic and in fact $(T^*f)^*\theta_1 = \theta_2$ where θ_i is the canonical one-form on Q_i, $i = 1,2$.*

Proof. Let $\pi_i : T^*Q_i \to Q_i$ be the cotangent bundle projection, $i = 1,2$. For w in the tangent space to T^*Q_2 at α_q, we have

$$\begin{aligned}(T^*f)^*\theta_1(\alpha_q)(w) &= \theta_1(T^*f(\alpha_q))(TT^*f\cdot w)\\ &= T^*f(\alpha_q)\cdot(T\pi_1\cdot TT^*f\cdot w)\\ &= T^*f(\alpha_q)\cdot(T(\pi_1\circ T^*f)\cdot w)\\ &= \alpha_q\cdot(T(f\circ\pi_1\circ T^*f)\cdot w\\ &= \alpha_q\cdot(T\pi_2\cdot w)\\ &= \theta_2(\alpha_q)\cdot w\end{aligned}$$

since, by construction, $f\circ\pi_1\circ T^*f = \pi_2$. ∎

In coordinates, if we write $f(q^1,\ldots,q^n) = (Q^1,\ldots,Q^n)$, then T^*f has the effect

$$(q^1,\ldots,q^n,p_1,\ldots,p_n) \mapsto (Q^1,\ldots,Q^n,P_1,\ldots,P_n),$$

where

$$p_j = \frac{\partial Q^i}{\partial q^j}P_i$$

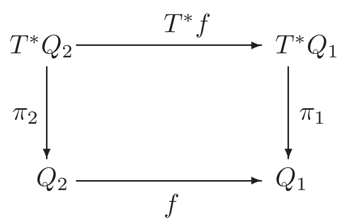
(evaluated at the corresponding points). That this transformation is always canonical and in fact preserves the canonical one-form may be verified directly:

$$P_i dQ^i = P_i \frac{\partial Q^i}{\partial q^k} dq^k = p_k dq^k.$$

Sometimes one refers to canonical transformations of this type as "point transformations" since they arise from general diffeomorphisms of Q_1 to Q_2. Notice that lifts of diffeomorphisms satisfy

$$f \circ \pi_2 = \pi_1 \circ T^* f;$$

that is, the following diagram commutes:



Notice also that

$$T^*(f \circ g) = T^* g \circ T^* f$$

and compare with

$$T(f \circ g) = Tf \circ Tg.$$

8.1.7 Corollary. *If Q_1 and Q_2 are Riemannian (or pseudo-Riemannian) manifolds and $f : Q_1 \to Q_2$ is an isometry, then $Tf : TQ_1 \to TQ_2$ is symplectic, and in fact $(Tf)^*\Theta_2 = \Theta_1$.*

Proof. This follows from the identity

$$Tf = {g_2}^{\#} \circ (T^* f)^{-1} \circ {g_1}^{\flat}.$$

All maps in this composition are symplectic and thus Tf is as well. ∎

Hamilton's Equations. So far no mention has been made of Hamilton's equations. Now we are ready to consider them.

8.1.8 Definition. *Let (P, ω) be a symplectic manifold. A vector field $X : P \to TP$ is called **Hamiltonian** if there is a C^1 function $H : P \to \mathbb{R}$ such that*

$$\mathbf{i}_X \omega = \mathbf{dH}.$$

*We say X is **locally Hamiltonian** if $\mathbf{i}_X\omega$ is closed.*

We write $X = X_H$ because usually in examples one is given H and then one constructs the Hamiltonian vector field X_H. If ω is only weakly nondegenerate, then given a smooth function $H : P \to \mathbb{R}$, X_H need not exist on all of P. Rather than being a pathology, this is quite essential in infinite dimensions, for the vector fields then correspond to partial differential equations and are only densely defined. The condition

$$\mathbf{i}_{X_H} \omega = \mathbf{d}H$$

is equivalent to

$$\omega_z(X_H(z), v) = \mathbf{d}H(z) \cdot v,$$

for $z \in P$ and $v \in T_zP$. Let us express this condition in canonical coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ on a $2n$-dimensional symplectic manifold P, that is, when $\omega = dq^i \wedge dp_i$. If $X = A^i\partial/\partial q^i + B^i\partial/\partial p_i$, then

$$\mathbf{i}_{X_H}\omega = \mathbf{i}_{X_H}(dq^i \wedge dq_i) = (\mathbf{i}_{X_H}dq^i)dp_i - (\mathbf{i}_{X_H}dp_i)dq^i = (A^i dp_i - B^i dq^i).$$

This equals

$$\mathbf{d}H = \frac{\partial H}{\partial q^i}dq^i + \frac{\partial H}{\partial p_i}dq_i$$

iff

$$A^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad B^i = -\frac{\partial H}{\partial q^i},$$

that is,

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \right).$$

If

$$\mathbb{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

where I is the $n \times n$ identity matrix, the formula for X_H can be expressed as

$$X_H = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i} \right) = \mathbb{J}\,\mathrm{grad}\, H.$$

More intrinsically, one can write $X_H = \omega^{\#}\mathbf{d}H$, so one sometimes says that X_H is the ***symplectic gradient*** of H. Note that the formula $X_H = \mathbb{J}\,\mathrm{grad}\,H$ is a little misleading in this respect, since no metric structure is actually needed and it is really the differential and not the gradient that is essential.

From the local expression for X_H we see that $(q^i(t), p_i(t))$ is an integral curve of X_H iff ***Hamilton's equations*** hold;

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

Properties of Hamiltonian systems. The proofs of the following properties are a bit more technical for densely defined vector fields, so for purposes of these theorems, we work with C^∞ vector fields.

8.1.9 Theorem. *Let X_H be a Hamiltonian vector field on the (weak) symplectic manifold (P, ω) and let F_t be the flow of X_H. Then*

(i) *F_t is symplectic, that is, $F_t^*\omega = \omega$, and*

(ii) *energy is conserved, that is, $H \circ F_t = F_t$.*

Proof. (i) Since $F_0 =$ identity, it suffices to show that $(d/dt)F_t^*\omega = 0$. But by the basic connection between Lie derivatives and flows (§5.4 and §6.4):

$$\begin{aligned} \frac{d}{dt}F_t^*\omega(x) &= F_t^*(\pounds_{X_H}\omega)(x) \\ &= F_t^*(\mathbf{d}\mathbf{i}_{X_H}\omega)(x) + F_t^*(\mathbf{i}_{X_H}\mathbf{d}\omega)(x). \end{aligned}$$

The first term is zero because it is $\mathbf{d}^2H = 0$ and the second is zero because ω is closed.

(ii) By the chain rule,

$$\begin{aligned}\frac{d}{dt}(H \circ F_t)(x) &= \mathbf{d}H(F_t(x)) \cdot X_H(F_t(x)) \\ &= \omega(F_t(x))(X_H(F_t(x)), X_H(F_t(x))).\end{aligned}$$

But this is zero in view of the skew symmetry of ω. ■

A corollary of (i) in finite dimensions is ***Liouville's theorem***: *F_t preserves the phase volume.* This is seen directly in canonical coordinates by observing that X_H is divergence-free.

Poisson Brackets. Define for any functions $f, g : U \to \mathbb{R}, U$ open in P, their ***Poisson bracket*** by

$$\{f, g\} = \omega(X_f, X_g).$$

Since

$$\begin{aligned}\pounds_{X_f} g &= \mathbf{i}_{X_f} \mathbf{d}g \\ &= \mathbf{i}_{X_f} \mathbf{i}_{X_g} \omega \\ &= \omega(X_g, X_f) = -\omega(X_f, X_g) = -\pounds_{X_g} f,\end{aligned}$$

we see that

$$\{f, g\} = \pounds_{X_g} f = -\pounds_{X_f} g.$$

If $\varphi : P_1 \to P_2$, is a diffeomorphism where (P_1, ω_1) and (P_2, ω_2) are symplectic manifolds, then by the property $\varphi^*(\pounds_X \alpha) = \pounds_{\varphi^* X} \varphi^* \alpha$ of pull–back, we have

$$\varphi^*\{f, g\} = \varphi^*(\pounds_{X_f} g) = \pounds_{\varphi^* X_f} \varphi^* g,$$

and

$$\{\varphi^* f, \varphi^* g\} = \pounds_{X_{\varphi^* f}} \varphi^* g.$$

Thus *φ preserves the Poisson bracket of any two functions defined on some open set of P_2 iff $\varphi^* X_f = X_{\varphi^* f}$ for all C^∞ functions $f : U \to \mathbb{R}$ where U is open in P_2.* This says that φ preserves the Poisson bracket iff it preserves Hamilton's equations. We have

$$\mathbf{i}_{X_{\varphi^* f}} \omega = \mathbf{d}(\varphi^* f) = \varphi^*(\mathbf{d}f) = \varphi^* \mathbf{i}_{X_f} \omega = \mathbf{i}_{\varphi^* X_f} \varphi^* \omega,$$

so that by the (weak) nondegeneracy of ω and the fact that any $v \in T_z P$ equals some $X_h(z)$ for a C^∞ function h defined in a neighborhood of z, we conclude that φ is symplectic iff $\varphi^* X_f = X_{\varphi^* f}$ for all C^∞ functions $f : U \to \mathbb{R}$, where U is open in P_2. We have thus proved the following.

8.1.10 Proposition. *Let (P_1, ω_1) and (P_2, ω_2) be symplectic manifolds and $\varphi : P_1 \to P_2$ a diffeomorphism. The following are equivalent*:

(i) *φ is symplectic.*

(ii) *φ preserves the Poisson bracket of any two locally defined functions.*

(iii) *$\varphi^* X_f = X_{\varphi^* f}$ for any local $f : U \to \mathbb{R}$, where U is open in P_2 (i.e., φ locally preserves Hamilton's equations).*

Conservation of energy is generalized in the following way.

8.1.11 Corollary. (i) *Let X_H be a Hamiltonian vector field on the (weak) symplectic manifold (P, ω) with (local) flow F_t. Then for any C^∞ function $f : U \to \mathbb{R}, U$ open in P, we have*

$$\frac{d}{dt}(f \circ F_t) = \{f, H\} \circ F_t = \{f \circ F_t, H\}.$$

(ii) *The curve $c(t)$ satisfies Hamilton's equations defined by H if and only if*

$$\frac{d}{dt} f(c(t)) = \{f, H\}(c(t))$$

for any C^∞ function $f : U \to \mathbb{R}$, where U is open in P.

Proof. (i) We compute as follows:

$$\frac{d}{dt}(f \circ F_t) = \frac{d}{dt} F_t^* f = F_t^* \pounds_{X_H} f = F_t^* \{f, H\} = \{F_t^* f, H\}$$

by the formula for Lie derivatives and the previous proposition.

(ii) Since $df(c(t))/dt = \mathbf{d}f(c(t)) \cdot (dc/dt)$ and

$$\{f, H\}(c(t)) = (\pounds_{X_H} f)(c(t)) = \mathbf{d}f(c(t)) \cdot X_H(c(t)),$$

the equation in the statement of the proposition holds iff $c'(t) = X_H(c(t))$ by the Hahn–Banach theorem and Corollary 4.2.14. ■

One writes $\dot{f} = \{f, H\}$ to stand for the equation in (ii). This equation is called the ***equation of motion in Poisson bracket formulation.***

Two functions $f, g : P \to \mathbb{R}$ are said to be ***in involution*** or to ***Poisson commute*** if $\{f, g\} = 0$. Any function Poisson commuting with the Hamiltonian of a mechanical system is, by Corollary 8.1.11, necessarily constant along on the flow of the Hamiltonian vector field. This is why such functions are called ***constants of the motion.*** A classical theorem of Liouville states that *in a mechanical system with a $2n$-dimensional phase space admitting k constants of the motion in involution and independent almost everywhere (i.e., the differentials are independent on an open dense set) one can reduce the dimension of the phase space to $2(n-k)$*. In particular, if $k = n$, the equations of motion can be "explicitly" integrated. In fact, under certain additional hypotheses, the trajectories of the mechanical system are straight lines on high-dimensional cylinders or tori. If the motion takes place on tori, the explicit integration of the equations of motion goes under the name of finding ***action-angle variables.*** See Arnol'd [1982], and Abraham and Marsden [1978, pp. 392–400] for details and Exercise 8.1-4 for an example. In infinite-dimensional systems the situation is considerably more complicated. A famous example is the Korteweg–deVries (KdV) equation; for this example we also refer to Abraham and Marsden [1978, pp. 462–72] and references therein. The following supplement gives some elementary but still interesting examples of infinite-dimensional Hamiltonian systems.

Supplement 8.1A
Two Infinite-Dimensional Examples

8.1.12 Example (The Wave Equation as a Hamiltonian System)**.** The wave equation for a function $u(x, t)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ is given by

$$\frac{d^2u}{dt^2} = \nabla^2 u + m^2 u, \quad \text{(where } m \geq 0 \text{ is a constant)},$$

with u and $\dot{u} = \partial u/\partial t$ given at $t = 0$. The energy is

$$H(u, \dot{u}) = \frac{1}{2}\left(\int |\dot{u}|^2 dx + \int \|\nabla u\|^2 dx\right).$$

We define H on pairs $(u, \dot{u})$ of finite energy by setting

$$P = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n),$$

where H^1 consists of functions in L^2 whose first (distributional) derivatives are also in L^2. (The Sobolev spaces H^s defined this way are Hilbert spaces that arise in many problems involving partial differential equations. We only treat them informally here.) Let $D = H^2 \times H^1$ and define $X_H : D \to P$ by

$$X_H(u, \dot{u}) = (\dot{u}, \nabla^2 u + m^2 u).$$

Let the symplectic form be associated with the L^2 metric as in the discussion following Definition 8.1.4, namely

$$\omega((u, \dot{u}), (v, \dot{v})) = \int \dot{v} u \, dx - \int \dot{u} v \, dx.$$

It is now an easy verification using integration by parts, to show that X_H, ω and H are in the proper relation, so in this sense the wave equation is Hamiltonian. That the wave equation has a flow on P follows from (the real form of) Stone's theorem (see Supplement 7.4A and **?**). ♦

8.1.13 Example (The Schrödinger Equation)**.** Let $P = \mathcal{H}$ a complex Hilbert space with $\omega = -2\,\mathrm{Im}\langle\, , \rangle$. Let H_{op} be a self-adjoint operator with domain D and let

$$X_H(\varphi) = iH_{\mathrm{op}} \cdot \varphi$$

and

$$H(\varphi) = \langle H_{\mathrm{op}}\varphi, \varphi\rangle, \quad \varphi \in D.$$

Again it is easy to check that ω, X_H and H are in the correct relation. Thus, X_H is Hamiltonian. Note that $\psi(t)$ is an integral curve of X_H if

$$\frac{1}{i}\frac{d\psi}{dt} = H_{\mathrm{op}}\psi,$$

which is the ***abstract Schrödinger equation*** of quantum mechanics. That X_H has a flow is a special case of Stone's theorem. We know from general principles that the flow $e^{itH_{\mathrm{op}}}$ will be symplectic. The additional structure needed for unitarity is exactly complex linearity. ♦

Turning our attention to geodesics and to Lagrangian systems, let M be a (weak) Riemannian manifold with metric $\langle\, , \rangle_x$ on the tangent space T_xM. The ***spray*** $S : TM \to T^2M$ of the metric $\langle\, , \rangle_x$ is the vector field on TM defined locally by $S(x, v) = ((x, v), (v, \gamma(x, v)))$, for $(x, v) \in T_xM$, where γ is defined by

$$\langle \gamma(x, v), w\rangle_x \equiv \frac{1}{2}\mathbf{D}_x\langle v, v\rangle_x \cdot \omega - \mathbf{D}_x\langle v, w\rangle_x \cdot v \tag{8.1.1}$$

and $\mathbf{D}_x, \langle v, v\rangle_x \cdot w$ means the derivative of $\langle v, v\rangle_x$ with respect to x in the direction of w. If M is finite dimensional, the ***Christoffel symbols*** are defined by putting $\gamma^i(x, v) = -\Gamma^i_{jk}(x)v^j v^k$. Equation (8.1.1) is equivalent to

$$-\Gamma^i_{jk}v^j v^k w_i = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}v^i v^j w^k - \frac{\partial g_{ij}}{\partial x^k}v^i w^j v^k;$$

that is,

$$\Gamma^i_{jk} = \frac{1}{2} g^{hi} \left(\frac{\partial g_{hk}}{\partial x^j} + \frac{\partial g_{jh}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^h} \right).$$

The verification that S is well-defined independent of the charts is not too difficult. Notice that γ is quadratic in v. We will show below that S is the Hamiltonian vector field on TM associated with the kinetic energy $\langle v, v\rangle/2$. The projection of the integral curves of S to M are called ***geodesics***. Their local equations are thus

$$\ddot{x} = \gamma(x, \dot{x}),$$

which in the finite-dimensional case becomes

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \quad i = 1, \ldots, n.$$

The definition of γ in equation (8.1.1) makes sense in the infinite as well as the finite-dimensional case, whereas the coordinate definition of Γ^i_{jk} makes sense only in finite dimensions. This provides a way to deal with geodesics in infinite-dimensional spaces.

Let $t \mapsto (x(t), v(t))$ be an integral curve of S. That is,

$$\dot{x}(t) = v(t) \quad \text{and} \quad \dot{v}(t) = \gamma(x(t), v(t)). \tag{8.1.2}$$

As we remarked, these will shortly be shown to be Hamilton's equations of motion in the absence of a potential. To include a potential, let $V : M \to \mathbb{R}$ be given. At each x, we have the differential of V, $\mathbf{d}V(x) \in T^*_x M$, and we define grad $V(x)$ by

$$\langle \operatorname{grad} V(x), w\rangle_x = \mathbf{d}V(x) \cdot w. \tag{8.1.3}$$

(In infinite dimensions, it is an extra assumption that grad V exists, since the map $T_xM \to T^*_x M$ induced by the metric is not necessarily bijective.)

The equations of motion in the potential field V are given by

$$\dot{x}(t) = v(t); \quad \dot{v}(t) = \gamma(x(t), v(t)) - \operatorname{grad} V(x(t))). \tag{8.1.4}$$

The total energy, kinetic plus potential, is given by

$$H(v_x) = \frac{1}{2}\|v_x\|^2 + V(x).$$

The vector field X_H determined by H relative to the symplectic structure on TM induced by the metric, is given by equation (8.1.4). This will be part of a more general derivation of Lagrange's equations given below.

Supplement 8.1B
Geodesics

Readers familiar with Riemannian geometry can reconcile the present approach to geodesics based on Hamiltonian mechanics to the standard one in the following way. Define the *covariant* ***derivative*** $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ locally by

$$(\nabla_X Y)(x) = \gamma_x(X(x), Y(x)) + \mathbf{D}Y(x) \cdot X(x),$$

where $X(x)$ and $Y(x)$ are the local representatives of X and Y in the model space $\mathbf{E}$ of M and $\gamma_x : \mathbf{E}\times\mathbf{E} \to \mathbf{E}$ denotes the symmetric bilinear continuous mapping defined by polarization of the quadratic form $\gamma(x, v)$. In finite dimensions, if $\mathbf{E} = \mathbb{R}^n$, then $\gamma(x, v)$ is an $\mathbb{R}^n$-valued quadratic form on $\mathbb{R}^n$ determined by the Christoffel symbols Γ^i_{jk}. The defining relation for $\nabla_X Y$ becomes

$$\nabla_X Y = X^j Y^k \Gamma^i_{jk} \frac{\partial}{\partial x^i} + X^j \frac{\partial Y^k}{\partial x^j} \frac{\partial}{\partial x^k},$$

where locally

$$X = X^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = Y^k \frac{\partial}{\partial x^k}.$$

It is a straightforward exercise to show that the foregoing definition of $\nabla_X Y$ is chart independent and that ∇ satisfies the following conditions defining an ***affine connection***:

(i) ∇ is $\mathbb{R}$-bilinear,

(ii) for $f : M \to \mathbb{R}$ smooth,

$$\nabla_{fX} Y = f \nabla_X Y \quad \text{and} \quad \nabla_X fY = f \nabla_x Y + X[f]Y,$$

(iii) $(\nabla_X Y - \nabla_Y X)(x) = \mathbf{D}Y(x) \cdot X(x) - \mathbf{D}X(x) \cdot Y(x) = [X, Y](x)$,

by the local formula for the Jacobi–Lie bracket of two vector fields. (The equivalence of sprays and affine connections was introduced by **?**.)

If $c(t)$ is a curve in M and $X \in \mathfrak{X}(M)$, the ***covariant derivative of*** X ***along*** c is defined by

$$\frac{DX}{dt} = \nabla_{\mathbf{c}} X,$$

where $\mathbf{c}$ is a vector field coinciding with $\dot{c}(t)$ at the points $c(t)$. Locally, using the chain rule, this becomes

$$\frac{DX}{dt}(c(t)) = -\gamma_{c(t)}(X(c(t)), X(c(t))) + \frac{d}{dt} X(c(t)),$$

which also shows that the definition of DX/dt depends only on $c(t)$ and not on how $\dot{c}$ is extended to a vector field. In finite dimensions, the coordinate form of the preceding equation is

$$\left(\frac{DX}{dt}\right)^i = \Gamma^i_{jk}(c(t)) X^j(c(t)) \dot{c}^k(t) + \frac{d}{dt} X^i(c(t)),$$

where $\dot{c}(t)$ denotes the tangent vector to the curve at $c(t)$.

The vector field X is called ***autoparallel*** or is ***parallel-transported along*** c if $DX/dt = \mathbf{0}$. Thus $\dot{c}$ is autoparallel along c iff in any coordinate system we have

$$\ddot{c}(t) - \gamma_{c(t)}(\dot{c}(t), \dot{c}(t)) = 0$$

or, in finite dimensions

$$\ddot{c}^i(t) + \Gamma^i_{jk}(c(t)) \dot{c}^j(t) \dot{c}^k(t) = 0.$$

That is, *$\dot{c}$ is autoparallel along c iff c is a geodesic.*

There is feedback between Hamiltonian systems and Riemannian geometry. For example, conservation of energy for geodesics is a direct consequence of their Hamiltonian character but can also be checked directly. Moreover, the fact that the flow of the geodesic spray on TM consists of canonical transformations is also useful in geometry, for example, in the study of closed geodesics (cf. Klingenberg [1978]). On the other hand, Riemannian geometry provides tools and concepts (such as parallel transport and curvature) that are useful in studying Hamiltonian systems.

Lagrangian Mechanics. We now generalize the idea of motion in a potential to that of a Lagrangian system; these are, however, still special types of Hamiltonian systems. We begin with a manifold M and a given function $L : TM \to \mathbb{R}$ called the ***Lagrangian***. In case of motion in a potential, take

$$L(v_x) = \frac{1}{2}\langle v_x, v_x\rangle - V(x),$$

which differs from the energy in that $-V$ is used rather than $+V$.

The Lagrangian L defines a map called the ***fiber derivative,*** $\mathbb{F}L : TM \to T^*M$ as follows: let $v, w \in T_xM$, and set

$$\mathbb{F}L(v) \cdot w \equiv \left.\frac{d}{dt}L(v + tw)\right|_{t=0}.$$

That is, $\mathbb{F}L(v) \cdot w$ is the derivative of L along the fiber in direction w. In the case of $L(v_x) = (1/2)\langle v_x, v_x\rangle_x - V(x)$, we see that $\mathbb{F}L(v_x) \cdot w_x = \langle v_x, w_x\rangle_x$, so we recover the usual map $g^\flat : TM \to T^*M$ associated with the bilinear form $\langle\, ,\rangle_x$.

Since T^*M carries a canonical symplectic form ω, we can use $\mathbb{F}L$ to obtain a closed two-form ω_L on TM :

$$\omega_L = (\mathbb{F}L)^*\omega.$$

A local coordinate computation yields the following local formula for ω_L : if M is modeled on a linear space $\mathbf{E}$, so locally TM looks like $U \times \mathbf{E}$ where $U \subset \mathbf{E}$ is open, then $\omega_L(u, e)$ for $(u, e) \in U \times \mathbf{E}$ is the skew symmetric bilinear form on $\mathbf{E} \times \mathbf{E}$ given by

$$\begin{aligned}\omega_L(u,e) &\cdot ((e_1, e_2), (f_1, f_2)) \\ &= \mathbf{D}_1(\mathbf{D}_2L(u,e) \cdot e_1) \cdot f_1 - \mathbf{D}_1(\mathbf{D}_2L(u,e) \cdot f_1) \cdot e_1 \\ &\quad + \mathbf{D}_2(\mathbf{D}_2L(u,e) \cdot e_1) \cdot f_2 - \mathbf{D}_2(\mathbf{D}_2L(u,e) \cdot f_1) \cdot e_2, \end{aligned} \tag{8.1.5}$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ denote the indicated partial derivatives of L. In finite dimensions this reads

$$\omega_L = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} dq^i \wedge d\dot{q}^i,$$

where $(q^i, \dot{q}^j)$ are the standard local coordinates on TQ.

The two form ω_L is (weakly) nondegenerate if $\mathbf{D}_2\mathbf{D}_2L(u, e)$ is (weakly) nondegenerate; in this case L is called (***weakly***) ***nondegenerate***. In the case of motion in a potential, nondegeneracy of ω_L amounts to nondegeneracy of the metric $\langle\, ,\rangle_x$. The ***action*** of L is defined by $A : TM \to \mathbb{R}$, $A(v) = \mathbb{F}L(v) \cdot v$, and the ***energy*** of L is $E = A - L$. In charts,

$$E(u, e) = \mathbf{D}_2L(u, e) \cdot e - L(u, e),$$

and in finite dimensions, E is given by the expression

$$E(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L(q, \dot{q}).$$

Given L, we say that a vector field Z on TM is a ***Lagrangian vector field*** or a ***Lagrangian system*** for L if the ***Lagrangian condition*** holds:

$$\omega_L(v)(Z(v), w) = \mathbf{d}E(v) \cdot w \tag{8.1.6}$$

for all $v \in T_qM$ and $w \in T_v(TM)$. Here $\mathbf{d}E$ denotes the differential of E. We shall see that for motion in a potential, this leads to the same equations of motion as we found before.

If ω_L were a weak symplectic form there would exist at most one such Z, which would be the Hamiltonian vector field for the Hamiltonian E. The dynamics is obtained by finding the integral curves of Z; that is, the curves $t \mapsto v(t) \in TM$ satisfying $(dv/dt)(t) = Z(v(t))$. From the Lagrangian condition it is easy to check that energy is conserved (even though L may be degenerate).

8.1.14 Proposition. *Let Z be a Lagrangian vector field for L and let $v(t) \in TM$ be an integral curve of Z. Then $E(v(t))$ is constant in t.*

Proof. By the chain rule,

$$\begin{aligned}\frac{d}{dt}E(v(t)) &= \mathbf{d}E(v(t)) \cdot v'(t) \\ &= \mathbf{d}E(v(t)) \cdot Z(v(t)) - \omega_L(v(t))(Z(v(t)), Z(v(t))) = 0\end{aligned}$$

by skew symmetry of ω_L. ■

We now generalize our previous local expression for the spray of a metric, and the equations of motion in the presence of a potential. In the general case the equations are called Lagrange's equations.

8.1.15 Proposition. *Let Z be a Lagrangian system for L and suppose Z is a second-order equation (i.e., in a chart $U \times \mathbf{E}$ for TM, $Z(u, e) = (u, e, e, Z_2(u, e))$ for some map $Z_2 : U \times \mathbf{E} \to \mathbf{E}$). Then in the chart $U \times \mathbf{E}$, an integral curve $(u(t), v(t)) \in U \times \mathbf{E}$ of Z satisfies **Lagrange's equations**: that is,*

$$\frac{du}{dt}(t) = v(t), \quad \frac{d}{dt}(\mathbf{D}_2 L(u(t), v(t)) \cdot w = \mathbf{D}_1 L(u(t), v(t)) \cdot w \tag{8.1.7}$$

for all $w \in \mathbf{E}$. If $\mathbf{D}_2\mathbf{D}_2 L$, or equivalently ω_L, is weakly nondegenerate, then Z is automatically second order.

In case of motion in a potential, equation (8.1.7) reduces to the equations (8.1.4).

Proof. From the definition of the energy E we have locally

$$\begin{aligned}\mathbf{D}E(u, e) \cdot (f_1, f_2) = \mathbf{D}_1(\mathbf{D}_2 L(u, e) \cdot e) \cdot f_1 \\ + \mathbf{D}_2(\mathbf{D}_2 L(u, e) \cdot e) \cdot f_2 - \mathbf{D}_1 L(u, e) \cdot f_1\end{aligned}$$

(a term $\mathbf{D}_2 L(u, e) \cdot f_2$ has canceled). Locally we may write

$$Z(u, e) = (u, e, Y_1(u, e), Y_2(u, e)).$$

Using formula (8.1.5) for ω_L, the condition on Z may be written

$$\begin{aligned}&\mathbf{D}_1(\mathbf{D}_2 L(u, e) \cdot Y_1(u, e)) \cdot f_1 - \mathbf{D}_1(\mathbf{D}_2 L(u, e) \cdot f_1) \cdot Y_1(u, e) \\ &+ \mathbf{D}_2(\mathbf{D}_2 L(u, e) \cdot Y_1(u, e)) \cdot f_2 - \mathbf{D}_2(\mathbf{D}_2 L(u, e) \cdot f_1) \cdot Y_2(u, e) \\ &\quad = \mathbf{D}_1(\mathbf{D}_2 L(u, e) \cdot e) \cdot f_1 - \mathbf{D}_1 L(u, e) \cdot f_1 \\ &\quad\quad + \mathbf{D}_2(\mathbf{D}_2 L(u, e) \cdot e) \cdot f_2.\end{aligned} \tag{8.1.8}$$

Thus if ω_L is a weak symplectic form, then $\mathbf{D}_2\mathbf{D}_2 L(u, e)$ is weakly nondegenerate, so setting $f_1 = 0$ we get $Y_1(u, e) = e$, that is, Z is a second-order equation. In any case, if we assume that Z is second order, then condition (8.1.8) becomes

$$\mathbf{D}_1 L(u, e) \cdot f_1 = \mathbf{D}_1(\mathbf{D}_2 L(u, e) \cdot f_1) \cdot e + \mathbf{D}_2(\mathbf{D}_2 L(u, e) \cdot f_1) \cdot Y_2(u, e)$$

for all $f_1 \in \mathbf{E}$. If $(u(t), v(t))$ is an integral curve of Z and using dots to denote time differentiation, then $\dot{u} = v$ and $\ddot{u} = Y_2(u, \dot{u})$, so

$$\begin{aligned}\mathbf{D}_1 L(u, \dot{u}) \cdot f_1 &= \mathbf{D}_1(\mathbf{D}_2 L(u, \dot{u}) \cdot f_1) \cdot \dot{u} + \mathbf{D}_2(\mathbf{D}_2 L(u, \dot{u}) \cdot f_1) \cdot \ddot{u} \\ &= \frac{d}{dt}\mathbf{D}_2 L(u, \dot{u}) \cdot f_1\end{aligned}$$

by the chain rule. ■

The condition of being second order is intrinsic; Z is second order iff $T\tau_M \circ Z =$ identity, where $\tau_M : TM \to M$ is the projection. See Exercise 8.1-4.

In finite dimensions Lagrange's equations (8.1.7) take the form

$$\frac{dq^i}{dt} = \dot{q}^i, \quad \text{and} \quad \frac{d}{dt}(\frac{\partial L}{\partial q^i}) = \frac{\partial L}{\partial q^i}, \quad i = 1, \ldots, n.$$

8.1.16 Proposition. *Assume $\varphi : Q \to Q$ is a diffeomorphism which leaves a weakly nondegenerate Lagrangian L invariant, that is, $L \circ T\varphi = L$. Then $c(t)$ is an integral curve of the Lagrangian vector field Z if and only if $T\varphi \circ c$ is also an integral curve.*

Proof. Invariance of L under φ implies $\mathbb{F}L \circ T\varphi = T^*\varphi^{-1} \circ \mathbb{F}L$ so that

$$\begin{aligned}(T\varphi)^*\omega_L &= (\mathbb{F}L \circ T\varphi)^*\omega = (T^*\varphi^{-1} \circ \mathbb{F}L)^*\omega = (\mathbb{F}L)^*(T^*\varphi^{-1})^*\omega \\ &= (\mathbb{F}L)^*\omega = \omega_L\end{aligned}$$

by Proposition 8.1.6. We also have for any $v \in TQ$,

$$A(T\varphi(v)) = \mathbb{F}L(T\varphi(v)) \cdot T\varphi(v) = \mathbb{F}L(v)$$

and thus relation (8.1.6) implies

$$\mathbf{d}E = (T\varphi)^*\mathbf{d}E = (T\varphi)^*\mathbf{i}_Z\omega_L = \mathbf{i}_{(T\varphi)^*Z}(T\varphi)^*\omega_L = \mathbf{i}(T\varphi)^*Z\omega_L.$$

Weak nondegeneracy of L yields then $(T\varphi)^*Z = Z$ which by Proposition 4.2.4 is equivalent to the statement in the proposition.

■

8.1.17 Example (Geodesics on the Poincaré Upper Half Plane). Let

$$Q = \{ (x, y) \in \mathbb{R}^2 \mid y > 0 \}$$

so that $TQ = Q \times \mathbb{R}^2$. Define the ***Poincaré metric*** g on Q by

$$g(x, y)((u^1, u^2), (v^1, v^2)) = \frac{u^1v^1 + u^2v^2}{y^2}$$

and consider the Lagrangian

$$L(x, y, v^1, v^2) = \frac{(v^1)^2 + (v^2)^2}{y^2}$$

defined by g. L is nondegenerate and thus by Proposition 8.1.15, the Lagrangian vector field Z defined by L is a second order equation. By local existence and uniqueness of integral curves, for every point $(x_0, y_0) \in Q$ and every vector $(v_0^1, v_0^2) \in T_{(x,y)}Q$, there is a unique geodesic $\gamma(t)$ satisfying $\gamma(0) = (x_0, y_0)$, $\gamma'(0) = (v_0^1, v_0^2)$. We shall determine the geodesics of g by taking advantage of invariance properties of L.

Note that the reflection $r : (x, y) \in Q \mapsto (-x, y) \in Q$ leaves L invariant. Furthermore, consider the homographies $h(z) = (az + b)/(cz + d)$ for $a, b, c, d \in \mathbb{R}$ satisfying $ad - bc = 1$, where $z = x + iy$. Since

$$\text{Im}[h(z)] = \text{Im}\left[\frac{z}{(cz + d)^2}\right],$$

it follows that $h(Q) = Q$ and since $T_zh(v) = v/(cz + d)^2$, where $v = v^1 + iv^2$, it follows that h leaves L invariant. Therefore, by Proposition **??**, γ is a geodesic if and only if $r \circ \gamma$ and $h \circ \gamma$ are. In particular, if $\gamma(0) = (0, y_0)$, $\gamma'(0) = (0, u_0)$, then

$$(r \circ \gamma)(0) = (0, y_0) \quad \text{and} \quad (r \circ \gamma)'(0) = (0, u_0),$$

that is, $\gamma = r \circ \gamma$ and thus γ is the semiaxis $y > 0$, $x = 0$. Since for any $q_0 \in Q$ and tangent vector v_0 to Q at q_0 there exists a homography h such that $h(iy_0) = q_0$, and the tangent of h at iy_0 in the direction iu_0 is v_0, it follows that the geodesics of g are images by h of the semiaxis $\{ (0, y) \mid y > 0 \}$. If $c = 0$ or $d = 0$, this image equals the ray $\{ (b/d, y) \mid y > 0 \}$ or the ray $\{ (a/c, y) \mid y > 0 \}$. If both $c \neq 0$, $d \neq 0$, then the image equals the arc of the circle centered at $((ad + bc)/2cd, 0)$ of radius $1/(2cd)$. Thus the *geodesics of the Poincaré upper half plane are either rays parallel to the y-axis or arcs of circles centered on the x-axis*. (See Figure 8.1.1.) The Poincaré upper half-plane is a model of the Lobatchevski geometry. Two geodesics in Q are called ***parallel*** if they do not intersect in Q. Given either a ray parallel to Oy or a semicircle centered on Ox and a point not on this geodesic, there are infinitely many semicircles passing through this point and not intersecting the geodesics, that is, *through a point not on a geodesic there are infinitely many geodesics parallel to it*. ♦

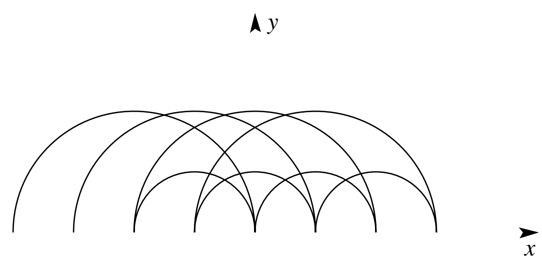


FIGURE 8.1.1. Geodesics in the Poincaré upper half plane.

We close with a result for Lagrangian systems generalizing Example 4.1.23B.

8.1.18 Definition. *A C^2 function $V_0 : [0, \infty] \to \mathbb{R}$ is called **positively complete** if it is decreasing and for any $e > \sup_{i \geq 0}\{V_0(t)\}$, satisfies*

$$\int_x^\infty [e - V_0(t)]^{-1/2} dt = +\infty, \quad \textit{where } x \geq 0.$$

The last condition is independent of e. Examples of positively complete functions are $-t^\alpha$, $-t[\log(1+t)]^\alpha$, $-t\log(1+t)[\,\log(\log(1+t)+1)]^\alpha$, etc.

8.1.19 Theorem (?). *Let Q be a complete weak Riemannian manifold, $V : Q \to \mathbb{R}$ be a C^2 function and let Z be the Lagrangian vector field for*

$$L(v) = \frac{1}{2}\|v\|^2 - V(\tau(v)),$$

where $\tau : TQ \to Q$ is the tangent bundle projection. Suppose there is a positively complete function v_0 and a point $q' \in Q$ such that $V(q) \geq V_0(d(q, q'))$ for all $q \in Q$. Then Z is complete.

Proof. Let $c(t)$ be an integral curve of Z and let $q(t) = (\tau \circ c)(t)$ be its projection in Q. Let $q_0 = q_{(0)}$ and consider the differential equation on $\mathbb{R}$

$$f''(t) = -\frac{dV_0}{df}(f(t)) \tag{8.1.9}$$

with initial conditions $f(0) = d(q', q_0)$, $f'(0) = \sqrt{2(\beta - V_0(f(0)))}$, where

$$\beta = E(c(t)) = E(c(0)) \geq V(q_0).$$

We can assume $\beta > V(q_0)$, for if $\beta = V(q_0)$, then $\dot{q}(0) = 0$; now if $\dot{q}(t) = 0$, the conclusion is trivially satisfied, so we need to work under the assumption that there exists a t_0 for which $\dot{q}(t_0) \neq 0$; by time translation we can assume $t_0 = 0$.

We show that the solution $f(t)$ of equation (8.1.9) is defined for all $t \geq 0$. Multiplying both sides of equation (8.1.9) by $f'(t)$ and integrating yields

$$\frac{1}{2} f'(t)^2 = \beta - V_0(f(t)), \quad \text{that is,} \quad t(s) = \int_{d(q', q_0)}^{s} [2(\beta - V_0(u))]^{-1/2} du.$$

By hypothesis, the integral on the right diverges and hence $t(s) \to +\infty$ as $s \to +\infty$. This shows that $f(t)$ exists for all $t \geq 0$.

For $t \geq 0$, conservation of energy and the estimate on the potential V imply

$$\begin{aligned} d(q(t), q') &\leq d(q(t), q_0) + d(q_0, q') \leq d(q', q_0) + \int_0^t \|\dot{q}(s)\| ds \\ &= d(q', q_0) + \int_0^t [2(\beta - V_0(q(s))]^{1/2} ds \\ &\leq d(q', q_0) + \int_0^t [2(\beta - V_0(d(q(s)), q'))]^{1/2} ds. \end{aligned}$$

Since

$$f(t) = d(q', q_0) + \int_0^1 [2(\beta - V_0(f(s))]^{1/2} ds,$$

it follows that $d(q(t), q') \leq f(t)$; see Exercise 4.1-9(v) or the reasoning in Example 4.1.23B plus an approximation of $d(q(t), q')$ by C^1 functions. Hence if Q is finite dimensional, $q(t)$ remains in a compact set for finite t-intervals, $t \geq 0$. Therefore $c(t)$ does as well, $V(q(t))$ being bounded below on such a finite t-interval. Proposition 4.1.19 implies that $c(t)$ exists for all $t \geq 0$. The proof in infinite dimensions is done in Supplement 8.1C. If F_t is the local flow of Z, from $\tau(F_{-t}(v)) = \tau(F_t(-v))$ (reversibility), it follows that $c(t)$ exists also for all $t \leq 0$ and so the theorem is proved. ■

Supplement 8.1C

Completeness of Lagrangian Vector Fields on Hilbert Manifolds

This supplement provides the proof of Theorem **??** for infinite dimensional Riemannian manifolds. We start with a few facts of general interest. Thm. 8.1.19?

Let (Q, g) be a Riemannian manifold and $\tau : TQ \to Q$ the tangent bundle projection. For $v \in T_qQ$, the subspace

$$V_v = \ker T_v\tau = T_v(T_qQ) \subset T_v(TQ)$$

is called the ***vertical subspace*** of $T_v(TQ)$. The local expression of the covariant derivative ∇ defined by g in Supplement 8.1B shows that $\nabla_Y X$ depends only on the point values of Y and thus it defines a linear map

$$(\nabla X)(q) : v \in T_qQ \mapsto (\nabla_Y X)(q) \in T_qQ,$$

where $Y \in \mathfrak{X}(Q)$ is any vector field satisfying $Y(q) = v$. Let $j_v : T_qQ \to T_v(T_qQ) = V_v$ denote the isomorphism identifying the tangent space to a linear space with the linear space itself and consider the map $j_v \circ (\nabla X)(q) : T_qQ \to V_v$. Define the horizontal map $h_v : T_qQ \to T_v(TQ)$ by

$$h_v = T_qX - j_v \circ (\nabla X)(q)$$

where $v \in T_qQ$ and $X \in \mathfrak{X}(Q)$ satisfies $X(q) = v$. Locally, if $\mathbf{E}$ is the model of Q, h_v has the expression

$$h_v : (x, u) \in U \times \mathbf{E} \mapsto (x, v, u - \gamma_x(u, v)) \in U \times \mathbf{E} \times \mathbf{E} \times \mathbf{E}.$$

This shows that h_v is a linear continuous injective map with split image. The image of h_v is called the ***horizontal subspace*** of $T_v(TQ)$ and is denoted by H_v. It is straightforward to check that $T_v(TQ) = V_v \oplus H_v$ and that

$$T_v\tau | H_v : H_v \to T_qQ, \quad j_v : T_qQ \to V_v$$

are Banach space isomorphisms. Declaring them to be isometries and H_v perpendicular to V_v gives a metric g^T on TQ. We have proved that if (Q, q) *is a* (*weak*) *Riemannian manifold, then* g *induces a metric* g^T *on* TQ[1]. The following result is taken from Ebin [1970].

8.1.20 Proposition. *If* (Q, g) *is a complete* (*weak*) *Riemannian manifold then so is* (TQ, g^T).

Proof. Let $\{v_n\}$ be a Cauchy sequence in TQ and let $q_n = \tau(v_n)$. Since τ is distance decreasing it follows that $\{q_n\}$ is a Cauchy sequence in Q and therefore convergent to $q \in Q$ by completeness of Q. If $\mathbf{E}$ is the model of Q, $\mathbf{E}$ is a Hilbert space, again by completeness of Q. Let (U, φ) be a chart at q and assume that U is a closed ball in the metric defined by g of radius 3ϵ. Also, assume that $T_vT\varphi : T_v(TM) \in \mathbf{E}\times\mathbf{E}$ is an isometry for all $v \to T_qQ$ which implies that for ϵ small enough there is a $C > 0$ such that $\|TT\varphi(w)\| \leq C\|w\|$ for all $w \in TTU$. This means that all curves in TU are stretched by $T\varphi$ by a factor of C. Let $V \subset U$ be the closed ball of radius ϵ centered at q and let n,m be large enough so the distance between v_n and v_m is smaller than ϵ and $v_n, v_m \in TV$. If γ is a path from v_n to v_m of length $< 2\epsilon$, then $\tau \circ \gamma$ is a path from q_n to q_m of length $< 2\epsilon$ and therefore $\tau \circ \gamma \subset U$, which in turn implies that $\gamma \subset TU$. Moreover, $T\varphi \circ \gamma$ has length $< 2C\epsilon$ and therefore the distance between $T\varphi(v_n)$ and $T\varphi(v_m)$ in $\mathbf{E}$ is at most $2C\epsilon$. This shows that $\{T\varphi(v_n)\}$ is a Cauchy sequence in $\mathbf{E}$ and hence convergent. Since $T\varphi$ is a diffeomorphism, $\{v_n\}$ is convergent. ■

In general, completeness of a vector field on M implies completeness of the first variation equation on TM.

Proof of Theorem 8.1.19 in infinite dimensions. Let $c :]a, b[\to TQ$ be a maximal integral curve of Z. We shall prove that $\lim_{t\uparrow b} c(t)$ exists in TQ which implies, by local existence and uniqueness, that c can be continued beyond b, that is, that $b = +\infty$. One argues similarly for a. We have shown that $q(t) = (\tau \circ c)(t)$ is bounded on finite t-intervals. Since $V(q(t))$ is bounded on such a finite t-interval, it follows that $\dot{q}(t) = c(t)$ is bounded in the metric defined by g^T on TQ. By the mean value inequality it follows that if $t_n \uparrow b$, then $\{q(t_n)\}$ is a Cauchy sequence and therefore convergent since Q is complete.

Next we show by the same argument that if $t_n \uparrow b$, then $\{c(t_n)\}$ is Cauchy, that is, we will show that $\dot{c}(t)$ is bounded on bounded t-intervals. Write $\dot{c}(t) = Z(c(t)) = S(c(t)) + V(c(t))$, where S is the spray of g and represents the horizontal part of Z and V is the vertical part of Z. Since $V(c(t))$ depends only on $q(t)$ and since $q(t)$ extends continuously to $q(b)$, it follows that $\|V(c(t))\|$ is bounded as $t \uparrow b$. Since

$$\|S(c(t))\| = \|c(t)\| \quad \text{and} \quad \|\dot{c}(t)\|^2 = \|S(c(t))\|^2 + \|V(c(t))\|^2$$

by the definition of the metric g^T, it follows that $\|\dot{c}(t)\|$ remains bounded on finite t-intervals. Therefore $\{c(t_n)\}$ is Cauchy and Proposition 8.1.20 implies that $c(t)$ can be continuously extended to $c(b)$. ■

[1]Sometimes this metric is called the Sasaki metric.

Remark. Note that completeness of Q, an estimate on the potential V, and conservation of the energy E, replaces "compactness" in Proposition 4.1.19 with "boundedness." ♦

Exercises

◇ **8.1-1.** Let (M, ω) be a symplectic manifold with $\omega = \mathbf{d}\theta$ and $f : M \to M$ a local diffeomorphism. Prove that f is a symplectic iff for every compact oriented two-manifold B with boundary, $B \subset M$, we have

$$\int_{\partial B} \theta = \int_{f(\partial B)} \theta.$$

◇ **8.1-2** (J. Moser)**.** Use the method of proof of Darboux theorem to prove that if M is a compact manifold, μ and ν are two volume forms with the same orientation, and

$$\int \mu = \int \nu,$$

then there is a diffeomorphism $f : M \to M$ such that $f^*\nu = \mu$.
Hint: Use the Lie transform method. Since

$$\int \mu = \int \nu, \quad \mu - \nu = \mathbf{d}\alpha$$

(see Supplement 7.5B); put $\nu_t = t\nu + (1-t)\mu$ and define X_t by letting the interior product of X_t with ν_t be α. Let φ_t be the flow of X_t and set $f = \varphi_1$.

◇ **8.1-3.** On $T^*\mathbb{R}^3$, consider the periodic three-dimensional ***Toda lattice*** Hamiltonian,

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\|\mathbf{p}\|^2 + e^{q_1 - q_2} + e^{q_2 - q_3} + e^{q_3 - q_1}.$$

(i) Write down Hamilton's equations.

(ii) Show that

$$\begin{aligned} f_1(\mathbf{q}, \mathbf{p}) &= p_1 + p_2 + p_3, \ f_2 = H, \text{ and} \\ f_3(\mathbf{q}, \mathbf{p}) &= \frac{1}{3}(p_1^3 + p_2^3 + p_3^3) + p_1(\exp(q^1 - q^2) + \exp(q^3 - q_1)) \\ &\quad + p_2(\exp(q^1 - q^2) + \exp(q^2 - q^3)) \\ &\quad + p_3(\exp(q^1 - q^2) + \exp(q^2 - q^3)) \end{aligned}$$

are in involution and are independent everywhere.

(iii) Prove the same thing for

$$g_1 = f_1, \quad g_2(\mathbf{q}, \mathbf{p}) = \exp(q^1 - q^2) + \exp(q^2 - q^3) + \exp(q^3 - q^1),$$

and

$$\begin{aligned} g_3(\mathbf{q}, \mathbf{p}) &= p_1p_2p_3 - p_1 \exp(q^2 - q^3) - p_2 \exp(q^3 - q^1) \\ &\quad - p_3 \exp(q^1 - q^2). \end{aligned}$$

(iv) Can you establish (iii) without explicitly computing the Poisson brackets?

HINT: Express g_1, g_2, g_3 as polynomials of f_1, f_2, f_3.

◇ **8.1-4.** ***A second-order equation*** on a manifold M is a vector field X on TM such that $T_{\tau_M} \circ X = \mathrm{Id}_{TM}$. Show that

(i) X is a second-order equation iff for all integral curves c of X in TM we have $(\tau_M \circ c)' = c$. One calls $\tau_M \circ c$ a ***base integral curve***.

(ii) X is a second-order equation iff in every chart the local representative of X has the form $(u, e) \mapsto (u, e, e, V(u, e))$.

(iii) If M is finite dimensional and X is a second-order equation, then the base integral curves satisfy

$$\frac{d^2 x(t)}{dt^2} = V(x(t), \dot{x}(t)),$$

where $(x, \dot{x})$ denotes standard coordinates on TM.

◇ **8.1-5** (Noether theorem)**.** Prove the following result for Lagrangian systems.

Theorem. *Let Z be a Lagrangian vector field for $L : TM \to \mathbb{R}$ and suppose Z is a second-order equation. Let Φ_t be a one-parameter group of diffeomorphisms of M generated by the vector field $Y : M \to TM$. Suppose that for each real number t, $L \circ T\Phi_t = L$. Then the function $P(Y) : TM \to \mathbb{R}$, defined by $P(Y)(v) = \mathbb{F}L(v) \cdot Y$ is constant along integral curves of Z .*

◇ **8.1-6.** Use Exercise 8.1-5 to show conservation of linear (resp., angular) momentum for the motion of a particle in $\mathbb{R}^3$ moving in a potential that has a translation (resp., rotational) symmetry.

◇ **8.1-7.** Consider $\mathbb{R}^{2n+2}$ with coordinates $(q^1, \dots, q^n, E, p_1, \dots, p_n, t)$ and define the symplectic form

$$\omega = dq^i \wedge dp_i + dE \wedge dt.$$

Consider the function $P(q, p, E, t) = H(q, p, t) - E$. Show that the vector field

$$X = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} + \dot{E} \frac{\partial}{\partial t} + \dot{t} \frac{\partial}{\partial E}$$

defined by $\mathbf{i}_X \omega = \mathbf{d}P$ reproduces familiar equations for $\dot{q}, \dot{p}, \dot{t}$ and $\dot{E}$.

◇ **8.1-8.** Show that the wave equation (see Supplement 8.1A) may be derived as a Lagrangian system.

◇ **8.1-9.** Refer to Example 8.1.13 on the Schrödinger equation. Let A and B be self adjoint operators on $\mathcal{H}$ and let $f_A : \mathcal{H} \to \mathbb{R}$ be given by $f_A(\psi) = \langle \psi, A\psi \rangle$ (the expectation value of A in the state ψ). Show that Poisson brackets and commutators are related by

$$f_{i[A,B]} = \{f_A, f_B\}.$$

◇ **8.1-10.** Show that the geodesic flow of a compact Riemannian manifold is complete. (Warning: Compact pseudo-Riemannian manifolds need not be complete; see **?** and Marsden [1973].)

◇ **8.1-11.** Show that any isometry of a weak pseudo-Riemannian manifold maps geodesics to geodesics. (A map $\varphi : Q \to Q$ is called an *isometry* if $\varphi^* g = g$, where g is the weak pseudo-Riemannian metric on Q.)

◇ **8.1-12.** Let (Q, g) be a weak Riemannian manifold.

(i) If F_t is the flow of the spray of g show that $\tau(F_t(sv)) = \tau(F_{st}(v))$, where $\tau : TQ \to Q$ is the projection.

(ii) Let U be any bounded set in T_qQ. Show that there is an $\epsilon > 0$ such that for any $v \in U$, the integral curve of the spray with initial condition v exists for a time $\geq \epsilon$.

Hint: Let $V \subset U$ be an open neighborhood of v such that all integral curves starting in V exist for time $\geq \delta$. Find $R > 0$ such that $R^{-1}U \subset V$ and use (i).

◇ **8.1-13.** A weak pseudo-Riemannian manifold (Q, g) is called *homogeneous* if for any $x, y \in Q$ there is an isometry φ such that $\varphi(x) = y$. Show that homogeneous weak Riemannian manifolds are complete by using Exercises 8.1-11 and 8.1-12.
Hint: Put the initial condition v in a ball B and choose ϵ as in Exercise 8.1-12(ii). Let $v(t)$ be the integral curve of S through v and let $q(t)$ be the corresponding geodesic. The geodesic starting at $q(\epsilon)$ in the direction $v(\epsilon)$, is φ applied to the geodesic through $q = \tau(v)$ in the direction $T\varphi^{-1}(v)(\epsilon)$; φ is the isometry sending q to $q(\epsilon)$. The latter geodesic lies in the ball B, so it exists for time $\geq \epsilon$.

8.2 Fluid Mechanics

We present a few of the basic ideas concerning the motion of an ideal fluid from the point of view of manifolds and differential forms. This is usually done in the context of Euclidean space using vector calculus. For the latter approach and additional details, the reader should consult one of the standard texts on the subject such as Batchelor [1967], Chorin and Marsden [1993], or Gurtin [1981]. The use of manifolds and differential forms can give additional geometric insight.

The present section is for expository reasons somewhat superficial and is intended only to indicate how to use differential forms and Lie derivatives in fluid mechanics. Once the basics are understood, more sophisticated questions can be asked, such as: in what sense is fluid mechanics an infinite-dimensional Hamiltonian system? For the answer, see Arnol'd [1982], Abraham and Marsden [1978], Marsden and Weinstein [1983], and **?**. For analogous topics in elasticity, see Marsden and Hughes [1983], and for plasmas, see §8.4 and Marsden and Weinstein [1982].

Let M be a compact, oriented finite-dimensional Riemannian n-manifold, possibly with boundary. Let the Riemannian volume form be denoted $\mu \in \Omega^n(M)$, and the corresponding volume element $d\mu$. Usually M is a bounded region with smooth boundary in two- or three-dimensional Euclidean space, oriented by the standard basis, and with the standard Euclidean volume form and inner product.

Imagine M to be filled with fluid and the fluid to be in motion. Our object is to describe this motion. Let $x \in M$ be a point in M and consider the particle of fluid moving through x at time $t = 0$. For example, we can imagine a particle of dust suspended in the fluid; this particle traverses a trajectory which we denote $\varphi_t(x) = \varphi(x, t)$. Let $u(x, t)$ denote the velocity of the particle of fluid moving through x at time t. Thus, for each fixed time, u is a vector field on M. See Figure 8.2.1. We call u the ***velocity field of the fluid***. Thus the relationship between u and φ_t is

$$\frac{d\varphi_t(x)}{dt} = u(\varphi_t(x), t);$$

that is, u is a time-dependent vector field with evolution operator φ_t in the same sense as was used in §4.1.

For each time t, we shall assume that the fluid has a well-defined mass density and we write $\rho_t(x) = \rho(x, t)$. Thus if W is any subregion of M, we assume that the mass of fluid in W at time t is given by

$$m(W, t) = \int_W \rho_t \, d\mu.$$

Our derivation of the equations is based on three basic principles, which we shall treat in turn:

1. Mass is neither created nor destroyed.

2. (**Newton's second law**) The rate of change of momentum of a portion of the fluid equals the force applied to it.

3. Energy is neither created nor destroyed.

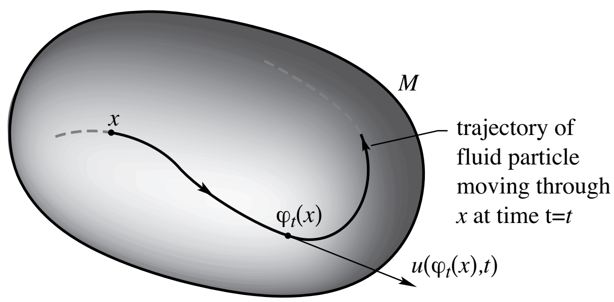


FIGURE 8.2.1. The trajectory of a fluid particle

1. Conservation of mass. This principle says that the total mass of the fluid, which at time $t = 0$ occupied a nice region W, remains unchanged after time t; that is,

$$\int_{\varphi_t(W)} \rho_t \, d\mu = \int_W \rho_0 \, d\mu.$$

(We call a region W "nice" when it is an open subset of M with smooth enough boundary to allow us to use Stokes' theorem.) Let us recall how to use the transport theorem 7.1.12 to derive the continuity equation. Using the change-of-variables formula, conservation of mass may be rewritten as

$$\int_W \varphi_t^*(\rho_t \mu) = \int_W \rho_0 \mu$$

for any nice region W in M, which is equivalent to

$$\varphi_t^*(\rho_t \mu) = \rho_0 \mu, \quad \text{or} \quad (\varphi_t^* \rho_t) J(\varphi_t) = \rho_0,$$

where $J(\varphi_t)$ is the Jacobian of φ_t. This in turn is equivalent to

$$\begin{aligned} 0 = \frac{d}{dt} \varphi_t^*(\rho_t, \mu) &= \varphi_t^* \left(\pounds_u(\rho_t, \mu) + \frac{\partial \rho}{\partial t} \mu \right) \\ &= \varphi_t^* \left\{ \left(u[\rho_t] + \rho_t \operatorname{div} u + \frac{\partial \rho}{\partial t} \right) \mu \right\} \\ &= \varphi_t^* \left\{ \left(\operatorname{div}(\rho_t u) + \frac{\partial \rho}{\partial t} \right) \mu \right\} \end{aligned}$$

by the Lie derivative formula and Proposition 6.5.17. Thus

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho_t u) = 0$$

is the differential form of the law of conservation of mass, also known as the ***continuity equation***.

Because of shock waves that could be present, ρ and u may not be smooth enough to justify the steps leading to the differential form of this law; the integral form will then be the one to use. Also note that the Riemannian metric has as yet played no role; only the volume element of M was needed.

2. Balance of momentum. Newton's second law asserts that the rate of change of momentum of a portion of the fluid equals the total force applied to it. To see how to apply this principle on a general manifold, let us discuss the situation $M \subset \mathbb{R}^3$ first. Here we follow the standard vector calculus conventions and write the velocity fields in boldface type. The momentum of a portion of the fluid at time t that at time $t = 0$ occupied the region W is

$$\int_{\varphi_t(W)} \rho \mathbf{u}\, d\mu.$$

Here and in what follows the integral is $\mathbb{R}^3$-valued, so we apply all theorems on integration componentwise.

For any continuum, forces acting on a piece of material are of two types. First there are ***forces of stress***, whereby the piece of material is acted on by forces across its surface by the rest of the continuum. Second, there are external, or ***body forces***, such as gravity or a magnetic field, which exert a force per unit volume on the continuum. The clear formulation of surface stress forces in a continuum is usually attributed to Cauchy. We shall assume that the body forces are given by a given force density $\mathbf{b}$, that is, the total body forces acting on W are $\int_W \rho \mathbf{b}\, d\mu$. In continuum mechanics the forces of stress are assumed to be of the form $\int_{\partial W} \sigma(x,t) \cdot \mathbf{n}\, da$, where da is the induced volume element on the boundary, $\mathbf{n}$ is the outward unit normal, and $\sigma(x,t)$ is a time-dependent contravariant symmetric two-tensor, called the ***Cauchy stress tensor***. The contraction $\sigma(t,x) \cdot \mathbf{n}$ is understood in the following way: if σ has components σ^{ij} and $\mathbf{n}$ has components n^k, then $\sigma \cdot \mathbf{n}$ is a vector with components $(\sigma \cdot \mathbf{n})^i = g_{jk}\sigma^{ij} n^k$, where g is the metric (in our case $g_{jk} = \delta_{jk}$). The vector $\sigma \cdot \mathbf{n}$, called the ***Cauchy traction vector***, measures the force of contact (per unit area orthogonal to $\mathbf{n}$) between two parts of the continuum. (A theorem of Cauchy states that if one postulates the existence of a continuous Cauchy traction vector field $\mathbf{T}(\mathbf{x}, t, \mathbf{n})$ satisfying balance of momentum, then it must be of the form $\sigma \cdot \mathbf{n}$, for a two-tensor, σ; moreover if balance of moment of momentum holds, σ must be symmetric. See Chorin and Marsden [1993], Gurtin [1981], or Marsden and Hughes [1983] for details.) ***Balance of momentum*** is said to hold when

$$\frac{d}{dt} \int_{\varphi_t(W)} \rho \mathbf{u}\, d\mu = \int_{\varphi_t(W)} \rho \mathbf{b}\, d\mu + \int_{\partial \varphi_t(W)} \sigma \cdot \mathbf{n}\, da$$

for any nice region W in $M = \mathbb{R}^3$. If $\operatorname{div} \sigma$ denotes the vector with components $(\operatorname{div}(\sigma^{1i}), \operatorname{div}(\sigma^{2i}), \operatorname{div}(\sigma^{3i}))$, then by Gauss' theorem

$$\int_{\partial \varphi_t(W)} \sigma \cdot \mathbf{n}\, da = \int_{\varphi_t(W)} (\operatorname{div} \sigma) d\mu.$$

By the change-of-variables formula and Lie derivative formula, we get

$$\begin{aligned} \frac{d}{dt} \int_{\varphi_t(W)} \rho u^i d\mu &= \int_W \frac{d}{dt} \varphi_t^* (\rho u^i d\mu) \\ &= \int_{\varphi_t(W)} \left(\frac{\partial (\rho u^i)}{\partial t} + (\pounds_u \rho) u^i + \rho \pounds_u u^i + \rho u^i \operatorname{div} u \right) d\mu, \end{aligned}$$

so that the balance of momentum is equivalent to

$$\frac{\partial \rho}{\partial t} u^i + \rho \frac{\partial u^i}{\partial t} + (\mathbf{d}\rho \cdot \mathbf{u}) u^i + \rho \pounds_u u^i + \rho u^i \operatorname{div} \mathbf{u} = \rho b^i + (\operatorname{div} \sigma)^i.$$

But $\mathbf{d}\rho \cdot \mathbf{u} + \rho \operatorname{div} \mathbf{u} = \operatorname{div}(\rho \mathbf{u})$ and by conservation of mass, $\partial \rho / \partial t + \operatorname{div}(\rho \mathbf{u}) = 0$. Also, $\pounds_{\mathbf{u}} u^i = (\partial u^i / \partial x^j) u^j = (\mathbf{u} \cdot \nabla) u^i$, so we get

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{b} + \frac{1}{\rho} \operatorname{div} \sigma,$$

which represents the basic ***equations of motion***. Here the quantity $\partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\mathbf{u}$ is usually called the ***material derivative*** and is denoted by $D\mathbf{u}/dt$. These equations are for any continuum, be it elastic or fluid.

An ***ideal fluid*** is by definition a fluid whose Cauchy stress tensor σ is given in terms of a function $p(x,t)$ called the ***pressure***, by $\sigma^{ij} = -pg^{ij}$. In this case, balance of momentum in differential form becomes the ***Euler equations for an ideal fluid***:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{b} - \frac{1}{\rho} \operatorname{grad} p.$$

The assumption on the stress σ in an ideal fluid means that if S is any fluid surface in M with outward unit normal $\mathbf{n}$, then the force of stress per unit area exerted across a surface element S at $\mathbf{x}$ with normal $\mathbf{n}$ at time t is $-p(\mathbf{x}, t)\mathbf{n}$ (see Figure 8.2.2).

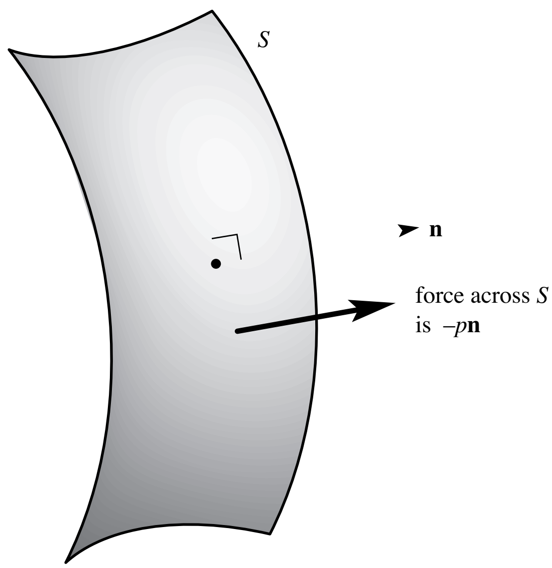


FIGURE 8.2.2. The stress in an ideal fluid is given by the pressure

Let us return to the context of a Riemannian manifold M. First, it is not clear what the vector-valued integrals should mean. But even if we could make sense out of this, using, say parallel transport, there is a more serious problem with the integral form of balance of momentum as stated. Namely, if one changes coordinates, then balance of momentum *does not look the same*. One says that the integral form of balance of momentum is *not covariant*. Therefore we shall concentrate on the differential form and from now on we shall deal only with ideal fluids. (For a detailed discussion of how to formulate the basic integral balance laws of continuum mechanics covariantly, see Marsden and Hughes [1983]. A genuine difficulty with shock wave theory is that the notion of weak solution is not a coordinate independent concept.)

Rewrite Euler's equations in $\mathbb{R}^3$ with indices down; that is, take the flat of these equations. Then the i-th equation, $i = 1, 2, 3$ is

$$\frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x^1} + u_2 \frac{\partial u_i}{\partial x^2} + u_3 \frac{\partial u_i}{\partial x^3} = b_i - \frac{1}{\rho} \frac{\partial p}{\partial x^i}.$$

We seek an invariant meaning for the sum of the last three terms on the left-hand side. For fixed i this expression is

$$u_j \frac{\partial u_i}{\partial x^j} = u_j \frac{\partial u_i}{\partial x^j} + u_j \frac{\partial u_j}{\partial x^i} - u_j \frac{\partial u_j}{\partial x^i} = (\pounds_u \mathbf{u}^b)_i - \left(\frac{1}{2}\mathbf{d}\|u\|^2\right)_i.$$

That is, Euler's equations can be written in the invariant form

$$\frac{\partial \mathbf{u}^\flat}{\partial t} + \pounds_u \mathbf{u}^\flat - \frac{1}{2}\mathbf{d}(\mathbf{u}^\flat(\mathbf{u})) = -\frac{1}{\rho}\mathbf{d}p + \mathbf{b}^\flat.$$

We postulate this equation as the balance of momentum in M for an ideal fluid. The reader familiar with Riemannian connections (see Supplement 8.1B) can prove that this form is equivalent to the form

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{u}}\mathbf{u} = -\frac{1}{\rho}\operatorname{grad} p + \mathbf{b}$$

by showing that

$$\pounds_{\mathbf{u}}\mathbf{u}^\flat = (\nabla_{\mathbf{u}}\mathbf{u})^\flat + \frac{1}{2}\mathbf{d}(\mathbf{u}^\flat(\mathbf{u})).$$

where $\nabla_{\mathbf{u}}\mathbf{u}$ is the covariant derivative of $\mathbf{u}$ along itself, with ∇ the Riemannian connection given by g.

The boundary conditions that should be imposed come from the physical significance of ideal fluid: namely, no friction should exist between the fluid and ∂M; that is, $\mathbf{u}$ is tangent to ∂M at points of ∂M. Summarizing, the *equations of motion* of an ideal fluid on a compact Riemannian manifold M with smooth boundary ∂M and outward unit normal n are

$$\frac{\partial u^\flat}{\partial t} + \pounds_{\mathbf{u}}\mathbf{u}^\flat - \frac{1}{2}\mathbf{d}(u^\flat(u)) = -\frac{1}{\rho}\mathbf{d}p + b^\flat \quad \text{and} \quad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0.$$

We also have the ***boundary conditions***

$$u \| \partial M, \quad \text{that is,} \quad u \cdot n = \text{on}\, \partial M;$$

and ***initial conditions***

$$u(x,0) = u_0(x) \quad \text{given on } M.$$

We shall assume $b = 0$ from now on for simplicity.

3. Conservation of energy. A basic problem of ideal fluid dynamics is to solve the initial-boundary-value problem. The unknowns are u, ρ, and p, that is, $n+2$ scalar unknowns. We have, however, only $n+1$ equations. Thus one might suspect that to specify the fluid motion, one more equation is needed. This is in fact true and the law of conservation of energy will supply the necessary extra equation in fluid mechanics. (The situation is similar for general continua; see Marsden and Hughes [1983].)

For a fluid moving in M with velocity field u, the ***kinetic energy*** of the fluid is

$$E_{\text{kinetic}} = \frac{1}{2}\int_M \rho \|u\|^2 d\mu$$

where $\|u\|^2 = \langle u, u \rangle$ is the square length of the vector function u. We assume that the total energy of the fluid can be written

$$E_{\text{total}} = E_{\text{kinetic}} + E_{\text{internal}},$$

where E_{internal} is the energy that relates to energy we cannot "see" on a macroscopic scale and derives from sources such as intermolecular potentials and molecular vibrations. If energy is pumped into the fluid or if we allow the fluid to do work, E_{total} will change. We describe two particular examples of energy equations that are useful.

A. Assume that $E_{\text{internal}} = \text{constant}$. Then we ought to have E_{kinetic} as a constant of the motion; that is,

$$\frac{d}{dt}(\frac{1}{2}\int_M \rho\|u\|^2 d\mu) = 0.$$

To deal with this equation it is convenient to use the following.

8.2.1 Theorem (Transport Theorem with Mass Density). *Let f be a time-dependent smooth function on M. Then if W is any (nice) open set in M,* Can't break.

$$\frac{d}{dt}\int_{\varphi_t(W)} \rho f\, d\mu = \int_{\varphi_t(W)} \rho \frac{Df}{dt} d\mu,$$

where $Df/dt = \partial f/\partial t + \pounds_u f$.

Proof. By the change of variables formula, the Lie derivative formula, $\operatorname{div}(\rho u) = u[\rho] + \rho \operatorname{div}(u)$, and conservation of mass, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\varphi_t(W)} \rho f\, d\mu &= \frac{d}{dt}\int_W \varphi_t^*(\rho f \mu) = \int_W \varphi_t^*\left(\frac{\partial(\rho f)}{\partial t}\mu + \pounds_u(\rho f \mu)\right) \\
&= \int_{\varphi_t(W)} \left(\frac{\partial \rho}{\partial t} f\mu + \rho\frac{\partial f}{\partial t}\mu + u[\rho] f\mu + \rho(\pounds_u f)\mu + \rho \pounds_u \mu\right) \\
&= \int_{\varphi_t(W)} \left[\left(\frac{\partial \rho}{\partial t} + u[\rho] + \rho \operatorname{div} u\right) f\mu + \rho\left(\frac{\partial f}{\partial t} + \pounds_u f\right)\mu\right] \\
&= \int_{\varphi_t(W)} \left[f\left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u)\right) + \rho\left(\frac{\partial f}{\partial t} + \pounds_u f\right)\right]\mu \\
&= \int_{\varphi_t(W)} \rho\left(\frac{\partial f}{\partial t} + \pounds_u f\right)\mu. \qquad \blacksquare
\end{aligned}$$

Making use of

$$\pounds_u\left(\|u\|^2\right) = \pounds_u\left(u^\flat(u)\right) = \left(\pounds_u u^\flat\right)(u) = \mathbf{d}\left(u^\flat(u)\right)(u),$$

the transport lemma, and Euler's equations, we get

$$\begin{aligned}
0 &= \frac{d}{dt}\left(\frac{1}{2}\int_M \rho\|u\|^2 d\mu\right) = \frac{1}{2}\int_M \rho\left(\frac{\partial\|u\|^2}{\partial t} + \pounds_u\|u\|^2\right) d\mu \\
&= \int_M \rho\frac{\partial u^\flat}{\partial t}\cdot u\, d\mu + \frac{1}{2}\int_M (\pounds_u u^\flat)\cdot u\, d\mu \\
&= \int_M \rho\frac{\partial u^\flat}{\partial t}\cdot u\, d\mu + \int_M \rho(\pounds_u u^\flat)\cdot u\, d\mu - \frac{1}{2}\int_M \rho\mathbf{d}(u^\flat(u))\cdot u\, d\mu \\
&= -\int_M \mathbf{d}p\cdot u\, d\mu = \int_M \{(\operatorname{div} u)p\mu - \pounds_u(p\mu)\} \text{ (by the Leibniz rule for } \pounds_u) \\
&= \int_M \{(\operatorname{div} u)p\mu - \mathbf{d}(\mathbf{i}_u p\mu)\} = \int_M (\operatorname{div} u)p\mu.
\end{aligned}$$

The last equality is obtained by Stokes' theorem and the boundary conditions $0 = (u\cdot n)da = \mathbf{i}_u\mu$. If we imagine this to hold for the same fluid in all conceivable motions, we are forced to postulate one of the additional equations

$$\operatorname{div} u = 0 \quad \text{or} \quad p = 0.$$

The case $\operatorname{div} u = 0$ is that of an ***incompressible fluid***. Thus in this case the Euler equations are

$$\begin{aligned} \frac{\partial u^\flat}{\partial t} + \pounds_u u^\flat - \frac{1}{2}\mathbf{d}\|u\|^2 &= -\frac{1}{\rho}\mathbf{d}p \\ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) &= 0 \\ \operatorname{div} u &= 0 \end{aligned}$$

with the boundary condition $\mathbf{i}_u\mu = 0$ on ∂M and initial condition $u(x,0) = u_0(x)$. The case $p = 0$ is also possible but is less interesting.

For a homogeneous incompressible fluid, with *constant* density ρ, Euler's equations can be reformulated in terms of the Hodge decomposition theorem (see §7.5). Nonhomogeneous incompressible flow requires a weighted Hodge decomposition (see Marsden [1976]). Recall that any one–form α can be written in a unique way as $\alpha = \mathbf{d}\beta + \gamma$, where $\delta\gamma = 0$. Define the linear operator

$$\mathbb{P} : \Omega^1(M) \to \left\{ \gamma \in \Omega^1(M) \mid \delta\gamma = 0 \right\} \quad \text{by} \quad \mathbb{P}(\alpha) = \gamma.$$

We are now in a position to reformulate Euler's equations. Let $\Omega^1_{\delta=0}$ be the set of C^∞ one-forms γ with $\delta\gamma = 0$ and γ tangent to ∂M; that is, $*\gamma|_{\partial M} = 0$. Let $T : \Omega^1_{\delta=0} \to \Omega^1_{\delta=0}$ be defined by

$$T(u^\flat) = \mathbb{P}(\pounds_u u^\flat).$$

Thus Euler's equations can be written as $\partial u^\flat/\partial t + T(u^\flat) = 0$, which is in the "standard form" for an evolution equation. Note that T is nonlinear. Another important feature of T is that it is *nonlocal*; this is because $\mathbb{P}(\alpha)(x)$ depends on the values of a on all of M and not merely those in the neighborhood of $x \in M$. ♦

B. We postulate an internal energy over the region W to be of the form

$$E_{\text{internal}} = \int_W \rho w \, d\mu,$$

where the function w is the internal energy density per unit mass.

We assume that ***energy is balanced*** in the sense that the rate of change of energy in a region equals the work done on it:

$$\frac{d}{dt}\left(\int_{\varphi_t(W)} \frac{1}{2}\|u\|^2 d\mu + \int_{\varphi_t(W)} \rho w \, d\mu \right) = -\int_{\partial\varphi_t(W)} p u \cdot n \, da.$$

By the transport theorem and arguing as in our previous results, this reduces to

$$0 = \int_{\varphi_t(W)} \left(p \operatorname{div} u + \rho \frac{Dw}{dt} \right) d\mu.$$

Since W is arbitrary,

$$p \operatorname{div} u + \rho \frac{Dw}{dt} = 0.$$

Now assume that w depends on the fluid motion through the density; that is, the internal energy depends only on how much the fluid is compressed. Such a fluid is called ***ideal isentropic*** or ***barotropic***. The preceding identity then becomes

$$\begin{aligned} 0 = p \operatorname{div} u + \rho \left(\frac{\partial w}{\partial t} + \mathbf{d}w \cdot u \right) &= p \operatorname{div} u + \rho \frac{\partial w}{\partial \rho} \frac{\partial \rho}{\partial t} + \rho \frac{\partial w}{\partial \rho} \mathbf{d}\rho \cdot u \\ &= p \operatorname{div} u + \rho \frac{\partial w}{\partial \rho} (-\rho \operatorname{div} u) \end{aligned}$$

using the equation of continuity. Since this is an identity and we are not restricting div u, we get

$$p = \rho^2 w'(\rho).$$

If p is a given function of ρ note that $w = -\int p\, d(1/\rho)$. In addition, $\mathbf{d}p/\rho = \mathbf{d}(w + p/\rho)$. This follows from $p = \rho^2 w'$ by a straightforward calculation in which p and w are regarded as functions of ρ. The quantity $w + p/\rho = w + \rho w'$ is called the ***enthalpy*** and is often denoted h.

Thus Euler's equations for compressible ideal isentropic flow are

$$\frac{\partial u^\flat}{\partial t} + \pounds_u u^\flat - \frac{1}{2} d(u^\flat(u)) + \frac{\mathbf{d}p}{\rho} = 0,$$
$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0,$$
$$u(x, 0) = u_0(x) \text{ on } M \quad \text{and} \quad u \cdot n = 0 \text{ on } \partial M.$$

where $p = \rho^2 w'(\rho)$ is a function of ρ, called an ***equation of state***, which depends on the particular fluid. It is known that these equations lead to a well-posed initial value problem (i.e., there is a local existence and uniqueness theorem) only if $p'(\rho) > 0$. This agrees with the common experience that increasing the surrounding pressure on a volume of fluid causes a decrease in occupied volume and hence an increase in density. Many gases can often be viewed as satisfying our hypotheses, with $p = A\rho^\gamma$ where A and γ are constants and $\gamma \geq 1$. ♦

Cases A and B above are rather opposite. For instance, if $\rho = \rho_0$ is a constant for an incompressible fluid, then clearly p cannot be an invertible function of ρ. However, the case $\rho =$ constant may be regarded as a limiting case $p'(\rho) \to \infty$. In Case B, p is an explicit function of ρ. In Case A, p is implicitly determined by the condition $\operatorname{div} u = 0$. Finally, notice that in neither Case A or B is the possibility of a loss of total energy due to friction taken into account. This leads to the subject of ***viscous fluids***, not dealt with here.

Given a fluid flow with velocity field $\mathbf{u}(x, t)$, a ***streamline*** at a fixed time t is an integral curve of u; that is, if $x(s)$ is a streamline parameterized by s at the instant t, then $x(s)$ satisfies

$$\frac{dx}{ds} = u(x(s), t), \quad t \text{ fixed}.$$

On the other hand, a ***trajectory*** is the curve traced out by a particle as time progresses, as explained at the beginning of this section; that is, is a solution of the differential equation

$$\frac{dx}{dt} = u(x(t), t)$$

with given initial conditions. If u is independent of t (i.e., $\partial u/\partial t = \mathbf{0}$), then, streamlines and trajectories coincide. In this case, the flow is called ***stationary*** or ***steady***. This condition means that the "shape" of the fluid flow is not changing. Even if each particle is moving under the flow, the global configuration of the fluid does not change. The following criteria for steady solutions for homogeneous incompressible flow is a direct consequence of Euler's equations, written in the form $\partial u^\flat/\partial t + \mathbb{P}(\pounds_u u^\flat) = 0$, where $\mathbb{P}$ is the Hodge projection to the co-closed 1-forms.

8.2.2 Proposition. *Let u_t be a solution to the Euler equations for homogeneous incompressible flow on a compact manifold M and φ_t its flow. The following are equivalent*:

(i) *u_t is a steady flow (i.e., $(\partial u/\partial t) = \mathbf{0}$).*

(ii) *φ_t is a one-parameter group: $\varphi_{t+s} = \varphi_t \circ \varphi_s$.*

(iii) *$\pounds_{u_0} u_0^\flat$ is an exact 1-form.*

(iv) $\mathbf{i}_{u_0}\mathbf{d}u_0^\flat$ *is and exact* 1*-form.*

It follows from (iv) that if u_0 is a *harmonic* vector field; that is, u_0 satisfies $\delta u_0^\flat = 0$ and $\mathbf{d}u_0^\flat = \mathbf{0}$, then it yields a stationary flow. Also, it is known that there are other steady flows. For example, on a closed two-disk, with polar coordinates (r, θ), $u = f(r)(\partial/\partial\theta)$ is the velocity field of a steady flow because

$$(u \cdot \nabla)u = -\nabla p, \quad \text{where } p(r, \theta) = \int_0^r f^2(s) s\, ds.$$

Clearly such a u need not be harmonic.

We saw that for compressible ideal isentropic flow, the total energy

$$\int_M \left(\frac{\|u\|^2}{2} + \rho w \right) d\mu$$

is conserved. We can refine this a little for stationary flows as follows.

8.2.3 Theorem (Bernoulli's Theorem)**.** *For stationary compressible ideal isentropic flow, with p a function of ρ,*

$$\frac{1}{2}\|u\|^2 + \int \frac{dp}{\rho} = \frac{1}{2}\|u\|^2 + w + \frac{p}{\rho}$$

is constant along streamlines where the enthalpy $\int \mathbf{d}p/\rho = w + p/\rho$ denotes a potential for the one form $\mathbf{d}p/\rho$. The same holds for stationary homogeneous (ρ = constant in space $= \rho_0$) incompressible flow with $\int \mathbf{d}p/\rho$ replaced by p/ρ_0. If body forces deriving from a potential U are present, that is, $b^\flat = -\mathbf{d}U$, then the conserved quantity is

$$\frac{1}{2}\|u\|^2 + \int \frac{dp}{\rho} = \frac{1}{2}\|u\|^2 + w + \frac{p}{\rho} + U.$$

Proof. Since $\pounds_u(u^\flat) \cdot u = \mathbf{d}(u^\flat(u)) \cdot u$, for stationary ideal compressible or incompressible homogeneous flows we have

$$\begin{aligned} 0 = \frac{\partial u^\flat}{\partial t} \cdot u &= -(\pounds_u u^\flat) \cdot u + \frac{1}{2}\mathbf{d}(u^\flat) \cdot u - \frac{\mathbf{d}p}{\rho} \cdot u \\ &= -\frac{1}{2}(\mathbf{d}\|u\|^2) \cdot u - \frac{1}{\rho}\mathbf{d}p \cdot u, \end{aligned}$$

so that

$$\begin{aligned} \left(\frac{1}{2}\|u\|^2 + \int \frac{\mathbf{d}p}{\rho} \right) \Bigg|_{x(s_1)}^{x(s_2)} &= \int_{s_1}^{s_2} \mathbf{d} \left(\frac{1}{2}\|u\|^2 + \int \frac{\mathbf{d}p}{\rho} \cdot u \right) \cdot x'(s) ds \\ &= \int_{s_1}^{s_2} \frac{\partial u^\flat}{\partial s} \cdot u(x(s)) ds = 0 \end{aligned}$$

since $x'(s) = u(x(s))$. ■

The two-form $\omega = \mathbf{d}u^\flat$ is called ***vorticity***, which, in $\mathbb{R}^3$ can be identified with curl $\mathbf{u}$. Our assumptions so far have precluded any tangential forces and thus any mechanism for starting or stopping rotation. Hence, intuitively, we might expect rotation to be conserved. Since rotation is intimately related to the vorticity, we can expect the vorticity to be involved. We shall now prove that this is so.

Let C be a simple closed contour in the fluid at $t = 0$ and let C_t be the contour carried along the flow. In other words, $C_t = \varphi_t(C)$ where φ_t is the fluid flow map. (See Figure 8.2.3.) The ***circulation*** around C_t is defined to be the integral

$$\Gamma_{C_t} = \int_{C_1} u^\flat,$$

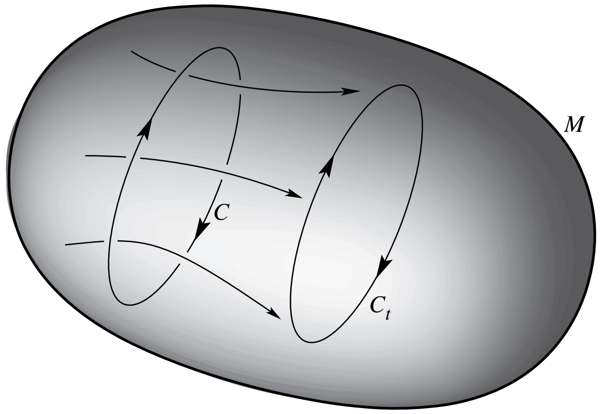


FIGURE 8.2.3. A loop advected by the flow

8.2.4 Theorem (Kelvin Circulation Theorem). *Let M be a manifold and $l \subset M$ a smooth closed loop, that is, a compact one-manifold. Let u_t solve the Euler equations on M for ideal isentropic compressible or homogeneous incompressible flow and $l(t)$ be the image of l at time t when each particle moves under the flow φ_t of u_t; that is, $l(t) = \varphi_t(l)$. Then the circulation is constant in time; that is,*

$$\frac{d}{dt}\int_{l(t)} u_t^\flat = 0$$

Proof. Let φ_t be the flow of u_t. Then $l(t) = \varphi_t(l)$, and so changing variables,

$$\frac{d}{dt}\int_{\varphi_t(l)} u_t^\flat = \int_l \left[\varphi_t^*(\pounds_u u^\flat) + \varphi_t^*\left(\frac{\partial u^\flat}{\partial t}\right)\right].$$

However, $\pounds_u u^\flat + \partial u^\flat/\partial t$ is exact from the equations of motion and the integral of an exact form over a closed loop is zero. ■

We now use Stokes' theorem, which will bring in the vorticity. If Σ is a surface (a two-dimensional submanifold of M) whose boundary is a closed contour C, then Stokes' theorem yields

$$\Gamma_C = \int_C u^\flat = \int_\Sigma \mathbf{d}u^\flat = \int_\Sigma \omega.$$

See Figure 8.2.4.

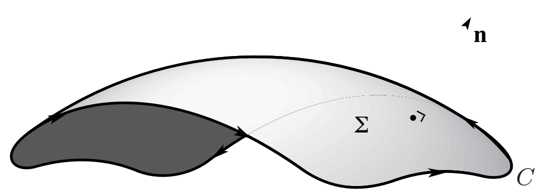


FIGURE 8.2.4. A surface and contour for Helmholtz' theorem

Thus, as a corollary of the circulation theorem, we can conclude:

8.2.5 Theorem (Helmholtz' Theorem). *Under the hypotheses of Theorem* 8.2.4*, the flux of vorticity across a surface moving with the fluid is constant in time.*

We shall now show that ω and $\eta = \omega/\rho$ are *Lie propagated by the flow.*

8.2.6 Proposition. *For isentropic or homogeneous incompressible flow, we have*

(i)
$$\frac{\partial \omega}{\partial t} + \pounds_u \omega = 0 \quad \text{and} \quad \frac{\partial \eta}{\partial t} + \pounds_u \eta - \eta \operatorname{div} u = 0$$

*called the **vorticity–stream equation** and*

(ii)
$$\varphi_t^* \omega = \omega_0 \quad \text{and} \quad \varphi_t^* \eta_t = J(\varphi_t)\eta_0$$

where $\eta_t(x) = \eta(x,t)$ and $J(\varphi_t)$ is the Jacobian of φ_t.

Proof. Applying $\mathbf{d}$ to Euler's equations for the two types of fluids we get the ***vorticity equation***:

$$\frac{\partial \omega}{\partial t} + \pounds_u \omega = 0.$$

Thus

$$\begin{aligned} \frac{\partial \eta}{\partial t} + \pounds_u \eta &= \frac{1}{\rho}\left(\frac{\partial \omega}{\partial t} + \pounds_u \omega\right) - \frac{\omega}{p^2}\left(\frac{\partial \rho}{\partial^t} + \mathbf{d}\rho \cdot u\right) \\ &= \frac{\eta}{\rho}\left(\frac{\partial \rho}{\partial t} + \mathbf{d}\rho \cdot u + \rho \operatorname{div} u\right) + \eta \operatorname{div} u = \eta \operatorname{div} u \end{aligned}$$

by conservation of mass.

From $\partial\omega/\partial t + \pounds_u \omega = 0$ it follows that $(\partial/\partial t)(\varphi_t^* \omega_t) = 0$, so $\varphi_t^* \omega_t = \omega_0$. Since $\varphi_t^* \rho_t = \rho_0/J(\varphi_t)$ we also get $\varphi_t^* \eta_t = J(\varphi_t)\eta_0$. ■

In three dimensions we can associate to η the vector field $\zeta = *\eta$ (or equivalently $\mathbf{i}_\zeta \mu = \eta$). Thus $\zeta = \operatorname{curl} \mathbf{u}/\rho$, if M is embedded in $\mathbb{R}^3$.

8.2.7 Corollary. *If $\dim M = 3$, then ζ is transported as a vector by φ_t; that is,*

$$\zeta_t = \varphi_{t*}\zeta_0 \quad \text{or} \quad \zeta_t(\varphi_t(x)) = T_x\varphi_t(\zeta_t(x)).$$

Proof. $\varphi_t^* \eta_t = J(\varphi_t)\eta_0$ by Proposition 8.2.6, so

$$\varphi_t^* \mathbf{i}_{\zeta_t} \mu = J(\varphi_t) \mathbf{i}_{\zeta_0} \mu.$$

But

$$\varphi_t^* \mathbf{i}_{\zeta_t} \mu = \mathbf{i}_{\varphi_t^* \zeta_t} \varphi_t^* \mu = \mathbf{i}_{\varphi_t^* \zeta_t} J(\varphi_t) \mu$$

Thus $\mathbf{i}_{\varphi_t^* \zeta_t} u = \mathbf{i}_{\zeta_0} \mu$, which gives $\varphi_t^* \zeta_t = \zeta_0$. ■

Notice that the vorticity as a *two-form* is Lie transported by the flow but as a *vector field* it is vorticity/ρ, which is Lie transported. Here is another instance where distinguishing between forms and vector fields makes an important difference.

The flow φ_t of a fluid plays the role of a configuration variable and the velocity field u plays the role of the corresponding velocity variable. In fact, to understand fluid mechanics as a Hamiltonian system in the sense of §8.1, a first step is to set up its phase space using the set of all diffeomorphisms $\varphi : M \to M$ (volume preserving for incompressible flow) as the configuration space. The references noted at the beginning of this section carry out this program (see also Exercise 8.2-9 and §8.4).

Exercises

◇ **8.2-1.** In classical texts on fluid mechanics, the identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) + (\nabla \times \mathbf{u}) \times \mathbf{u}$$

is often used. To what identity does this correspond in this section?

◇ **8.2-2.** A flow is called ***potential flow*** if $u^\flat = \mathbf{d}\varphi$ for a function φ. For (not necessarily stationary) homogeneous incompressible or isentropic flow prove ***Bernoulli's law*** in the form

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\|u\|^2 + \int \frac{\mathbf{d}p}{\rho} = \text{constant on a streamline.}$$

◇ **8.2-3.** Complex variables texts "show" that the gradient of $\varphi(r, \theta) = (r + 1/r)\cos\theta$ describes stationary ideal incompressible flow around a cylinder in the plane. Verify this in the context of this section.

◇ **8.2-4.** Translate Proposition 8.2.2 into vector analysis notation in $\mathbb{R}^3$ and give a direct proof.

◇ **8.2-5.** Let $\dim M = 3$, and assume the vorticity ω has a one-dimensional kernel.

(i) Using Frobenius' theorem, show that this distribution is integrable.

(ii) Identify the one-dimensional leaves with integral curves of ζ (see Corollary 8.2.7)—these are called ***vortex lines***.

(iii) Show that vortex lines are propagated by the flow.

◇ **8.2-6.** Assume $\dim M = 3$. A ***vortex tube*** T is a closed oriented two-manifold in M that is a union of vortex lines. The ***strength*** of the vortex tube is the flux of vorticity across a surface Σ inside T whose boundary lies on T and is transverse to the vortex lines. Show that vortex tubes are propagated by the flow and have a strength that is constant in time.

◇ **8.2-7.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a linear function and $g : S^2 \to \mathbb{R}$ be its restriction to the unit sphere. Show that $\mathbf{d}g$ gives a stationary solution of Euler's equations for flow on the two-sphere.

◇ **8.2-8.** Stream Functions

(i) For incompressible flow in $\mathbb{R}^2$, show that there is a function ψ such that $u^1 = \partial\psi/\partial y$ and $u^2 = -\partial\psi/\partial x$. One calls φ the ***stream function*** (as in Batchelor [1967]).

(ii) Show that if we let ${}^*\psi = \psi\, dx \wedge dy$ be the associated two form, then $\mathbf{u}^\flat = \delta^*\psi$.

(iii) Show that $\mathbf{u}$ is a Hamiltonian vector field (see §8.1) with energy ψ directly in $\mathbb{R}^2$ and then for arbitrary two-dimensional Riemannian manifolds M.

(iv) do stream functions exist for arbitrary fluid flow on $\mathbb{T}^2$? On S^2?

(v) Show that the vorticity is $\omega = \Delta^*\psi$.

◇ **8.2-9** (Clebsch Variables; Clebsch [1859])**.** Let $\mathcal{F}$ be the space of functions on a compact manifold M with the dual space $\mathcal{F}^*$, taken to be densities on M; the pairing between $f \in \mathcal{F}$ and $\rho \in \mathcal{F}^*$ is $\langle f, \rho \rangle = \int_M f\rho$.

(i) On the symplectic manifold $\mathcal{F} \times \mathcal{F}^* \times \mathcal{F} \times \mathcal{F}^*$ with variables $(\alpha, \lambda, \mu, \rho)$, show that Hamilton's equations for a given Hamiltonian H are

$$\dot\alpha = \frac{\delta H}{\delta \lambda}, \quad \dot\mu = \frac{\delta H}{\delta \rho}, \quad \dot\lambda = -\frac{\delta H}{\delta \alpha}, \quad \dot\rho = -\frac{\delta H}{\delta \mu},$$

where $\delta H/\delta\lambda$ is the ***functional derivative*** of H defined by

$$\left\langle \frac{\delta H}{\delta \lambda}, \dot\lambda \right\rangle = \mathbf{D}H(\lambda) \cdot \dot\lambda.$$

(ii) In the ideal isentropic compressible fluid equations, set $\mathbf{M} = \rho u^\flat$, the ***momentum density***, where dx denotes the Riemannian volume form on M. Identify the density $\sigma(x)dx \in \mathcal{F}^*$ with the function $\sigma(x) \in \mathcal{F}$ and write $\mathbf{M} = -(\rho \mathbf{d}\mu + \lambda \mathbf{d}\alpha)dx$. For momentum densities of this form show that Hamilton's equations written in the variables $(\alpha, \lambda, \mu, \rho)$ imply Euler's equation and the equation of continuity.

8.3 Electromagnetism

Classical electromagnetism is governed by Maxwell's field equations. The form of these equations depends on the physical units chosen, and changing these units introduces factors like 4π, $c =$ the speed of light, $\epsilon_0 =$ the dielectric constant and $\mu_0 =$ the magnetic permeability. The discussion in this section assumes that ϵ_0, μ_0 are constant; the choice of units is such that the equations take the simplest form; thus $c = \epsilon_0 = \mu_0 = 1$ and factors 4π disappear. We also do not consider Maxwell's equations in a material, where one has to distinguish $\mathbf{E}$ from $\mathbf{D}$, and $\mathbf{B}$ from $\mathbf{H}$.

Let $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{J}$ be time dependent C^1-vector fields on $\mathbb{R}^3$ and $\rho : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ a scalar. These are said to satisfy ***Maxwell's equations*** with ***charge density*** ρ and ***current density*** $\mathbf{J}$ when the following hold:

$$\operatorname{div} \mathbf{E} = \rho \quad (\textit{Gauss's law}) \tag{8.3.1}$$

$$\operatorname{div} \mathbf{B} = 0 \quad (\textit{no magnetic sources}) \tag{8.3.2}$$

$$\operatorname{curl} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \quad (\textit{Faraday's law of induction}) \tag{8.3.3}$$

$$\operatorname{curl} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \quad (\textit{Ampère's law}) \tag{8.3.4}$$

$\mathbf{E}$ is called the ***electric field*** and $\mathbf{B}$ the ***magnetic field***.

The quantity $\int_\Omega \rho \, dV = Q$ is called the ***charge*** of the set $\Omega \subset \mathbb{R}^3$. By Gauss' theorem, equation (8.3.1) is equivalent to

$$\int_{\partial\Omega} \mathbf{E} \cdot \mathbf{n} \, dS = \int_\Omega \rho \, dV = Q \tag{8.3.5}$$

for any (nice) open set $\Omega \subset \mathbb{R}^3$; that is, the *electric flux out of a closed surface equals the total charge inside the surface*. This generalizes Gauss' law for a point charge discussed in §7.3. By the same reasoning, equation (8.3.2) is equivalent to

$$\int_{\partial\Omega} \mathbf{B} \cdot \mathbf{n} \, dS = 0. \tag{8.3.6}$$

That is, the *magnetic flux out of any closed surface is zero.* In other words there are no magnetic sources inside any closed surface.

By Stokes' theorem, equation (8.3.3) is equivalent to

$$\int_{\partial S} \mathbf{E} \cdot \mathbf{ds} = \int_S (\operatorname{curl} \mathbf{E}) \cdot \mathbf{n} \, dS = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{n} \, dS \tag{8.3.7}$$

for any closed loop ∂S bounding a surface S. The quantity $\int_{\partial S} \mathbf{E} \cdot \mathbf{ds}$ is called the ***voltage*** around ∂S. Thus, Faraday's law of induction equation (8.3.3), says that *the voltage around a loop equals the negative of the rate of change of the magnetic flux through the loop.*

Finally, again by the classical Stokes' theorem, equation (8.3.4) is equivalent to

$$\int_{\partial S} \mathbf{B} \cdot \mathbf{ds} = \int_S (\operatorname{curl} \mathbf{B}) \cdot \mathbf{n} \, dS = \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot \mathbf{n} \, dS + \int_S \mathbf{J} \cdot \mathbf{n} \, dS. \tag{8.3.8}$$

Since $\int_S \mathbf{J} \cdot \mathbf{n}\, dS$ has the physical interpretation of ***current***, Ampère's law states that *if* $\mathbf{E}$ *is constant in time, then the magnetic potential difference* $\int_{\partial S} \mathbf{B} \cdot \mathbf{ds}$ *around a loop equals the current through the loop*. In general, if $\mathbf{E}$ varies in time, Ampère's law states that the *magnetic potential difference around a loop equals the total current in the loop plus the rate of change of electric flux through the loop.*

We now show how to express Maxwell's equations in terms of differential forms. Let $M = \mathbb{R}^4 = \{(x, y, z, t)\}$ with the Lorentz metric g on $\mathbb{R}^4$ having diagonal form $(1, 1, 1, -1)$ in standard coordinates (x, y, z, t).

8.3.1 Proposition. *There is a unique two-form F on $\mathbb{R}^4$, called the **Faraday two-form** such that*

$$\mathbf{E}^\flat = -\mathbf{i}_{\partial/\partial t} F; \tag{8.3.9}$$

$$\mathbf{B}^\flat = -\mathbf{i}_{\partial/\partial t} * F. \tag{8.3.10}$$

(*Here the $^\flat$ is associated with the Euclidean metric in $\mathbb{R}^3$ and the $*$ is associated with the Lorentzian metric in $\mathbb{R}^4$.*)

Proof. If

$$\begin{aligned} F &= F_{xy} dx \wedge dy + F_{zx} dz \wedge dx + F_{yz} dy \wedge dz \\ &\quad + F_{xt} dx \wedge dt + F_{yt} dy \wedge dt + F_{zt} dz \wedge dt, \end{aligned}$$

then (see Example 6.2.14E),

$$\begin{aligned} *F &= F_{xy} dz \wedge dt + F_{zx} dy \wedge dt + F_{yz} dx \wedge dt \\ &\quad - F_{xt} dy \wedge dz - F_{yt} dz \wedge dx - F_{zt} dx \wedge dy \end{aligned}$$

and so

$$-\mathbf{i}_{\partial/\partial t} \mathbf{F} = F_{xt} dx + F_{yt} dy + F_{zt} dz$$

and

$$-\mathbf{i}_{\partial/\partial t} * \mathbf{F} = F_{xy} dz + F_{zx} dy + F_{yz} dx.$$

Thus, $\mathbf{F}$ is uniquely determined by equations (8.3.9) and (8.3.10), namely

$$\begin{aligned} F &= E^1 dx \wedge dt + E^2 dy \wedge dt + E^3 dz \wedge dt \\ &\quad + B^3 dx \wedge dy + B^2 dz \wedge dx + B^1 dy \wedge dz. \end{aligned}$$

■

We started with $\mathbf{E}$ and $\mathbf{B}$ and used them to construct F, but one can also take F as the primitive object and construct $\mathbf{E}$ and $\mathbf{B}$ from it using equations (8.3.9) and (8.3.10). Both points of view are useful.

Similarly, out of ρ and $\mathbf{J}$ we can form the ***source one-form*** $j = -\rho\, dt + J_1 dx + J_2 dy + J_3 dz$; that is, j is uniquely determined by the equations $-\mathbf{i}_{\partial/\partial t} j = \rho$ and $\mathbf{i}_{\partial/\partial t} * j = *\mathbf{J}^\flat$; in the last relation, $\mathbf{J}$ is regarded as being defined on $\mathbb{R}^4$.

8.3.2 Proposition. *Maxwell's equations* (8.3.1)–(8.3.4) *are equivalent to the equations*

$$\mathbf{d}F = 0 \quad \textit{and} \quad \delta F = j$$

on the manifold $\mathbb{R}^4$ endowed with the Lorentz metric.

Proof. A straightforward computation shows that

$$\begin{aligned}\mathbf{d}F &= \left(\operatorname{curl}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right)_x dy \wedge dz \wedge dt + \left(\operatorname{curl}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right)_y dz \wedge dx \wedge dt \\ &\quad + \left(\operatorname{curl}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right)_z dx \wedge dy \wedge dt + (\operatorname{div}\mathbf{B}) dx \wedge dy \wedge dz.\end{aligned}$$

Thus $\mathbf{d}F = 0$ is equivalent to equations (8.3.2) and (8.3.3).

Since the index of the Lorentz metric is 1, we have $\delta = *\mathbf{d}*$. Thus,

$$\begin{aligned}\delta F = *\mathbf{d} * F &= *\mathbf{d}\left(-E^1 dy \wedge dz - E^2 dz \wedge dx - E^3 dx \wedge dy \right. \\ &\qquad \left. + B^1 dx \wedge dt + B^2 dy \wedge dt + B^3 dz \wedge dt\right) \\ &= *\left[-(\operatorname{div}\mathbf{E}) dx \wedge dy \wedge dz + \left(\operatorname{curl}\mathbf{B} - \frac{\partial \mathbf{B}}{\partial t}\right)_x dy \wedge dz \wedge dt + \right. \\ &\qquad \left. + \left(\operatorname{curl}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}\right)_y dz \wedge dx \wedge dt + \left(\operatorname{curl}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}\right)_z dx \wedge dy \wedge dt\right] \\ &= \left(\operatorname{curl}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}\right)_x dx + \left(\operatorname{curl}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}\right)_y dy \\ &\qquad + \left(\operatorname{curl}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}\right)_z dz - (\operatorname{div}\mathbf{E})\, dt.\end{aligned}$$

Thus $\delta F = j$ iff equations (8.3.1) and (8.3.4) hold. ■

As a skew matrix, we can represent F as follows

$$F = \begin{bmatrix} 0 & B^3 & -B^2 & E^1 \\ -B^3 & 0 & B^1 & E^2 \\ B^2 & -B^1 & 0 & E^3 \\ -E^1 & -E^2 & -E^3 & 0 \end{bmatrix}.$$

Recall from §6.5 and Exercise 7.5-7, the formula

$$(\delta F)^i = |\det[g_{lj}]|^{-1/2}(F^{ik}|\det[g_{lj}]|^{1/2})_{,k}$$

Since $|\det[g_{kl}]| = 1$, Maxwell's equations can be written in terms of the Faraday two-form F in components as

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \tag{8.3.11}$$

and

$$F^{ik}_{,k} = -j^i, \tag{8.3.12}$$

where $F_{ij,k} = \partial F_{ij}/\partial x^k$, etc. Since $\delta^2 = 0$, we obtain

$$\begin{aligned}0 = \delta^2 F = \delta j = *\mathbf{d} * j &= *\mathbf{d}(-\rho\, dx \wedge dy \wedge dz + (*\mathbf{J}^\flat) \wedge dt) \\ &= *\left[\left(\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{J}\right) dx \wedge dy \wedge dz \wedge dt\right] = \frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{J};\end{aligned}$$

that is, $\partial\rho/\partial t + \operatorname{div}\mathbf{J} = 0$, which is the ***continuity equation*** (see §8.2). Its integral form is, by Gauss' theorem,

$$\frac{dQ}{dt} = \frac{d}{dt}\int_\Omega \rho\, dV = \int_{\partial\Omega} \mathbf{J} \cdot \mathbf{n}\, dS$$

for any bounded open set Ω. Thus the continuity equation says that the *flux of the current density out of a closed surface equals the rate of change of the total charge inside the surface.*

Next we show that Maxwell's equations are Lorentz invariant, that is, are special-relativistic. The ***Lorentz group*** $\mathcal{L}$ is by definition the orthogonal group with respect to the Lorentz metric g, that is,

$$\mathcal{L} = \left\{ A \in \mathfrak{GL}(\mathbb{R}^4) \mid g(Ax, Ay) = g(x, y) \text{ for all } x, y \in \mathbb{R}^4 \right\}.$$

Lorentz invariance means that F satisfies Maxwell's equations with j iff A^*F satisfies them with A^*F, for any $A \in \mathcal{L}$. But due to Proposition 8.3.1 this is clear since pull-back commutes with $\mathbf{d}$ and orthogonal transformations commute with the Hodge operator (see Exercise 6.2-4) and thus they commute with δ.

As a 4×4 matrix, the Lorentz transformation A acts on F by $F \mapsto A^*F = AFA^T$. Let us see that the action of $A \in \mathcal{L}$ mixes up $\mathbf{E}$'s and $\mathbf{B}$'s. (This is the source of statements like: "A moving observer sees an electric field partly converted to a magnetic field.")

Proposition 8.3.1 defines $\mathbf{E}$ and $\mathbf{B}$ intrinsically in terms of F. Thus, if one performs a Lorentz transformation A on F, the new resulting electric and magnetic fields $\mathbf{E}'$ and $\mathbf{B}'$ with respect to the Lorentz unit normal $A^*(\partial/\partial t)$ to the image $A(\mathbb{R}^3 \times 0)$ in $\mathbb{R}^4$ are given by

$$(\mathbf{E}')^\flat = -\mathbf{i}_{A^*\partial/\partial t} A^*F, \quad (\mathbf{B}')^\flat = -\mathbf{i}_{A^*\partial/\partial t} A^*F.$$

For a Lorentz transformation of the form

$$x' = \frac{x - vt}{\sqrt{1 - v^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - vx}{\sqrt{1 - v^2}}$$

(the special-relativistic analogue of an observer moving uniformly along the x-axis with velocity v) we get

$$\mathbf{E}' = \left(E', \frac{E^2 - vB^3}{\sqrt{1 - v^2}}, \frac{E^3 + vB^2}{\sqrt{1 - v^2}} \right)$$

and

$$\mathbf{B}' = \left(B', \frac{B^2 - vE^3}{\sqrt{1 - v^2}}, \frac{B^3 + vE^2}{\sqrt{1 - v^2}} \right).$$

We leave the verification to the reader.

By the way we have set things up, note that Maxwell's equations make sense on any ***Lorentz manifold***; that is, a four-dimensional manifold with a pseudo-Riemannian metric of signature $(+, +, +, -)$.

Maxwell's vacuum equations (i.e., $j = 0$) will now be shown to be ***conformally invariant*** on any Lorentz manifold (M, g). A diffeomorphism $\varphi : (M, g) \to (M, g)$ is said to be ***conformal*** if $\varphi^*g = f^2 g$ for a nowhere vanishing function f. (See Fulton, Rohrlich, and Witten [1962] for a review of conformal invariance in physics and the original literature references.)

8.3.3 Proposition. *Let $F \in \Omega^2(M)$ where (M, g) is a Lorentz manifold, satisfy $\mathbf{d}F = 0$ and $\delta F = j$. Let φ be a conformal diffeomorphism. Then φ^*F satisfies*

$$\mathbf{d}\varphi^*F = 0 \quad \text{and} \quad \delta\varphi^*F = f^2 \varphi^* j.$$

Hence Maxwell's vacuum equations (with $j = 0$) are conformally invariant; that is, if F satisfies them, so does φ^*F.

Proof. Since φ^* commutes with $\mathbf{d}$, $\mathbf{d}F = 0$ implies $\mathbf{d}\varphi^*F = 0$. The second equation implies $\varphi^*\delta F = \varphi^* j$. By Exercise 7.5-8, we have $\delta_{\varphi^*g}\varphi^*\beta = \varphi^*\delta\beta$. Hence $\delta F = j$ implies $\delta_{\varphi^*g}\varphi^*F = \varphi^*j = \delta_{f^2 g}\varphi^*F$ since φ is conformal. The local formula for δF, namely

$$(\delta F)_i = |\det[g_{ks}]|^{-1/2} g_{ir} \frac{\partial}{\partial x^l} (g^{ra} g^{lb} F_{ab} |\det[g_{ks}]|^{-1/2})$$

shows that when one replaces g by $f^2 g$, we get

$$\delta_{f^2 g}\varphi^* F = f^{-2}\varphi^* F,$$

and so

$$\delta\varphi^* F = f^2 \varphi^* j. \qquad \blacksquare$$

Let us now discuss the energy equation for the electromagnetic field. Introduce the ***energy density*** of the field

$$\tfrac{1}{2}\mathcal{E} = (\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B})$$

and the ***Poynting energy-flux vector***

$$\mathbf{S} = \mathbf{E}\times\mathbf{B}.$$

Poynting's theorem states that

$$-\frac{\partial\mathcal{E}}{\partial t} = \operatorname{div}\mathbf{S} + \mathbf{E}\cdot\mathbf{J}.$$

This is a straightforward computation using equations (8.3.3) and (8.3.4). We shall extend this result to $\mathbb{R}^4$ and, at the same time, shall rephrase it in the framework of forms.

Introduce the ***stress–energy–momentum tensor*** (or the ***Maxwell stress tensor***) T by

$$T^{ij} = F^{ik}F^j_k - \tfrac{1}{4}g^{ij}F_{pq}F^{pq} \tag{8.3.13}$$

(or intrinsically,

$$T = F\cdot F - \tfrac{1}{4}\langle F, F\rangle g,$$

where $F\cdot F$ denotes a single contraction of F with itself). A straightforward computation shows that the divergence of T equals

$$T^{ij}_{,j} = F^{ik}_{,j}F^j_k + F^{ik}F^j_{k,j} - \tfrac{1}{2}F^{,i}_{pq}F^{pq}$$

where $F^{,i}_{pq} = (\partial F_{pq}/\partial x^k)g^{ik}$. Taking into account $\delta F = j$ written in the form (8.3.12), it follows that

$$T^{il}_{,l} = F^{ik}j_k. \tag{8.3.14}$$

For $i = 4$, the relation (8.3.14) becomes Poynting's theorem.[2] It is clear that T is a symmetric 2-tensor. As a symmetric matrix,

$$T = \begin{bmatrix} \sigma & \mathbf{E}\times\mathbf{B} \\ (\mathbf{E}\times\mathbf{B})^T & \mathcal{E} \end{bmatrix},$$

[2]Poynting's theorem can also be understood in terms of a Hamiltonian formulation; see Example 8.4.2 below. The Poynting energy-flux vector is the Noether conserved quantity for the action of the diffeomorphism group of $\mathbb{R}^3$ on $T^*\mathcal{A}$, where $\mathcal{A}$ is the space of vector potentials $\mathcal{A}$ defined in the following paragraph, and Poynting's theorem is just conservation of momentum (Noether's theorem). We shall not dwell upon these aspects and refer the interested reader to Abraham and Marsden [1978] and Marsden and Ratiu [1999].

where σ is the ***stress tensor*** and $\mathcal{E}$ is the energy density. The symmetric 3×3 matrix σ has the following components

$$\begin{aligned}
\sigma^{11} &= \tfrac{1}{2}[-(E^1)^2 - (B^1)^2 + (E^2)^2 + (B^2)^3 + (E^3)^2 + (B^3)^2] \\
\sigma^{22} &= \tfrac{1}{2}[(E^1)^2 + (B^1)^2 - (E^2)^2 - (B^2)^2 + (E^3)^3 + (B^3)^2] \\
\sigma^{33} &= \tfrac{1}{2}[(E^1)^2 + (B^1)^2 + (E^2)^2 + (B^2)^2 - (E^3)^2 - (B^3)^2] \\
\sigma^{12} &= E^1E^2 + B^1B^2 \\
\sigma^{13} &= E^1E^3 + B^1B^3 \\
\sigma^{23} &= E^2E^3 + B^2B^3.
\end{aligned}$$

We close this section with a discussion of Maxwell's equations in terms of vector potentials. We first do this directly in terms of $\mathbf{E}$ and $\mathbf{B}$. Since $\operatorname{div}\mathbf{B} = 0$, if $\mathbf{B}$ is smooth on all of $\mathbb{R}^3$, there exists a vector field $\mathbf{A}$, called the ***vector potential***, such that $\mathbf{B} = \operatorname{curl}\mathbf{A}$, by the Poincaré lemma. This vector field $\mathbf{A}$ is not unique and one could also use $\mathbf{A}' = \mathbf{A} + \operatorname{grad} f$ for some (possibly time–dependent) function $f : \mathbb{R}^3 \to \mathbb{R}$. This freedom in the choice of $\mathbf{A}$ is called ***gauge freedom***. For any such choice of $\mathbf{A}$ we have by equation (8.3.3)

$$0 = \operatorname{curl}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}\mathbf{E} + \frac{\partial}{\partial t}\operatorname{curl}\mathbf{A} = \operatorname{curl}\left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right),$$

so that again by the Poincaré lemma there exists a (time-dependent) function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\operatorname{grad}\varphi. \tag{8.3.15}$$

Recall that the Laplace–Beltrami operator on functions is defined by $\nabla^2 f = \operatorname{div}(\operatorname{grad} f)$. On vector fields in $\mathbb{R}^3$ this operator may be defined componentwise. Then it is easy to check that

$$\operatorname{curl}(\operatorname{curl}\mathbf{A}) = \operatorname{grad}(\operatorname{div}\mathbf{A}) - \nabla^2\mathbf{A}.$$

Using this identity, (8.3.15), and $\mathbf{B} = \operatorname{curl}\mathbf{A}$ in (8.3.4), we get

$$\begin{aligned}
\mathbf{J} = \operatorname{curl}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \operatorname{curl}(\operatorname{curl}\mathbf{A}) - \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad}\varphi\right) \\
&= \operatorname{grad}(\operatorname{div}\mathbf{A}) - \nabla^2\mathbf{A} + \frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{\partial}{\partial t}(\operatorname{grad}\varphi),
\end{aligned}$$

and thus

$$\nabla^2\mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mathbf{J} + \operatorname{grad}\left(\operatorname{div}\mathbf{A} + \frac{\partial \varphi}{\partial t}\right). \tag{8.3.16}$$

From equation (8.3.1) we obtain as before

$$\rho = \operatorname{div}\mathbf{E} = \operatorname{div}\left(-\frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad}\varphi\right) = -\nabla^2\varphi - \frac{\partial}{\partial t}(\operatorname{div}\mathbf{A}),$$

that is,

$$\nabla^2\varphi = -\rho - \frac{\partial}{\partial t}(\operatorname{div}\mathbf{A}),$$

or subtracting $\partial^2\varphi/\partial t^2$ from both sides,

$$\nabla^2\varphi - \frac{\partial^2 \varphi}{\partial t^2} = -\rho - + \frac{\partial}{\partial t}\left(\operatorname{div}\mathbf{A} + \frac{\partial \varphi}{\partial t}\right). \tag{8.3.17}$$

It is apparent that equations (8.3.16) and (8.3.17) can be considerably simplified if one could choose, using the gauge freedom, the vector potential $\mathbf{A}$ and the function φ such that

$$\operatorname{div}\mathbf{A} + \frac{\partial\varphi}{\partial t} = 0.$$

Assume one has chosen $\mathbf{A}_0, \varphi_0$ and one seeks a function f such that $\mathbf{A} = \mathbf{A}_0 + \operatorname{grad} f$ and $\varphi = \varphi_0 - \partial f/\partial t$ satisfy $\operatorname{div}\mathbf{A} + \partial\varphi/\partial t = 0$. This becomes, in terms of f,

$$0 = \operatorname{div}(\mathbf{A}_0 + \operatorname{grad} f) + \frac{\partial}{\partial t}\left(\varphi_0 - \frac{\partial f}{\partial t}\right) = \operatorname{div}\mathbf{A}_0 + \frac{\partial\varphi_0}{\partial t} + \nabla^2 f - \frac{\partial^2 f}{\partial t^2};$$

that is,

$$\nabla^2 f - \frac{\partial^2 f}{\partial t^2} = -\left(\operatorname{div}\mathbf{A}_0 + \frac{\partial\varphi_0}{\partial t}\right). \tag{8.3.18}$$

This equation is the classical ***inhomogeneous wave equation***. The homogeneous wave equation (right-hand side equals zero) has solutions

$$f(t, x, y, z) = \psi(x - t)$$

for any function ψ. This solution propagates the graph of ψ like a wave—hence the name wave equation.

Now we we can draw some conclusions regarding Maxwell's equations. In terms of the vector potential $\mathbf{A}$ and the function φ, equations (8.3.1) and (8.3.4) become

$$\nabla^2\varphi - \frac{\partial^2\varphi}{\partial t^2} = -\rho, \quad \nabla^2\mathbf{A} - \frac{\partial^2\mathbf{A}}{\partial t^2} = -\mathbf{J}, \tag{8.3.19}$$

which again are inhomogeneous wave equations. Conversely, if $\mathbf{A}$ and φ satisfy the foregoing equations and $\operatorname{div}\mathbf{A} + \partial\varphi/\partial t = 0$, then $\mathbf{E} = -\operatorname{grad}\varphi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \operatorname{curl}\mathbf{A}$ satisfy Maxwell's equations.

Thus in $\mathbb{R}^4$, this procedure reduces the study of Maxwell's equations to the wave equation, and hence solutions of Maxwell's equations can be expected to be wavelike.

We now repeat the foregoing constructions on $\mathbb{R}^4$ using differential forms. Since $\mathbf{d}F = 0$, on $\mathbb{R}^4$ we can write $F = \mathbf{d}G$ for a one-form G. Note that F is unchanged if we replace G by $G + \mathbf{d}f$. This again is the *gauge freedom*. Substituting $F = \mathbf{d}G$ into $\delta F = j$ gives $\delta\mathbf{d}G = j$. Since $\Delta = \mathbf{d}\delta + \delta\mathbf{d}$ is the Laplace–deRham operator in $\mathbb{R}^4$, we get

$$\Delta G = j - \mathbf{d}\delta G. \tag{8.3.20}$$

Suppose we try to choose G so that $\delta G = 0$ (a gauge condition). To do this, given an initial G_0, we can let $G = G_0 + \mathbf{d}f$ and demand that

$$0 = \delta G = \delta G_0 + \delta\mathbf{d}f = \delta G_0 + \Delta f$$

so f must satisfy $\Delta f = -\delta G_0$. Thus, if the gauge condition

$$\Delta f = -\delta G_0 \tag{8.3.21}$$

holds, then Maxwell's equations become

$$\Delta G = j. \tag{8.3.22}$$

Equation (8.3.21) is equivalent to (8.3.18) and (8.3.22) to (8.3.19) by choosing $G = \mathbf{A}^\flat + \varphi\, dt$ (where $^\flat$ is Euclidean in $\mathbb{R}^3$).

Exercises

◇ **8.3-1.** Assume that the Faraday two-form F depends only on $t - x$.

(i) Show that $\mathbf{d}F = 0$ is then equivalent to $B^3 = E^2$, $B^2 = -E^3$, $B^1 = 0$.

(ii) Show that $\delta F = 0$ is then equivalent to $B^3 = E^2$, $B^2 = -E^3$, $E^1 = 0$. These solutions of Maxwell's equations are called ***plane electromagnetic*** waves; they are determined only by E^2, E^3 or B^2, B^3, respectively.

◇ **8.3-2.** Let $u = \partial/\partial t$. Show that the Faraday two-form $F \in \Omega^2(\mathbb{R}^4)$ is given in terms of E and $B \in \mathfrak{X}(\mathbb{R}^4)$ by $F = u^\flat \wedge E^\flat - *(u^\flat \wedge B^\flat)$.

◇ **8.3-3.** Show that the Poynting vector satisfies

$$S^\flat = *(B^\flat \wedge E^\flat \wedge u^\flat)$$

where $u = \partial/\partial t$ and $E, B \in \mathfrak{X}(\mathbb{R}^4)$.

◇ **8.3-4.** Let (M, g) be a Lorentzian four-manifold and $u \in \mathfrak{X}(M)$ a timelike unit vector field on M; that is, $g(u, u) = -1$.

(i) Show that any $\alpha \in \Omega^2(M)$ can be written in the form

$$\alpha = (\mathbf{i}_u\alpha) \wedge u^\flat - *((\mathbf{i}_u * \alpha) \wedge u^\flat).$$

(ii) Show that if $\mathbf{i}_u\alpha = 0$, where $\alpha \in \Omega^2(M)$ ("α is orthogonal to u"), then $*\alpha$ is decomposable, that is, $*\alpha$ is the wedge product of two one-forms. Prove that α is also locally decomposable.

Hint: Use the Darboux theorem.

◇ **8.3-5.** The field of a stationary point charge is given by

$$E = \frac{e\mathbf{r}}{4\pi r^3}, \quad \mathbf{B} = 0,$$

where $\mathbf{r}$ is the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in $\mathbb{R}^3$ and r is its length. Use this and a Lorentz transformation to show that the electromagnetic field produced by a charge e moving along the x-axis with velocity $\mathbf{v}$ is

$$E = \frac{e}{4\pi} \frac{(1 - v^2)\mathbf{r}}{[x^2 + (1 - v^2)(y^2 + z^2)]^{exc:3.2-27}}$$

and, using spherical coordinates with the x-axis as the polar axis,

$$B_r = 0, \quad B_\theta = 0, \quad B_\varphi = \frac{e(1 - v^2)v \sin\theta}{4\pi r^2(1 - v^2 \sin^2\theta)^{exc:3.2-27}}$$

(the magnetic field lines are thus circles centered on the polar axis and lying in planes perpendicular to it).

◇ **8.3-6** (Misner, Thorne, and Wheeler [1973]). The following is the Faraday two-form for the field of an electric dipole of magnitude p_1 oscillating up and down parallel to the z-axis.

$$\begin{aligned} F = \operatorname{Re}\Bigg\{p_1 e^{i\omega r - i\omega t} \Bigg[&2\cos\theta\left(\frac{1}{r^2} - \frac{i\omega}{r^2}\right) dr \wedge dt \\ &+ \sin\theta\left(\frac{1}{r^3} - \frac{i\omega}{r^2} - \frac{\omega^2}{r}\right) r d\theta \wedge dt \\ &+ \sin\theta\left(-\frac{i\omega}{r^2} - \frac{\omega^2}{r}\right) dr \wedge r\, d\theta\Bigg]\Bigg\}. \end{aligned}$$

Verify that $\mathbf{d}F = 0$ and $\delta F = 0$, except at the origin.

£?

◇ **8.3-7.** Let the Lagrangian for electromagnetic theory be

$$\pounds = |F|^2 = -\frac{1}{2} F_{ij} F_{kl} g^{ik} g^{jl} \sqrt{-\det g}.$$

Check that $\partial \pounds / \partial g_{ij}$ is the stress–energy–momentum tensor T^{ij} (see Hawking and Ellis [1973, Section 3.3]).

8.4 The Lie–Poisson Bracket in Continuum Mechanics and Plasma Physics

This section studies the equations of motion for some Hamiltonian systems in Poisson bracket formation. As opposed to §8.1, the emphasis is placed here on the Poisson bracket rather than on the underlying symplectic structure. This naturally leads to a generalization of Hamiltonian mechanics to systems whose phase space is a "Poisson manifold." We do not intend to develop here the theory of Poisson manifolds but only to illustrate it with the most important example, the Lie–Poisson bracket. See Marsden and Ratiu [1999] for further details.

If (P, ω) is a (weak) symplectic manifold, $H : P \to \mathbb{R}$ a smooth Hamiltonian with Hamiltonian vector field $X_H \in \mathfrak{X}(P)$ whose flow is denoted by φ_t, recall from Corollary 8.1.11 that

$$\frac{d\varphi_t}{dt}(p) = X_H(\varphi_t(p)) \tag{8.4.1}$$

is equivalent to

$$\frac{d}{dt}(F \circ \varphi_t) = \{F \circ \varphi_t, H \circ \varphi_t\} \tag{8.4.2}$$

for any smooth locally defined function $f : U \to \mathbb{R}$, where U is open in P. In (8.4.2), $\{\,,\}$ denotes the Poisson bracket defined by ω, that is,

$$\{F, G\} = \omega(X_F, X_G) = X_G[F] = -X_F[G]. \tag{8.4.3}$$

Finally, recall that the Poisson bracket is an antisymmetric bilinear operation on $\mathcal{F}(P)$ which satisfies the ***Jacobi identity***

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0, \tag{8.4.4}$$

that is, $(\mathcal{F}(P), \{\,,\})$ is a Lie algebra. In addition, the multiplicative ring structure and the Lie algebra structure of $\mathcal{F}(P)$ are connected by the ***Leibniz rule***

$$\{FG, H\} = F\{G, H\} + G\{F, H\}, \tag{8.4.5}$$

that is, $\{\,,\}$ is a derivation in each argument. These observations naturally lead to the following generalization of the concept of symplectic manifolds.

8.4.1 Definition. *A smooth manifold P is called a* ***Poisson manifold*** *if $\mathcal{F}(P)$, the ring of functions on P, admits a Lie algebra structure which is a derivation in each argument. The bracket operation on $\mathcal{F}(P)$ is called a* ***Poisson bracket*** *and is usually denoted by $\{\,,\}$.*

From the remarks above, we see that any (weak) symplectic manifold is a Poisson manifold. One of the purposes of this section is to show that there are physically important Poisson manifolds which are not symplectic. But even in the symplectic context it is sometimes easier to compute the Poisson bracket than the symplectic form, as the following example shows.

Fix title

8.4.2 Example (Maxwell's Vacuum Equations as an Infinite Dimensional Hamiltonian System)**.** We shall indicate how the dynamical pair of Maxwell's vacuum equations (8.3.3) and (8.3.4) of the previous section with current $\mathbf{J} = 0$ are a Hamiltonian system.

As the configuration space for Maxwell's equations, we take the space $\mathcal{A}$ of vector potentials. (In more general situations, one should replace $\mathcal{A}$ by the set of connections on a principal bundle over configuration space.) The corresponding phase space is then the cotangent bundle $T^*\mathcal{A}$ with the canonical symplectic structure. Elements of $T^*\mathcal{A}$ may be identified with pairs $(\mathbf{A}, \mathbf{Y})$ where $\mathbf{Y}$ is a vector field density on $\mathbb{R}^3$. (We do not distinguish $\mathbf{Y}$ and $\mathbf{Y}\, d^3x$.) The pairing between $\mathbf{A}$'s and $\mathbf{Y}$'s is given by integration, so the canonical symplectic structure ω on $T^*\mathcal{A}$ is

$$\omega((\mathbf{A}_1, \mathbf{Y}_1), (\mathbf{A}_2, \mathbf{Y}_2)) = \int_{\mathbb{R}^3} (\mathbf{Y}_2 \cdot \mathbf{A}_1 - \mathbf{Y}_1 \cdot \mathbf{A}_2) d^3x, \tag{8.4.6}$$

with associated Poisson bracket

$$\{F, G\}(\mathbf{A}, \mathbf{Y}) = \int_{\mathbb{R}^3} \left(\frac{\delta F}{\delta \mathbf{A}} \cdot \frac{\partial G}{\delta \mathbf{Y}} - \frac{\delta F}{\delta \mathbf{Y}} \cdot \frac{\delta G}{\delta \mathbf{A}} \right) d^3x, \tag{8.4.7}$$

where $\delta F/\delta \mathbf{A}$ is the vector field defined by

$$\mathbf{D_A} F(\mathbf{A}, \mathbf{Y}) \cdot \mathbf{A}' = \int \frac{\delta F}{\delta \mathbf{A}} \cdot \mathbf{A}' d^3x.$$

with the vector field $\delta F/\delta \mathbf{Y}$ defined similarly. With the Hamiltonian

$$H(\mathbf{A}, \mathbf{Y}) = \frac{1}{2} \int \|\mathbf{Y}\|^2 d^3x + \frac{1}{2} \|\operatorname{curl} \mathbf{A}\|^2 d^3x, \tag{8.4.8}$$

Hamilton's equations are easily computed to be

$$\frac{\partial \mathbf{Y}}{\partial t} = -\operatorname{curl}\operatorname{curl} \mathbf{A} \quad \text{and} \quad \frac{\partial \mathbf{A}}{\partial t} = \mathbf{Y}. \tag{8.4.9}$$

If we write $\mathbf{B}$ for $\operatorname{curl} \mathbf{A}$ and $\mathbf{E}$ for $-\mathbf{Y}$, the Hamiltonian becomes the field energy

$$\frac{1}{2} \int \|\mathbf{E}\|^2 d^3x + \frac{1}{2} \|\mathbf{B}\|^3 d^3x. \tag{8.4.10}$$

Equation (8.4.9) implies Maxwell's equations

$$\frac{\partial \mathbf{E}}{\partial t} = \operatorname{curl} \mathbf{B} \quad \text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = -\operatorname{curl} \mathbf{E}, \tag{8.4.11}$$

and the Poisson bracket of two functions $F(\mathbf{A}, \mathbf{E}), G(\mathbf{A}, \mathbf{E})$ is

$$\{F, G\}(\mathbf{A}, \mathbf{E}) = -\int_{\mathbb{R}^3} \left(\frac{\delta F}{\delta \mathbf{A}} \cdot \frac{\delta G}{\delta \mathbf{E}} - \frac{\delta G}{\delta \mathbf{A}} \cdot \frac{\delta F}{\delta \mathbf{E}} \right) d^3x. \tag{8.4.12}$$

We can express this Poisson bracket in terms of $\mathbf{E}$ and $\mathbf{B} = \operatorname{curl} \mathbf{A}$. To do this, we consider functions $\tilde{F} : \mathcal{V} \times \mathfrak{X}(\mathbb{R}^3) \to \mathbb{R}$, where

$$\mathcal{V} = \left\{ \operatorname{curl} \mathbf{Z} \mid \mathbf{Z} \in \mathfrak{X}(\mathbb{R}^3) \right\}.$$

We pair $\mathcal{V}$ with itself relative to the L^2-inner product. This is a weakly non-degenerate pairing since by the Hodge–Helmholtz decomposition

$$\int_{\mathbb{R}^3} \operatorname{curl} \mathbf{Z}_1 \cdot \operatorname{curl} \mathbf{Z}_2 \, d^3x = 0$$

for all $\mathbf{Z}_2 \in \mathfrak{X}(\mathbb{R}^3)$ implies that curl $\mathbf{Z}_1 = \nabla f$, whence

$$\Delta f = \operatorname{div} \nabla f = \operatorname{div} \operatorname{curl} \mathbf{Z}_1 = 0,$$

and so, $f =$ constant by Liouville's theorem. Therefore $\operatorname{curl} \mathbf{Z}_1 = \nabla f = 0$, as was to be shown.

We compute $\delta F / \delta \mathbf{A}$ in terms of the functional derivative of an arbitrary extension $\hat{F}$ of $\tilde{F}$ to $\mathfrak{X}(\mathbb{R}^3)$, where $F(\mathbf{A}, \mathbf{E}) = \tilde{F}(\mathbf{B}, \mathbf{E})$, for $\mathbf{B} = \operatorname{curl} \mathbf{A}$. Let L be the linear map $L(\mathbf{A}) = \operatorname{curl} \mathbf{A}$ so that

$$F = \tilde{F} \circ (L \times \text{Identity}) = \hat{F} \circ (L \times \text{Identity}).$$

By the chain rule, we have for any $\delta \mathbf{A} \in \mathfrak{X}(\mathbb{R}^3)$,

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{\delta F}{\delta \mathbf{A}} \cdot \delta \mathbf{A} d^3 x &= \mathbf{D} F(\mathbf{A}) \cdot \delta \mathbf{A} = (\mathbf{D} \hat{F}(\mathbf{B}) \circ \mathbf{D} L(\mathbf{A})) \cdot \delta \mathbf{A} \\ &= \int_{\mathbb{R}^3} \frac{\delta \hat{F}}{\delta \mathbf{B}} \cdot \operatorname{curl} \delta \mathbf{A} \, d^3 x = \int_{\mathbb{R}^3} \operatorname{curl} \frac{\delta \hat{F}}{\delta \mathbf{B}} \cdot \delta \mathbf{A} d^3 x, \end{aligned}$$

since $\mathbf{D} L(\mathbf{A}) = L$ and $\int_{\mathbb{R}^3} \mathbf{X} \cdot \operatorname{curl} \mathbf{Y} d^3 x = \int_{\mathbb{R}^3} \mathbf{Y} \cdot \operatorname{curl} \mathbf{X} \, d^3 x$. Therefore

$$\frac{\delta F}{\delta \mathbf{A}} = \operatorname{curl} \frac{\delta \tilde{F}}{\delta \mathbf{B}}. \tag{8.4.13}$$

This formula seems to depend on the extension $\hat{F}$ of $\tilde{F}$. However, this is not the case. More precisely, let $K : \mathfrak{X}(\mathbb{R}^3) \times \mathfrak{X}(\mathbb{R}^3) \to \mathbb{R}$ be such that $K | (\mathcal{V} \times \mathfrak{X}(\mathbb{R})) \equiv 0$. We claim that if $\mathbf{B} \in \mathcal{V}$, then $\delta K / \delta \mathbf{B}$ is a gradient. Granting this statement, this shows that equation (8.4.13) is independent of the extension, since any two extensions of $\tilde{F}$ coincide on $\mathcal{V} \times \mathfrak{X}(\mathbb{R}^3)$ and since $\operatorname{curl} \circ \operatorname{grad} = 0$. To prove the claim, note that for any $\mathbf{Z} \in \mathfrak{X}(\mathbb{R}^3)$,

$$0 = \mathbf{D} K(\mathbf{B}) \cdot \operatorname{curl} \mathbf{Z} = \int_{\mathbb{R}^3} \frac{\delta K}{\delta \mathbf{B}} \cdot \operatorname{curl} \mathbf{Z} \, d^3 x = \int_{\mathbb{R}^3} \operatorname{curl} \frac{\delta K}{\delta \mathbf{B}} \cdot \mathbf{Z} \, d^3 x$$

whence $\operatorname{curl} \delta K / \delta \mathbf{B} = 0$, that is, $\delta K / \delta \mathbf{B}$ is a gradient. Thus, equation(8.4.13) implies

$$\frac{\delta F}{\delta \mathbf{A}} = \operatorname{curl} \frac{\delta \tilde{F}}{\delta \mathbf{B}}, \tag{8.4.14}$$

where on the right-hand side $\delta \tilde{F} / \delta \mathbf{B}$ is understood as the functional derivative relative to $\mathbf{B}$ of an arbitrary extension of $\tilde{F}$ to $\mathfrak{X}(\mathbb{R}^3)$. Since $\delta \tilde{F} / \delta \mathbf{E} = \delta F / \delta \mathbf{E}$, the Poisson bracket (8.4.12) becomes

$$\{\tilde{F}, \tilde{G}\}(\mathbf{B}, \mathbf{E}) = \int_{\mathbb{R}^3} \left(\frac{\delta \tilde{F}}{\delta \mathbf{E}} \cdot \operatorname{curl} \frac{\delta \tilde{G}}{\delta \mathbf{B}} - \frac{\delta \tilde{G}}{\delta \mathbf{E}} \cdot \operatorname{curl} \frac{\delta \tilde{F}}{\delta \mathbf{B}} \right) d^3 x \tag{8.4.15}$$

This bracket was found by Born and Infeld [1935] by a different method.

Using the Hamiltonian

$$H(\mathbf{B}, \mathbf{E}) = \frac{1}{2} \int_{\mathbb{R}^3} (\|\mathbf{B}\|^2 + \|\mathbf{E}\|^2) d^3 x, \tag{8.4.16}$$

equations (8.4.11) are equivalent to the Poisson bracket equations

$$\dot{\tilde{F}} = \{\tilde{F}, \tilde{H}\}. \tag{8.4.17}$$

Indeed, since $\delta\tilde{H}/\delta\mathbf{E} = \mathbf{E}$, $\operatorname{curl}(\delta\tilde{H}/\delta\mathbf{B}) = \operatorname{curl}\mathbf{B}$, we have

$$\begin{aligned}\{\tilde{F}, \tilde{H}\} &= \int_{\mathbb{R}^3} \left(\frac{\delta\tilde{F}}{\delta\mathbf{E}} \cdot \operatorname{curl}\mathbf{B} - \mathbf{E} \cdot \operatorname{curl} \frac{\delta\tilde{F}}{\delta\mathbf{B}} \right) d^3x \\ &= \int_{\mathbb{R}^3} \left(\frac{\delta\tilde{F}}{\delta\mathbf{E}} \cdot \operatorname{curl}\mathbf{B} - \mathbf{E} \cdot \operatorname{curl} \frac{\delta\tilde{F}}{\delta\mathbf{B}} \right) d^3x\end{aligned}$$

Moreover,

$$\begin{aligned}\tilde{F} &= \mathbf{D}\tilde{F}(\mathbf{B}) \cdot \dot{\mathbf{B}} + \mathbf{D}\tilde{F}(\mathbf{E}) \cdot \dot{\mathbf{E}} \\ &= \int_{\mathbb{R}^3} \left(\frac{\delta'\tilde{F}}{\delta\mathbf{B}} \cdot \dot{\mathbf{B}} + \frac{\delta\tilde{F}}{\delta\mathbf{E}} \cdot \dot{\mathbf{E}} \right) d^3x \\ &= \int_{\mathbb{R}^3} \left(\frac{\delta\tilde{F}}{\delta\mathbf{B}} \cdot \dot{\mathbf{B}} + \frac{\delta\tilde{F}}{\delta\mathbf{E}} \cdot \dot{\mathbf{E}} \right) d^3x,\end{aligned}$$

where $\delta'\tilde{F}/\delta\mathbf{B}$ denotes the functional derivative of $\tilde{F}$ relative to $\mathbf{B}$ in $\mathcal{V}$, that is,

$$\mathbf{D}\tilde{F}(\mathbf{B}) \cdot \delta\mathbf{B} = \int_{\mathbb{R}^3} \frac{\delta'\tilde{F}}{\delta\mathbf{B}} \cdot \delta\mathbf{B}\, d^3x. \tag{8.4.18}$$

The last equality in the formula for $\dot{\tilde{F}}$ is proved in the following way. Recall that $\delta\tilde{F}/\delta\mathbf{B}$ is the functional derivative of an arbitrary extension of $\tilde{F}$ computed at $\mathbf{B}$, that is,

$$\mathbf{D}\tilde{F}(\mathbf{B}) \cdot \mathbf{Z} = \int \frac{\delta\tilde{F}}{\delta\mathbf{B}} \cdot \mathbf{Z}\, d^3x \quad \text{for any } \mathbf{Z} \in \mathfrak{X}(\mathbb{R}^3);$$

therefore since $\delta\mathbf{B}$ is a curl, this implies $\delta'\tilde{F}/\delta\mathbf{B}$ and $\delta\tilde{F}/\delta\mathbf{B}$ differ by a gradient which is L^2-orthogonal to $\dot{\mathbf{B}}$, since $\dot{\mathbf{B}}$ is divergence free (again by the Helmholtz–Hodge decomposition). Therefore equation (8.4.11) holds if and only if equation (8.4.17) does. ♦

The Lie–Poisson bracket. We next turn to the most important example of a Poisson manifold which is not symplectic. Let $\mathfrak{g}$ denote a Lie algebra that is, a vector space with a pairing $[\xi, \eta]$ of elements of $\mathfrak{g}$ that is bilinear, antisymmetric and satisfies Jacobi's identity. Let $\mathfrak{g}^*$ denote its "dual", that is, a vector space weakly paired with $\mathfrak{g}$ via $\langle\,,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$. If $\mathfrak{g}$ is finite dimensional, we take this pairing to be the usual action of forms on vectors.

8.4.3 Definition. *For $F, G : \mathfrak{g}^* \to \mathbb{R}$, define the $(\pm)$* ***Lie–Poisson brackets*** *by*

$$\{F, G\}_\pm(\mu) = \pm \left\langle \mu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu} \right] \right\rangle \tag{8.4.19}$$

where $\mu \in \mathfrak{g}^*$ and $\delta F/\delta\mu$, $\delta G/\delta\mu \in \mathfrak{g}$ are the functional derivatives of F and G, that is, $\mathbf{D}F(\mu) \cdot \delta\mu = \langle \delta\mu, \delta F/\delta\mu \rangle$.

If $\mathfrak{g}$ is finite dimensional with a basis ξ_i, and the structure constants are defined by

$$[\xi_i, \xi_j] = c_{ij}^k \xi_k,$$

the Lie–Poisson bracket is

$$\{F, G\} = \pm \mu_i^j c_{jk}^i \frac{\delta F}{\delta\mu_j} \frac{\delta G}{\delta\mu_k}.$$

8.4.4 Theorem (Lie–Poisson Theorem). *The dual space $\mathfrak{g}^*$ with the $(\pm)$ Lie–Poisson bracket is a Poisson manifold.*

Proof. Clearly $\{\,,\}_\pm$ is bilinear and skew symmetric. To show $\{\,,\}_\pm$ is a derivation in each argument, we show that

$$\frac{\delta(FG)}{\delta\mu} = F(\mu)\frac{\delta G}{\delta\mu} + G(\mu)\frac{\delta F}{\delta\mu}. \tag{8.4.20}$$

To prove (8.4.20), let $\delta\mu \in \mathfrak{g}^*$ be arbitrary. Then

$$\begin{aligned}\left\langle \delta\mu, \frac{\delta(FG)}{\delta\mu}\right\rangle &= \mathbf{D}(FG)(\mu)\cdot\delta\mu \\ &= F(\mu)\mathbf{D}G(\mu)\cdot\delta\mu + G(\mu)\mathbf{D}F(\mu)\cdot\delta\mu \\ &= \left\langle \delta\mu, F(\mu)\frac{\delta G}{\delta\mu} + G(\mu)\frac{\delta F}{\delta\mu}\right\rangle.\end{aligned}$$

Finally, we prove the Jacobi identity. We start by computing the derivative of the map $\mu \in \mathfrak{g}^* \mapsto \delta F/\delta\mu \in \mathfrak{g}$. We have for every $\lambda, \nu \in \mathfrak{g}^*$

$$\mathbf{D}\left(\left\langle \nu, \frac{\delta F}{\delta\mu}\right\rangle\right)(\mu)\cdot\lambda = \mathbf{D}(\mathbf{D}F(\cdot)\cdot\nu)(\mu)\cdot\lambda = \mathbf{D}^2F(\mu)(\nu,\lambda),$$

that is,

$$\mathbf{D}\left(\frac{\delta F}{\delta\mu}\right)(\mu)\cdot\lambda = \mathbf{D}^2F(\mu)(\lambda,\cdot). \tag{8.4.21}$$

Therefore the derivative of $\mu \mapsto \left[\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}\right]$ is

$$\mathbf{D}\left[\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}\right](\mu)\cdot\nu = \left[\mathbf{D}^2F(\mu)(\nu,\cdot), \frac{\delta G}{\delta\mu}\right] + \left[\frac{\delta F}{\delta\mu}, \mathbf{D}^2G(\mu)(\nu,\cdot)\right] \tag{8.4.22}$$

where $\mathbf{D}^2F(\mu)(\nu,\cdot) \in L(\mathfrak{g}^*,\mathbb{R})$ is assumed to be represented via $\langle\,,\rangle$ by an element of $\mathfrak{g}$. Therefore by (8.4.19) and (8.4.22)

$$\begin{aligned}\left\langle \nu, \frac{\delta}{\delta\mu}\{F,G\}\right\rangle &= \mathbf{D}\{F,G\}(\mu)\cdot\nu \\ &= \left\langle \nu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}\right]\right\rangle + \left\langle \mu, \left[\mathbf{D}^2F(\mu)(\nu,\cdot), \frac{\delta G}{\delta\mu}\right]\right\rangle \\ &\quad + \left\langle \mu, \left[\frac{\delta F}{\delta\mu}, \mathbf{D}^2G(\mu)(\nu,\cdot)\right]\right\rangle \\ &= \left\langle \nu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}\right]\right\rangle - \left\langle \operatorname{ad}\left(\frac{\delta G}{\delta\mu}\right)^*\mu, \mathbf{D}^2F(\mu)(\nu,\cdot)\right\rangle \\ &\quad + \left\langle \operatorname{ad}\left(\frac{\delta F}{\delta\mu}\right)^*\mu, \mathbf{D}^2G(\mu)(\nu,\cdot)\right\rangle,\end{aligned}$$

where $\operatorname{ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$ is the linear map $\operatorname{ad}(\xi)\cdot\eta = [\xi,\eta]$ and $\operatorname{ad}(\xi)^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is its dual defined by

$$\langle \operatorname{ad}(\xi)^*\mu, \eta\rangle = \langle \mu, [\xi,\nu]\rangle, \quad \eta \in \mathfrak{g},\ \mu \in \mathfrak{g}^*.$$

Therefore,

$$\begin{aligned}\left\langle \nu, \frac{\delta}{\delta \nu}\{F,G\}\right\rangle &= \left\langle \nu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle - \left\langle \nu, \mathbf{D}^2 F(\mu)\left(\operatorname{ad}\left(\frac{\delta G}{\delta \mu}\right)^* \mu, \cdot\right)\right\rangle \\ &\quad + \left\langle \nu, \mathbf{D}^2 G(\mu)\left(\operatorname{ad}\left(\frac{\delta F}{\delta \mu}\right)^* \mu, \cdot\right)\right\rangle \\ \frac{\delta}{\delta \nu}\{F,G\} &= \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] - \mathbf{D}^2 F(\mu)\left(\operatorname{ad}\left(\frac{\delta G}{\delta \mu}\right)^* \mu, \cdot\right) \\ &\quad + \mathbf{D}^2 G(\mu)\left(\operatorname{ad}\left(\frac{\delta F}{\delta \mu}\right)^* \mu, \cdot\right),\end{aligned} \tag{8.4.23}$$

which in turn implies

$$\begin{aligned}\{\{F,G\},(\mu)H\} &= \left\langle \mu, \left[\frac{\delta}{\delta \mu}\{F,G\}, \frac{\delta H}{\delta \mu}\right]\right\rangle = \left\langle \mu, \left[\left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right], \frac{\delta H}{\delta \mu}\right]\right\rangle \\ &\quad + \mathbf{D}^2 F(\mu)\left(\operatorname{ad}\left(\frac{\delta G}{\delta \mu}\right)^* \mu, \operatorname{ad}\left(\frac{\delta H}{\delta \mu}\right)^* \mu\right) \\ &\quad - \mathbf{D}^2 G(\mu)\left(\operatorname{ad}\left(\frac{\delta F}{\delta \mu}\right)^* \mu, \operatorname{ad}\left(\frac{\delta H}{\delta \mu}\right)^* \mu\right),\end{aligned}$$

The two cyclic permutations in F, G, H added to the above formula sum up to zero: all six terms involving second derivatives cancel and the three first terms add up to zero by the Jacobi identity for the bracket of $\mathfrak{g}$. ■

8.4.5 Example (The Free Rigid Body)**.** The equations of motion of the free rigid body described by an observer fixed on the moving body are given by ***Euler's equation***

$$\dot{\Pi} = \Pi \times \omega, \tag{8.4.24}$$

where Π, $\omega \in \mathbb{R}^3$, $\Pi_i = I_i \omega_i$, $i = 1,2,3$, $I = (I_1, I_2, I_3)$ are the principal moments of inertia, the coordinate system in the body is chosen so that the axes are the principal axes, ω is the angular velocity in the body, and Π is the angular momentum in the body. It is straightforward to check that the kinetic energy

$$H(\Pi) = \frac{1}{2}\Pi \cdot \omega \tag{8.4.25}$$

is a conserved quantity for equation (8.4.24).

We shall prove below that (8.4.24) *are Hamilton's equations with Hamiltonian* (8.4.25) *relative to a* $(-)$ *Lie–Poisson structure* on $\mathbb{R}^3$.

The vector space $\mathbb{R}^3$ is in fact a Lie algebra with respect to the bracket operation given by the cross product, that is, $[\mathbf{x}, \mathbf{y}] = \mathbf{x} \times \mathbf{y}$. (This is the structure that it inherits from the rotation group.) We pair $\mathbb{R}^3$ with itself using the usual dot-product, that is, $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x} \cdot \mathbf{y}$. Therefore, if $F : \mathbb{R}^3 \to \mathbb{R}$, $\delta F/\delta \Pi = \nabla F(\Pi)$. Thus, the $(-)$ Lie–Poisson bracket is given via equation (8.4.19) by the triple product

$$\{F,G\}(\Pi) = -\Pi \cdot (\nabla F(\Pi) \times \nabla G(\Pi)). \tag{8.4.26}$$

Since $\delta H/\delta \Pi = \omega$, we see that for any $F : \mathbb{R}^3 \to \mathbb{R}$,

$$\begin{aligned}\frac{d}{dt}(F(\Pi)) &= DF(\Pi) \cdot \dot{\Pi} = \dot{\Pi} \cdot \nabla F(\Pi) = -\Pi \cdot (\nabla F(\Pi) \times \omega) \\ &= \nabla F(\Pi) \cdot (\Pi \times \omega)\end{aligned}$$

so that $\dot{F} = \{F, H\}$ for any $F : \mathbb{R}^3 \to \mathbb{R}$ is equivalent to Euler's equations of motion (8.4.24).

The following result, due to Pauli [1953], Martin [1959], Arnol'd [1966], **?** summarizes the situation.

8.4.6 Proposition. *Euler's equations* (8.4.24) *for a free rigid body are a Hamiltonian system in* $\mathbb{R}^3$ *relative to the* $(-)$ *Lie Poisson bracket* (8.4.24) *and Hamiltonian function* (8.4.25).

♦

8.4.7 Example (Ideal Incompressible Homogeneous Fluid Flow)**.** In §8.2 we have shown that the equations of motion for an ideal incompressible homogeneous fluid in a region $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$ are given by ***Euler's equations of motion***

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p \tag{8.4.27a}$$

$$\operatorname{div} \mathbf{v} = 0 \tag{8.4.27b}$$

$$\mathbf{v}(t, x) \in T_x(\partial\Omega) \text{ for } x \in \partial\Omega \tag{8.4.27c}$$

with initial condition $\mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x})$, a given vector field on Ω. Here $\mathbf{v}(t, \mathbf{x})$ is the Eulerian or spatial velocity, a time dependent vector field on Ω. The pressure p is a function of $\mathbf{v}$ and is uniquely determined by $\mathbf{v}$ (up to a constant) by the Neumann problem (take div and the dot product with $\mathbf{n}$ of the first equation in (8.4.27))

$$\Delta p = -\operatorname{div}((\mathbf{v} \cdot \nabla)\mathbf{v}) \tag{8.4.28a}$$

$$\frac{\partial p}{\partial \mathbf{n}} = \nabla p \cdot \mathbf{n} = -((\mathbf{v} \cdot \nabla)\mathbf{v}) \cdot \mathbf{n} \text{ on } \partial\Omega, \tag{8.4.28b}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. The kinetic energy

$$H(\mathbf{v}) = \frac{1}{2} \int_\Omega \|\mathbf{v}\|^2 d^3x \tag{8.4.29}$$

has been shown in §8.2 to be a conserved quantity for (8.4.27). We shall prove below that the first equation in (8.4.27) is Hamiltonian relative to a $(+)$ Lie–Poisson bracket with Hamiltonian function given by (8.4.29).

Consider the Lie algebra $\mathfrak{X}_{\text{div}}(\Omega)$ of divergence free vector fields on Ω tangent to $\partial\Omega$ with bracket given by *minus* the bracket of vector fields, that is, for $\mathbf{u}, \mathbf{v} \in \mathfrak{X}_{\text{div}}(\Omega)$ define

$$[\mathbf{u}, \mathbf{v}] = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v}. \tag{8.4.30}$$

(The reason for this strange choice comes from the fact that the usual Lie bracket for vector fields is the right Lie algebra bracket of the diffeomorphism group of Ω.) Now pair $\mathfrak{X}_{\text{div}}(\Omega)$ with itself via the L^2-pairing. As in Example 8.4.2, using the Hodge–Helmholtz decomposition, it follows that this pairing is weakly nondegenerate. In particular

$$\frac{\delta H}{\delta \mathbf{v}} = \mathbf{v}. \tag{8.4.31}$$

The $(+)$ Lie–Poisson bracket on $\mathfrak{X}_{\text{div}}(\Omega)$ is

$$\{F, G\}(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot \left[\left(\frac{\delta G}{\delta \mathbf{v}} \cdot \nabla \right) \frac{\delta F}{\delta \mathbf{v}} - \left(\frac{\delta F}{\delta \mathbf{v}} \cdot \nabla \right) \frac{\delta G}{\delta \mathbf{v}} \right] d^3x. \tag{8.4.32}$$

Therefore, for any $\mathfrak{X}_{\text{div}}(\Omega) \to \mathbb{R}$, we have

$$\begin{aligned} \frac{d}{dt}(F(\mathbf{v})) = \mathbf{D}F(\mathbf{v}) \cdot \dot{\mathbf{v}} &= \int_\Omega \frac{\delta F}{\delta \mathbf{v}} \cdot \dot{\mathbf{v}} d^3x \\ &= \int_\Omega \mathbf{v} \cdot \left[(\mathbf{v} \cdot \nabla) \frac{\delta F}{\delta \mathbf{v}} - \left(\frac{\delta F}{\delta \mathbf{v}} \cdot \nabla \right) \mathbf{v} \right] d^3x \\ &= \int_\Omega \mathbf{v} \cdot \left((\mathbf{v} \cdot \nabla) \frac{\delta F}{\delta \mathbf{v}} \right) d^3x - \int_\Omega \frac{\partial F}{\partial \mathbf{v}} \cdot \nabla \left(\frac{1}{2} \|\mathbf{v}\|^2 \right) d^3x. \end{aligned}$$

To handle the first integral, observe that if

$$f, g : \Omega \to \mathbb{R} \quad \text{then} \quad \operatorname{div}(fg\mathbf{v}) = f \operatorname{div}(g\mathbf{v}) + g\mathbf{v} \cdot \nabla f,$$

so that by Stokes' theorem and $\mathbf{v} \cdot \mathbf{n} = 0$, $\operatorname{div} \mathbf{v} = 0$, we get

$$\begin{aligned}\int_\Omega g\mathbf{v} \cdot \nabla f d^3x &= \int_{\partial\Omega} fg\, \mathbf{v} \cdot n\, dS - \int_\Omega f \operatorname{div}(g\, \mathbf{v}) d^3x \\ &= -\int_\Omega f\mathbf{v} \cdot \nabla g\, d^3x.\end{aligned}$$

Applying the above relation to $g = v^i$, $f = \delta F/\delta \mathbf{v}^i$, and summing over $i = 1, 2, 3$ we get

$$\int_\Omega \mathbf{v} \cdot \left((\mathbf{v} \cdot \nabla) \frac{\delta F}{\delta v} \right) d^3x = -\int_\Omega \frac{\delta F}{\delta \mathbf{v}} \cdot ((\mathbf{v} \cdot \nabla)\mathbf{v}) d^3x$$

so that $\dot{F} = \{F, H\}$ reads

$$\int_\Omega \frac{\delta F}{\delta \mathbf{v}} \cdot \dot{\mathbf{v}}\, d^3x = -\int_\Omega \frac{\delta F}{\delta \mathbf{v}} \cdot \left[(\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{2} \nabla \|\mathbf{v}\|^2 \right] d^3x \tag{8.4.33}$$

for any $F : \mathfrak{X}_{\mathrm{div}}(\Omega) \to \mathbb{R}$. One would like to conclude from here that the coefficients of $\delta F/\delta \mathbf{v}$ on both sides of equation (8.4.33) are equal. This conclusion, however, is incorrect, since

$$(\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{2} \nabla \|\mathbf{v}\|^2$$

is not divergence free. Thus, applying the Hodge–Helmholtz decomposition, write

$$(\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{2} \nabla \|\mathbf{v}\|^2 = \mathbf{X} - \nabla f \tag{8.4.34}$$

where $X \in \mathfrak{X}_{\mathrm{div}}(\Omega)$ and f is determined by

$$\begin{aligned}\Delta \left(f + \frac{1}{2} \|\mathbf{v}\|^2 \right) &= -\operatorname{div}((\mathbf{v} \cdot \nabla)\mathbf{v}), \text{ and} \\ \frac{\partial}{\partial \mathbf{n}} \left(f + \frac{1}{2} \|\mathbf{v}\|^2 \right) &= -((\mathbf{v} \cdot \nabla) \cdot \mathbf{v}) \cdot \mathbf{n}\end{aligned}$$

which coincides with equation (8.4.28), that is,

$$f + \frac{1}{2} \|\mathbf{v}\|^2 = \mathbf{p} + \text{constant}. \tag{8.4.35}$$

Moreover, since

$$\int_\Omega \frac{\delta F}{\delta \mathbf{v}} \cdot \nabla f d^3x = \int_{\partial\Omega} f \frac{\partial F}{\partial \mathbf{v}} \cdot \mathbf{n}\, dS - \int_\Omega f \operatorname{div} \frac{\delta F}{\delta \mathbf{v}} d^3x = 0$$

we have from equations (8.4.33), (8.4.34), (8.4.35)

$$\frac{\dot{\partial} \mathbf{v}}{\partial t} = -\mathbf{X} = -(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla p$$

which is the first equation in equation (8.4.27). We have thus proved the following result (see Arnol'd [1966], Marsden and Weinstein [1983]).

8.4.8 Proposition. *Euler's equations* (8.4.27) *are a Hamiltonian system on* $\mathfrak{X}_{\rm div}(\Omega)$ *relative to the* (+) *Lie–Poisson bracket* (8.4.32) *and Hamiltonian function given by* (8.4.29).

♦

8.4.9 Example (The Poisson–Vlasov Equation)**.** We consider a collisionless plasma consisting (for notational simplicity) of only one species of particles with charge q and mass m moving in Euclidean space $\mathbb{R}^3$ with positions $\mathbf{x}$ and velocities $\mathbf{v}$. Let $f(\mathbf{x}, \mathbf{v}, t)$ be the plasma density in the plasma space at time t. In the Coulomb or electrostatic case in which there is no magnetic field, the motion of the plasma is described by the Poisson–Vlasov equations which are the (collisionless) Boltzmann equations for the density function f and the Poisson equation for the scalar potential φ_f:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{q}{m} \frac{\partial \varphi_f}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \tag{8.4.36}$$

$$\Delta \varphi_f = -q \int f(\mathbf{x}, \mathbf{v}) d^3\mathbf{v} = \rho_f, \tag{8.4.37}$$

where $\partial/\partial\mathbf{x}$ and $\partial/\partial\mathbf{v}$ denote the gradients in $\mathbb{R}^3$ relative to the $\mathbf{x}$ and $\mathbf{v}$ variables, ρ_f is the charge density in physical space, and Δ is the Laplacian. Equation (8.4.36) can be written in "Hamiltonian" form

$$\frac{\partial f}{\partial t} + \{f, \mathcal{H}\} = 0, \tag{8.4.38}$$

where $\{\,,\}$ is the canonical Poisson bracket on phase space, namely,

$$\{f, g\} = \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\partial g}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{x}} = \frac{1}{m} \left[\frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\partial g}{\partial \mathbf{v}} - \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{\partial g}{\partial \mathbf{x}} \right], \tag{8.4.39}$$

where $\mathbf{p} = m\mathbf{v}$ and

$$\mathcal{H} = \mathcal{H}_f = m\|\mathbf{v}\|^2 + q\varphi_f$$

is the single particle energy, called the *self–consistent Hamiltonian.* Indeed,

$$\begin{aligned}\{\mathcal{H}_f, f\} &= \frac{1}{m}\left(\frac{\partial \mathcal{H}_f}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} - \frac{\partial \mathcal{H}_f}{\partial \mathbf{v}} \cdot \frac{\partial f}{\partial \mathbf{x}} \right) = \frac{1}{m}\left(q\frac{\partial \varphi_f}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} - m\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} \right) \\ &= \frac{q}{m} \frac{\partial \varphi_f}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} - \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} = \frac{\partial f}{\partial t}\end{aligned}$$

according to equation (8.4.36). There is another very useful way to think of the evolution of f. If $F(f)$ is any functional of the density function f and f evolves according to the Poisson–Vlasov equations (8.4.36) (or (8.4.38) equivalently) then F evolves in time by

$$\dot{F} = \{F, H\}_+$$

where $\{\,,\}_+$ is a (+) Lie–Poisson bracket (to be defined) of functionals and H is the total energy. Let us state this more precisely. Let

$$V = \left\{ f \in C^k(\mathbb{R}^6) \mid f \to 0 \,\text{as}\, \|\mathbf{x}\| \to \infty,\ \|\mathbf{v}\| \to \infty \right\}$$

with the L^2-pairing $\langle\,,\rangle : V \times V \to \mathbb{R}$;

$$\langle f, g\rangle = \int f(\mathbf{x}, \mathbf{v}) g(\mathbf{x}, \mathbf{v}) d^3x d^3v.$$

If $F : V \to \mathbb{R}$ is differentiable at $f \in V$, the functional $\delta F/\delta f$ is, by definition, the unique element $\delta F/\delta f \in V$ such that

$$\mathbf{D}F(f) \cdot g = \left\langle \frac{\delta F}{\delta f}, g \right\rangle = \int \frac{\delta F}{\delta f}(\mathbf{x}, \mathbf{v}) g(\mathbf{x}, \mathbf{v})\, d^3x\, d^3v.$$

The vector space V is a Lie algebra relative to the canonical Poisson bracket (8.4.39) on $\mathbb{R}^6$. For two functionals $F, G : V \to \mathbb{R}$ their (+) Lie–Poisson bracket $\{F, G\}_+ : V \to \mathbb{R}$ is then given by

$$\{F, G\}_+(\mathbf{x}, \mathbf{v}) = \int f(\mathbf{x}, \mathbf{v}) \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} (\mathbf{x}, \mathbf{v})\, d^3x\, d^3v,$$

where $\{\,,\}$ is the canonical bracket (8.4.39). For any $f, g, h \in V$ we have the formula

$$\int f\{g, h\}\, d^3x\, d^3v = \int g\{h, f\}\, d^3x\, d^3v. \tag{8.4.40}$$

Indeed by integration by parts, we get

$$\begin{aligned}
\int f\{g, h\}\, d^3x\, d^3v &= \frac{1}{m} \int f \frac{\partial g}{\partial \mathbf{x}} \cdot \frac{\partial h}{\partial \mathbf{v}}\, d^3x\, d^3v - \frac{1}{m} \int f \frac{\partial h}{\partial \mathbf{x}} \cdot \frac{\partial g}{\partial \mathbf{v}}\, d^3x\, d^3v \\
&= -\frac{1}{m} \int \frac{\partial f}{\partial \mathbf{x}} \cdot g \frac{\partial h}{\partial \mathbf{v}}\, d^3x\, d^3v + \frac{1}{m} \int g \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{\partial h}{\partial \mathbf{x}} g\, d^3x\, d^3v \\
&= \frac{1}{m} \int g \left(\frac{\partial h}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} - \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\partial h}{\partial \mathbf{v}} \right) d^3x\, d^3v \\
&= \int g\{h, f\}\, d^3x\, d^3v.
\end{aligned}$$

We have the following results of Iwínski and Turski [1976], **?** and Morrison [1980].

8.4.10 Proposition. *Densities $f \in V$ evolve according to the Poisson–Vlasov equation* (8.4.38) *if and only if any differentiable function $F : V \to \mathbb{R}$ having functional derivative $\delta F/\delta f$ evolves by the* (+) *Lie–Poisson equation*

$$\dot{F}(f) = \{F, H\}_+(f) \tag{8.4.41}$$

with the Hamiltonian $H : V \to \mathbb{R}$ equal to the total energy

$$H(f) = \frac{1}{2} \int m\|\mathbf{v}\|^2 f(\mathbf{x}, \mathbf{v})\, d^3x\, d^3v + \int \frac{1}{2} \varphi_f(\mathbf{x})\, d^3x.$$

Proof. First we compute $\delta H/\delta f$ using the definition

$$\mathbf{D}H(f) \cdot \delta f = \int \frac{\delta H}{\delta f} \delta f.$$

Note that the first term of $H(f)$ is linear in f and the second term is

$$\frac{1}{2} \int \varphi_f \rho_f\, d^3x = \frac{1}{2} \int \|\nabla \varphi_f\|^2\, d^3x$$

since $\Delta \varphi_f = -\rho_f$. Using the chain rule and integration by parts, we get

Break overflow?

$$
\begin{aligned}
\mathbf{D}H(f)\cdot\delta f &= \frac{1}{2}\int m\|\mathbf{v}\|^2\delta f\, d^3x\, d^3v + \int(\nabla\varphi_f)(\mathbf{D}(\nabla\varphi_f))\delta f\, d^3x\\
&= \frac{1}{2}\int m\|\mathbf{v}\|^2\delta f\, d^3x\, d^3v - \int \varphi_f \mathbf{D}(\Delta\varphi_f))\delta f\, d^3x\\
&= \frac{1}{2}\int m\|\mathbf{v}\|^2\delta f\, d^3x\, d^3v + \int \varphi_f\left(\mathbf{D}\left(q\int f\, d^3v\right)\right)(f)\delta f\, d^3x\\
&= \frac{1}{2}\int m\|\mathbf{v}\|^2\delta f\, d^3x\, d^3v + \int \varphi_f q\delta f\, d^3v.
\end{aligned}
$$

Therefore,

$$\frac{\delta H}{\delta f} = \frac{1}{2}m\|\mathbf{v}\|^2 + q\varphi_f = \mathcal{H}_f.$$

We have

$$\dot F(f) = \mathbf{D}F(f)\cdot \dot f = \int \frac{\delta H}{\delta f}\dot f\, d^3x\, d^3v$$

and

$$
\begin{aligned}
\int \frac{\delta F}{\delta f}\{\mathcal{H}_f, f\}\, d^3x\, d^3v &= \int \frac{\delta F}{\delta f}\left\{\frac{\delta H}{\delta f}, f\right\} d^3x\, d^3v\\
&= \int f\left\{\frac{\delta F}{\delta f}, \frac{\delta H}{\delta f}\right\} d^3x\, d^3v = \{F,H\}_+(f).
\end{aligned}
$$

by (8.4.40). Thus (8.4.41), for any F having functional derivatives, is equivalent to (8.4.38). ■

♦

8.4.11 Example (The Maxwell–Vlasov Equations)**.** We consider a plasma consisting of particles with charge q_1 and mass m moving in Euclidean space $\mathbb{R}^3$ with positions $\mathbf{x}$ and velocities $\mathbf{v}$. For simplicity we consider only one species of particle; the general case is similar. Let $f(\mathbf{x},\mathbf{v},t)$ be the plasma density at time t, $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ be the electric and magnetic fields. The ***Maxwell–Vlasov equations*** are:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f}{\partial \dot{\mathbf{v}}} = 0, \tag{8.4.42a}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\operatorname{curl}\mathbf{E}, \tag{8.4.42b}$$

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \operatorname{curl}\mathbf{B} - \frac{q}{c}\int \mathbf{v} f(\mathbf{x},\mathbf{v},t)\, d^3v, \tag{8.4.42c}$$

together with the non evolutionary equations

$$\operatorname{div}\mathbf{E} = \rho_f,\ \text{ where } \rho_f = q\int f(\mathbf{x},\mathbf{v},t)\, d^3v, \tag{8.4.43a}$$

$$\operatorname{div}\mathbf{B} = 0. \tag{8.4.43b}$$

Letting $c\to\infty$ leads to the *Poisson–Vlasov equation* (8.4.36)

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - \frac{q}{m}\frac{\partial \varphi_f}{\partial \mathbf{x}}\cdot\frac{\partial f}{\partial \mathbf{v}} = 0,$$

where $\Delta\varphi_f = -\rho_f$. In what follows we shall set $q = m = c = 1$. The Hamiltonian for the Maxwell–Vlasov system is

$$H(f, \mathbf{E}, \mathbf{B}) = \int \frac{1}{2}\|\mathbf{v}\|^2 f(\mathbf{x}, \mathbf{v}, t)\, dx\, dv + \int \frac{1}{2}[\|E(\mathbf{x}, t)\|^2 + \|B(\mathbf{x}, t)\|^2]\, d^3x. \tag{8.4.44}$$

Let $\mathcal{V} = \{\operatorname{curl}\mathbf{Z} \mid \mathbf{Z} \in \mathfrak{X}(\mathbb{R}^3)\}$. We have the following result of Iwínski and Turski [1976], Morrison [1980], and Marsden and Weinstein [1982]

8.4.12 Theorem. (i) *The manifold $\mathcal{F}(\mathbb{R}^6) \times \mathfrak{X}(\mathbb{R}^3) \times \mathcal{V}$ is a Poisson manifold relative to the bracket*

$$\begin{aligned}\{F, G\}(f, \mathbf{E}, B) = &\int f \left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} d^3x\, d^3v \\ &+ \int \left(\frac{\delta G}{\delta \mathbf{B}} \cdot \operatorname{curl} \frac{\delta G}{\delta \mathbf{B}} - \frac{\delta G}{\delta \mathbf{E}} \cdot \operatorname{curl} \frac{\delta F}{\delta \mathbf{B}}\right) d^3x \\ &+ \int \left(\frac{\delta F}{\delta \mathbf{E}} \cdot \frac{\partial f}{\partial \mathbf{v}} \frac{\delta G}{\delta f} - \frac{\delta G}{\delta \mathbf{E}} \cdot \frac{\partial f}{\partial \mathbf{v}} \frac{\delta F}{\delta f}\right) d^3x d^3v \\ &+ \int f\mathbf{B} \cdot \left(\frac{\partial}{\partial \mathbf{v}} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial \mathbf{v}} \frac{\delta G}{\delta f}\right) d^3x\, d^3v.\end{aligned} \tag{8.4.45}$$

(ii) *The equations of motion* (8.4.42) *are equivalent to*

$$\dot{F} = \{F, H\} \tag{8.4.46}$$

where F is any locally defined function with functional derivatives and $\{\,,\}$ is given by 8.4.44.

Proof. Part (i) follows from general considerations on reduction (see Marsden and Weinstein [1982]). The direct verification is laborious but straightforward, if one recognizes that the first two terms are the Poisson bracket for the Poisson–Vlasov equation and the Born–Infeld bracket respectively.

(ii) Since

$$\frac{\delta H}{\delta f} = \frac{1}{2}\|\mathbf{v}\|^2, \quad \frac{\delta H}{\delta \mathbf{E}} = \mathbf{E}, \quad \text{and} \quad \operatorname{curl} \frac{\delta H}{\delta \mathbf{B}} = \operatorname{curl} \mathbf{B},$$

we have, by equation (8.4.40) and integration by parts in the fourth integral,

$$\begin{aligned}\{F, H\} &= \int f \left\{\frac{\delta F}{\delta f}, \frac{1}{2}\|\mathbf{v}\|^2\right\} d^3x\, d^3v + \int \left(\frac{\delta F}{\delta \mathbf{B}} \cdot \operatorname{curl} \mathbf{B} - \mathbf{E} \cdot \operatorname{curl} \frac{\delta F}{\delta \mathbf{B}}\right) d^3x \\ &\quad + \int \left(\frac{\delta F}{\delta \mathbf{E}} \cdot \frac{\partial f}{\partial \mathbf{v}} \frac{1}{2}\|\mathbf{v}\|^2 - \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} \frac{\delta F}{\delta \mathbf{E}}\right) d^3x\, d^3v \\ &\quad + \int f\mathbf{B} \left(\frac{\partial}{\partial \mathbf{v}} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial \mathbf{v}} \left(\frac{1}{2}\|\mathbf{v}\|^2\right)\right) d^3x\, d^3v \\ &= \int \frac{\delta F}{\delta f} \left[\left\{\frac{1}{2}\|\mathbf{v}\|^2, f\right\} - F \cdot \frac{\delta f}{\delta \mathbf{v}} - \operatorname{div}_{\mathbf{v}}(\mathbf{v} \times f\mathbf{B})\right] d^3x\, d^3v \\ &\quad + \int \left(\frac{\delta F}{\delta \mathbf{E}} \cdot \left[\operatorname{curl} \mathbf{B} + \int \frac{\partial f}{\partial \mathbf{v}} \frac{1}{2}\|\mathbf{v}\|^2 d^3v\right]\right) d^3x - \int \mathbf{E} \cdot \frac{\delta F}{\delta \mathbf{B}} \operatorname{curl} d^3x\end{aligned}$$

where $\operatorname{div}_{\mathbf{v}}$ denotes the divergence only with respect to the $\mathbf{v}$-variable. Since

$$\left\{\frac{1}{2}\|\mathbf{v}\|^2, f\right\} = -\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}}, \quad \operatorname{div}_{\mathbf{v}}(\mathbf{v} \times f\mathbf{B}) = \frac{\partial f}{\partial \mathbf{v}} \cdot (\mathbf{v} \times \mathbf{B}),$$
$$\int \frac{\partial f}{\partial \mathbf{v}} \frac{1}{2}\|\mathbf{v}\|^2 d^3x = -\int \mathbf{v} f d^3x,$$

and

$$\int \mathbf{E} \cdot \operatorname{curl} \frac{\delta F}{\delta \mathbf{B}} d^3x = \int \frac{\delta F}{\delta \mathbf{B}} \cdot \operatorname{curl} \mathbf{E}\, d^3x,$$

we get

$$\begin{aligned} \{F, H\} = & \int \frac{\delta F}{\delta f} \left[-\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} \right] d^3x\, d^3v \\ & + \int \frac{\delta F}{\delta \mathbf{E}} \cdot \left(\operatorname{curl} \mathbf{B} - \int \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) d^3v \right) d^3v \\ & - \int \left(\frac{\delta F}{\delta \mathbf{B}} \cdot \operatorname{curl} \mathbf{E} \right) d^3x, \end{aligned} \tag{8.4.47}$$

and since

$$\dot{F} = \int \frac{\delta F}{\delta f} \cdot \dot{f}\, d^3x\, d^3v + \int \frac{\delta F}{\delta \mathbf{E}} \cdot \dot{\mathbf{E}}\, d^3x + \int \frac{\delta' F}{\delta \mathbf{B}} \cdot \dot{\mathbf{B}}\, d^3x$$

taking into account that $\delta F/\delta \mathbf{B}, \delta' F/\delta \mathbf{B}$ differ by a gradient (by equation (8.4.15)) which is L^2-orthogonal to $\mathcal{V}$ (of which both $\operatorname{curl} \mathbf{E}$ and $\mathbf{B}$ are a member), it follows from equation (8.4.47) that the equations (8.4.42)–(8.4.43) (with $q = c = m = 1$) are equivalent to (8.4.46). ■

♦

Exercises

◇ **8.4-1.** Find the symplectic form equivalent to the Born–Infeld bracket (8.4.16) on $\mathcal{V} \times \mathfrak{X}(\mathbb{R}^3)$.

◇ **8.4-2.** Show that the Hamiltonian vector field $X_H \in \mathfrak{X}(\mathfrak{g}^*)$ relative to the $(\pm)$ Lie–Poisson bracket is given by $X_H(\mu) = \mp \operatorname{ad}(\delta H/\delta\mu)^*\mu$.

◇ **8.4-3 (?).** Let $V = \{\, f : \mathbb{R} \to \mathbb{R} \mid f \text{ is } C^\infty,\ \lim_{|x| \to \infty} f(x) = 0 \,\}$.

(i) Show that the prescription

$$\{F, G\}(f) = \int_{-\infty}^{+\infty} \frac{\delta F}{\delta f} \frac{d}{dx} \frac{\delta G}{\delta f} dx$$

defines a Poisson bracket on V for appropriate functions F and G (be careful about what hypotheses you put on F and G).

(ii) Show that the Hamiltonian vector field of $H : V \to \mathbb{R}$ is given by

$$X_H(f) = \frac{d}{dx} \frac{\delta H}{\delta f}.$$

(iii) Let $H(f') = \int_{-\infty}^{+\infty} (f^3 + (1/2) f_x^2) dx$. Show that the differential equation for X_H is the ***Korteweg–deVries equation***:

$$f_t - 6 f f_x + f_{xxx} = 0.$$

◇ **8.4-4 (?).** Let $\mathfrak{g}$ be a Lie algebra and $\epsilon \in \mathfrak{g}^*$ be fixed. Show that the prescription

$$\{F, G\}_\epsilon(\mu) = \left\langle \epsilon, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle$$

defines a Poisson bracket on $\mathfrak{g}^*$.
Hint: Look at the formulas in the proof of Theorem 8.4.4.

◇ **8.4-5.** (i) (Pauli [1953], **?**). Let P be a finite dimensional Poisson manifold satisfying the following condition: If $f\{F, G\} = 0$ for any *locally* defined F implies $G =$ constant. Show that there exists an open dense set U in P such that the Poisson bracket restricted to U comes from a symplectic form on U.

Hint: Define $B : T^*P \times T^*P \to \mathbb{R}$ by $\mathbf{B}(\mathbf{d}F, \mathbf{d}G) = \{F, G\}$. Show first that

$$U = \{\, p \in P \mid B_p(\alpha, \beta) = 0 \text{ for all } \alpha \in T_p^*P \text{ implies } \beta_p = 0\,\}$$

is open and dense in P. Then show that B can be inverted at points in U.

(ii) Show that, in general, $U \neq P$ by the following example. On $\mathbb{R}^2$ define

$$\{F, G\}(x, y) = y\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x}\right)$$

Show that U in (i) is $\mathbb{R}^2\backslash\{\mathrm{O}x\text{-axis}\}$. Show that on U, the symplectic form generating the above Poisson bracket is $dx \wedge dy/y$.

8.5 Constraints and Control

The applications in this final section all involve the Frobenius theorem. Each example is necessarily treated briefly, but hopefully in enough detail so the interested reader can pursue the subject further by utilizing the given references.

Constaints. We start with the subject of *holonomic constraints in Hamiltonian systems*. A Hamiltonian system as discussed in §8.1 can have a condition imposed that limits the available points in phase space. Such a condition is a *constraint*. For example, a ball tethered to a string of unit length in $\mathbb{R}^3$ may be considered to be constrained only to move on the unit sphere S^2 (or possibly interior to the sphere if the string is collapsible). If the phase space is T^*Q and the constraints are all derivable from constraints imposed only on the configuration space (the q's), the constraints are called *holonomic*. For example, if there is one constraint $f(q) = 0$ for $f : Q \to \mathbb{R}$, the constraints on T^*Q can be simply obtained by differentiation: $\mathbf{d}f = 0$ on T^*Q. If the phase space is TQ, then the constraints are holonomic iff the constraints on the velocities are saying that the velocities are tangent to some constraint manifold of the positions. A constraint then can be thought of in terms of velocities as a subset $E \subset TM$. If it is a subbundle, this *constraint is thus holonomic iff it is integrable in the sense of Frobenius' theorem*.

Constraints that are not holonomic, are naturally called *nonholonomic constraints*. Holonomic constraints can be dealt with in the sense that one understands how to modify the equations of motion when the constraints are imposed, by adding *forces of constraint*, such as centrifugal force. See, for example Goldstein [1980, Chapter 1], and Abraham and Marsden [1978, Section 3.7]. We shall limit ourselves to the discussion of two examples of nonholonomic constraints. See for an extensive discussion and background.

A classical example of a nonholonomic system is a disk rolling without slipping on a plane. The disk of radius a is constrained to move without slipping on the (x, y)-plane. Let us fix a point P on the disk and call θ the angle between the radius at P and the contact point Q of the disk with the plane, as in Figure 8.5.1. Let (x, y, a) denote the coordinates of the center of the disk. Finally, if θ denotes the angle between the tangent line to the disk at Q and the x-axis, the position of the disk in space is completely determined by (x, y, θ, φ). These variables form elements of our configuration space $M = \mathbb{R}^2 \times S^1 \times S^1$. The condition that there is no slipping at Q means that the velocity at Q is zero; that is,

$$\frac{dx}{dt} + a\frac{d\theta}{dt}\cos\varphi = 0, \quad \frac{dy}{dt} + a\frac{d\theta}{dt}\sin\varphi = 0$$

(total velocity = velocity of center plus the velocity due to rotation by angular velocity $d\theta/dt$).

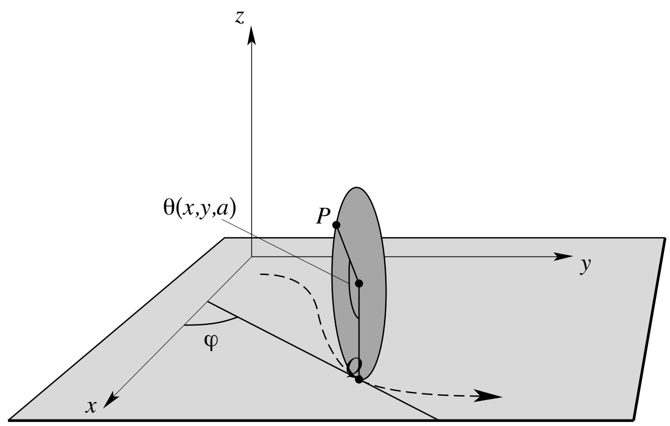


FIGURE 8.5.1. A rolling disk

These constraints may be written in terms of differential forms as $\omega_1 = 0$, $\omega_2 = 0$, where

$$\omega_1 = dx + a\cos\varphi d\theta \quad \text{and} \quad \omega_2 = dy + a\sin\varphi\, d\theta.$$

We compute that

$$\begin{aligned} \omega = \omega_1 \wedge \omega_2 &= dx \wedge dy + a\cos\varphi\, d\theta \wedge dy + a\sin\varphi\, dx \wedge d\theta, \\ \mathbf{d}\omega_1 &= -a\sin\varphi\, d\varphi \wedge d\theta, \\ \mathbf{d}\omega_2 &= a\cos\varphi\, d\varphi \wedge d\theta, \\ \mathbf{d}\omega_1 \wedge \omega &= -a\sin\varphi\, d\varphi \wedge d\theta \wedge dx \wedge dy, \\ \mathbf{d}\omega_2 \wedge \omega &= a\cos\varphi\, d\varphi \wedge d\theta \wedge dx \wedge dy. \end{aligned}$$

These do not vanish identically. Thus, according to Corollary 6.4.20, this system is not integrable and hence these constraints are nonholonomic.

A second example of constraints is due to Nelson [1967]. Consider the motion of a car and denote by (x, y) the coordinates of the center of the front axle, φ the angle formed by the moving direction of the car with the horizontal, and θ the angle formed by the front wheels with the car (Figure 8.5.2).

The configuration space of the car is $\mathbb{R}^2 \times \mathbb{T}^2$, parameterized by (x, y, φ, θ). We shall prove that the constraints imposed on this motion are nonholonomic. Call the vector field $X = \partial/\partial\theta$ ***steer***. We want to compute a vector field Y corresponding to ***drive***. Let the car be at the configuration point (x, y, φ, θ) and assume that it moves a small distance h in the direction of the front wheels. Notice that the car moves forward and simultaneously turns. Then the next configuration is

$$(x + h\cos(\varphi + \theta) + o(h), y + h\sin(\varphi + \theta) + o(h), \varphi + h\sin\theta + o(h), \theta).$$

Thus the "drive" vector field is

$$Y = \cos(\varphi + \theta)\frac{\partial}{\partial x} + \sin(\varphi + \theta)\frac{\partial}{\partial y} + \sin\theta\frac{\partial}{\partial\varphi}.$$

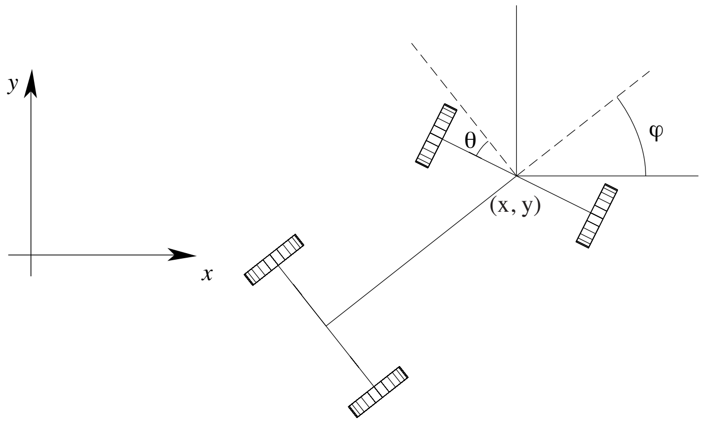


FIGURE 8.5.2. Automobile maneuvers

A direct computation shows that the vector field ***wriggle***,

$$W = [X, Y] = -\sin(\varphi + \theta)\frac{\partial}{\partial x} + \cos(\varphi + \theta)\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial \varphi},$$

and ***slide***,

$$S = [W, Y] = -\sin\varphi\frac{\partial}{\partial x} + \cos\varphi\frac{\partial}{\partial y},$$

satisfy

$$[X, W] = -Y, \quad [S, X] = 0, \quad [S, Y] = \sin\theta\cos\varphi\frac{\partial}{\partial x} + \sin\theta\sin\varphi\frac{\partial}{\partial y},$$

and

$$[S, W] = \cos\theta\cos\varphi\frac{\partial}{\partial x} + \cos\theta\sin\varphi\frac{\partial}{\partial y}.$$

Define the vector fields Z_1 and Z_2 by

$$Z_1 = [S, Y] = -W + (\cos\theta)S + \cos\theta\frac{\partial}{\partial \varphi},$$

$$Z_2 = [S, W] = Y - (\sin\theta)S - \sin\theta\frac{\partial}{\partial \varphi}.$$

A straightforward calculation shows that

$$[X, Z_1] = Z_2, \quad [X, Z_2] - Z_1, \quad [S, Z_1] = 0, \quad [S, Z_2] = 0, \quad [Z_1, Z_2] = 0,$$

that is, $\{X, Z_1, Z_2, S\}$ span a four dimensional Lie algebra $\mathfrak{g}$ with one dimensional center spanned by S. In addition, its derived Lie algebra $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$, equals span $\{Z_1, Z_2\}$ and is therefore abelian and two dimensional. Thus $\mathfrak{g}$ has no nontrivial non-abelian Lie subalgebras.

In particular the subbundle of $T(\mathbb{R}^2 \times \mathbb{T}^2)$ spanned by X and Y is not involutive and thus not integrable. By the Frobenius theorem, the field of two-dimensional planes spanned by X and Y is not tangent to a family of two-dimensional integral surfaces. Thus the motion of the car, subjected *only* to the constraints of "steer" and "drive" is nonholonomic. On the other hand, the motion of the car subjected to the constraints of "steer", "drive" and "wriggle" is holonomic. Moreover, since the Lie algebra generated by these three vector fields is abelian, the motion of the car with these constraints can be described by applying these three vector fields in any order.

Control. Next we turn our attention to *some elementary aspects of control theory*. We shall restrict our attention to a simple version of a local controllability theorem. For extensions and many additional results, we recommend consulting the book of and a few of the important papers and notes such as **?** [**?**, **?**], Sussmann [1977], Hermann and Krener [1977], Russell [1979], Hermann [1980], and Ball, Marsden, and Slemrod [1982] and references therein.

Consider a system of differential equations of the form

$$\dot{w}(t) = X(w(t)) + p(t)Y(w(t)) \tag{8.5.1}$$

on a time interval $[0, T]$ with initial conditions $w(0) = w_0$ where w takes values in a Banach manifold M, X and Y are smooth vector fields on M and $p : [0, T] \to \mathbb{R}$ is a prescribed function called a ***control***.

The existence theory for differential equations guarantees that equation (8.5.1) has a flow that depends smoothly on w_0 and on p lying in a suitable Banach space Z of maps of $[0, T]$ to $\mathbb{R}$, such as the space of C^1 maps. Let the flow of (8.5.1) be denoted

$$F_t(w_0, p) = w(t, p, w_0). \tag{8.5.2}$$

We consider the curve $w(t, 0, w_0) = w_0(t)$; that is, an integral curve of the vector field X. We say that (8.5.1) is ***locally controllable*** (at time T) if there is a neighborhood U of $w_0(T)$ such that for any point $h \in U$, there is a $p \in Z$ such that $w(T; p, w_0) = h$. In other words, we can alter the endpoint of $w_0(t)$ in a locally arbitrary way by altering p (Figure **??**).

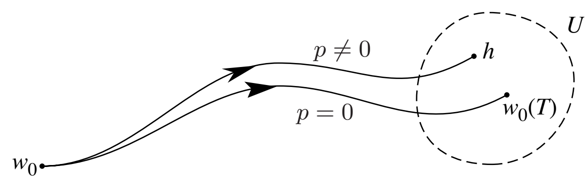


FIGURE 8.5.3. Controllability

To obtain a condition under which local controllability can be guaranteed, we fix T and w_0 and consider the map

$$P : Z \to M; \quad p \mapsto w(T, p, w_0). \tag{8.5.3}$$

The strategy is to apply the inverse function theorem to P. The derivative of $F_t(w_0, p)$ with respect to p in the direction $\rho \in Z$ is denoted

$$\mathbf{D}_p F_t(w_0, 0) \cdot \rho = L_t \rho \in T_{F_t(w_0, 0)} M.$$

Differentiating

$$\frac{d}{dt} w(t, p) = X(w(t, p)) + p(t)Y(w(t, p))$$

with respect to p at $p = 0$, we find that in $T^2 M$

$$\frac{d}{dt} L_t \rho = X(w_0(t)) \cdot L_t \rho + (\rho Y(w_0(t)))_{\text{vertical lift}}. \tag{8.5.4}$$

To simplify matters, let us assume $M = E$ is a Banach space and that X is a *linear operator*, so equation (8.5.4) becomes

$$\frac{d}{dt} L_t \rho = X \cdot L_t \rho + \rho Y(w_0(t)). \tag{8.5.5}$$

Equation (8.5.5) has the following solution given by the variation of constants formula

$$L_T\rho = \int_0^T e^{(T-s)X}\rho(s)Y(e^{sX}w_0)ds \tag{8.5.6}$$

since $w_0(t) = e^{tX}w_0$ for linear equations.

8.5.1 Proposition. *If the linear map $L_T : Z \to E$ given by equation* (8.5.6) *is surjective, then equation* (8.5.1) *is locally controllable* (*at time T*).

Proof. This follows from the "local onto" form of the implicit function theorem (see Theorem 2.5.9) applied to the map P. Solutions exist for time T for small p since they do for $p = 0$; see Corollary 4.1.25. ■

8.5.2 Corollary. *Suppose $E = \mathbb{R}^n$ and Y is linear as well. If*

$$\dim \operatorname{span}\{Y(w_0), [X,Y](w_0), [X,[X,Y]](w_0), \ldots\} = n,$$

then equation (8.5.1) *is locally controllable.*

Proof. We have the Baker–Campbell–Hausdorff formula

$$e^{-sX}Ye^{sX} = Y + s[X,Y] + \frac{s^2}{2}[X,[X,Y]] + \cdots,$$

obtained by expanding $e^{sX} = I + sX + (s^2/2)X^2 + \cdots$ and gathering terms. Substitution into equation (8.5.6) shows that L_T is surjective. ■

For the case of nonlinear vector fields and the system equation (8.5.1) on finite-dimensional manifolds, controllability hinges on the dimension of the space obtained by replacing the foregoing commutator brackets by Lie brackets of vector fields, n being the dimension of M. This is related to what are usually called ***Chow's theorem*** in control theory (see Chow [1947]).

To see that some condition involving brackets is necessary, suppose that the span of X and Y forms an involutive distribution of TM. Then by the Frobenius theorem, w_0 lies in a unique maximal two-dimensional leaf $\mathcal{L}(w_0)$ of the corresponding foliation. But then the solution of equation (8.5.1) can never leave $\mathcal{L}(w_0)$, no matter how p is chosen. Hence in such a situation, equation (8.5.1) would not be locally controllable; rather, one would only be able to move in a two-dimensional subspace. If repeated bracketing with X increases the dimension of vectors obtained then the attainable states increase in dimension accordingly.

Exercises

◇ **8.5-1.** Check that the system in Figure 8.5.1 is nonholonomic by verifying that there are two vector fields X, Y on M spanning the subset E of TM defined by the constraints

$$\dot{x} = a\,\dot{\theta}\cos\varphi = 0 \quad \text{and} \quad \dot{y} + a\,\dot{\theta}\sin\varphi = 0$$

such that $[X,Y]$ is not in E; that is, use Frobenius' theorem directly rather than using Pfaffian systems.

◇ **8.5-2.** Justify the names *wriggle* and *slide* for the vector fields W and S in the example of Figure 8.5.2 using the product formula in Exercise 4.2-4. Use these formulas to explain the following statement of Nelson [1967, p. 35]: "the Lie product of "steer" and "drive" is equal to "slide" and "rotate" ($= \partial/\partial\varphi$) on $\theta = 0$ and generates a flow which is the simultaneous action of sliding and rotating. This motion is just what is needed to get out of a tight parking spot."

◇ **8.5-3.** The word *holonomy* arises not only in mechanical constraints as explained in this section but also in the theory of connections (Kobayashi and Nomizu [1963, Volume II, Sections 7 and 8]). What is the relation between the two uses, if any?

◇ **8.5-4.** In linear control theory equation (8.5.1) is replaced by

$$\dot{w}(t) = X \cdot w(t) + \sum_{i=1}^{N} p_i(t) Y_i,$$

where X is a linear vector field on $\mathbb{R}^n$ and Y_i are *constant* vectors. By using the methods used to prove Proposition 8.5.1, rediscover for yourself the ***Kalman criterion*** for local controllability, namely, the set

$$\left\{ X^k Y_i \mid k = 0, 1, \ldots, n-1,\ i = 1, \ldots, N \right\}$$

spans $\mathbb{R}^n$.

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