7
Integration on Manifolds

The integral of an n-form on an n-manifold is defined by piecing together integrals over sets in $\mathbb{R}^n$ using a partition of unity subordinate to an atlas. The change of variables theorem guarantees that the integral is well defined, independent of the choice of atlas and partition of unity. Two basic theorems of integral calculus, the change of variables theorem and Stokes' theorem, are discussed in detail along with some applications.

7.1 The Definition of the Integral

The aim of this section is to define the integral of an n-form on an oriented n-manifold M and prove a few of its basic properties. We begin with a summary of the basic results in $\mathbb{R}^n$.

Integration on $\mathbb{R}^n$. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and has compact support. Then $\int f\, dx^1 \dots dx^n$ is defined to be the Riemann integral over any rectangle containing the support of f.

7.1.1 Definition. *Let $U \subset \mathbb{R}^n$ be open and $\omega \in \Omega^n(U)$ have compact support. If, relative to the standard basis of $\mathbb{R}^n$,*

$$\omega(x) = \frac{1}{n!}\omega_{i_1\dots i_n}(x)dx^{i_1} \wedge \dots \wedge dx^{i_n} = \omega_{1\dots n}(x)dx^1 \wedge \dots \wedge dx^n,$$

where the components of ω are given by

$$\omega_{i_1\dots i_n}(x) = \omega(x)(e_{i_1}, \dots, e_{i_n}),$$

then we define

$$\int_U \omega = \int_{\mathbb{R}^n} \omega_{1\dots n}(x)dx^1 \cdots dx^n.$$

Recall that if ζ is any integrable function and $f : \mathbb{R}^n \to \mathbb{R}^n$ is any diffeomorphism, the ***change of variables*** theorem states that $\zeta \circ f$ is integrable and

$$\int_{\mathbb{R}^n} \zeta(x^1, \ldots, x^n) dx^1 \cdots dx^n = \int_{\mathbb{R}^n} |J_\Omega f(x^1, \ldots, x^n)| (\zeta \circ f)(x^1, \ldots, x^n) dx^1 \cdots dx^n, \tag{7.1.1}$$

where $\Omega = dx^1 \wedge \cdots \wedge dx^n$ is the standard volume form on $\mathbb{R}^n$ and $J_\Omega f$ is the Jacobian determinant of f relative to Ω. This change of variables theorem can be rephrased in terms of pull backs in the following form.

7.1.2 Theorem (Change of Variables in $\mathbb{R}^n$). *Let U and V be open subsets of $\mathbb{R}^n$ and suppose $f : U \to V$ is an orientation-preserving diffeomorphism. If $\omega \in \Omega^n(V)$ has compact support, then $f^*\omega \in \Omega^n(U)$ has compact support as well and*

$$\int_U f^*\omega = \int_V \omega \tag{7.1.2}$$

Proof. If $\omega = \omega_{1\ldots n} dx^1 \wedge \cdots \wedge dx^n$, then $f^*\omega = (\omega_{1\ldots n} \circ f)(J_\Omega f)\Omega$, where the n-form $\Omega = dx^1 \wedge \cdots \wedge dx^n$ is the standard volume form on $\mathbb{R}^n$. As discussed in §6.5, $J_\Omega f > 0$. Since f is a diffeomorphism, the support of $f^*\omega$ is $f^{-1}(\operatorname{supp} \omega)$, which is compact. Then from equation (7.1.1),

$$\int_U f^*\omega = \int_{\mathbb{R}^n} (\omega_{1\ldots n} \circ f)(J_\Omega f) dx^1 \cdots dx^n = \int_{\mathbb{R}^n} \omega_{1\ldots n} dx^1 \cdots dx^n = \int_V \omega.$$

■

Integration on a Manifold. Suppose that (U, φ) is a chart on a manifold M and $\omega \in \Omega^n(M)$ has compact support. If $\operatorname{supp}(\omega) \subset U$, we may form $\omega|U$, which has the same support. Then $\varphi_*(\omega|U)$ has compact support, so we may state the following.

7.1.3 Definition. *Let M be an orientable n-manifold with orientation $[\Omega]$. Suppose $\omega \in \Omega^n(M)$ has compact support $C \subset U$, where (U, φ) is a positively oriented chart. Then we define*

$$\int_{(\varphi)} \omega = \int \varphi_*(\omega|U).$$

7.1.4 Proposition. *Suppose $\omega \in \Omega^n(M)$ has compact support $C \subset U \cap V$, where (U, φ), and (V, ψ) are two positively oriented charts on the oriented manifold M. Then*

$$\int_{(\varphi)} \omega = \int_{(\psi)} \omega.$$

Proof. By Theorem 7.1.2,

$$\int \varphi_*(\omega|U) = \int \left(\psi \circ \varphi^{-1}\right)_* \varphi_*(\omega|U).$$

Hence $\int \varphi_*(\omega|U) = \int \psi_*(\omega|U)$. (Recall that for diffeomorphisms, we have $f_* = (f^{-1})^*$ and $(f \circ g)_* = f_* \circ g_*$.) ■

Thus, we may define $\int_U \omega = \int_{(\varphi)} \omega$, where (U, φ) is any positively oriented chart containing the compact support of ω. More generally, we can define $\int_M \omega$ where ω has compact support not necessarily lying in a single chart as follows.

7.1.5 Definition. *Let M be an oriented manifold and $\mathcal{A}$ an atlas of positively oriented charts. Let $P = \{(U_\alpha, \varphi_\alpha, g_\alpha)\}$ be a partition of unity subordinate to $\mathcal{A}$. Define $\omega_\alpha = g_\alpha \omega$ (so ω_α has compact support in some U_i) and let*

$$\int_P \omega = \sum_\alpha \int \omega_\alpha. \tag{7.1.3}$$

7.1.6 Proposition. (i) *The sum* (7.1.3) *contains only a finite number of nonzero terms.*

(ii) *For any other atlas of positively oriented charts and subordinate partition of unity Q we have*

$$\int_P \omega = \int_Q \omega.$$

*The common value is denoted $\int_M \omega$, and is called the **integral** of $\omega \in \Omega^n(M)$.*

Proof. For any $m \in M$, there is a neighborhood U such that only a finite number of g_α are nonzero on U. By compactness of supp ω, a finite number of such neighborhoods cover the support of ω. Hence only a finite number of g_α are nonzero on the union of these U. For (ii), let $P = \{(U_\alpha, \varphi_\alpha, g_\alpha)\}$ and $Q = \{(V_\beta, \psi_\beta, h_\beta)\}$ be two partitions of unity with positively oriented charts. Then the functions $\{g_\alpha h_\beta\}$ satisfy $g_\alpha h_\beta(m) = 0$ except for a finite number of indices (α, β), and $\Sigma_\alpha \Sigma_\beta g_\alpha h_\beta(m) = 1$, for all $m \in M$. Since $\Sigma_\beta h_\beta = 1$, we get

$$\int_P \omega = \sum_\alpha \int g_\alpha \omega = \sum_\beta \sum_\alpha \int h_\beta g_\alpha \omega = \sum_\alpha \sum_\beta \int g_\alpha h_\beta \omega = \int_Q \omega. \qquad \blacksquare$$

Global Change of Variables. This result can now be formulated very elegantly as follows.

7.1.7 Theorem (Change of Variables Theorem)**.** *Suppose M and N are oriented n-manifolds and $f : M \to N$ is an orientation-preserving diffeomorphism. If $\omega \in \Omega^n(N)$ has compact support, then $f^*\omega$ has compact support and*

$$\int_N \omega = \int_M f^*\omega. \tag{7.1.4}$$

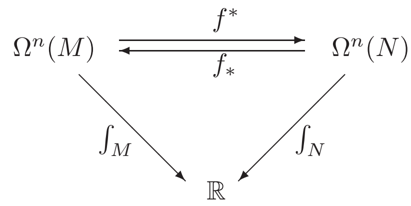
Proof. First, note that

$$\operatorname{supp}(f^*\omega) = f^{-1}(\operatorname{supp}(\omega)),$$

which is compact. To prove equation (7.1.4), let $\{(U_i, \varphi_i)\}$ be an atlas of positively oriented charts of M and let $P = \{g_i\}$ be a subordinate partition of unity. Then $\{(f(U_i), \varphi_i \circ f^{-1})\}$ is an atlas of positively oriented charts of N and $Q = \{g_i \circ f^{-1}\}$ is a partition of unity subordinate to the covering $\{f(U_i)\}$. By Proposition 7.1.6,

$$\begin{aligned}
\int_M f^*\omega &= \sum_i \int_M g_i f^*\omega = \sum_i \int_{\mathbb{R}^n} \varphi_{i*}(g_i f^*\omega) \\
&= \sum_i \int_{\mathbb{R}^n} \varphi_{i*}(f^{-1})_*(g_i \circ f^{-1})\omega \\
&= \sum_i \int_{\mathbb{R}^n} (\varphi_i \circ f^{-1})_*(g_i \circ f^{-1})\omega \\
&= \int_N \omega. \qquad \blacksquare
\end{aligned}$$

This result is summarized by the following commutative diagram:



7.1.8 Definition. *Let (M, μ) be a volume manifold. Suppose $f \in \mathcal{F}(M)$ and f has compact support. Then we call $\int_M f\mu$ the **integral of f with respect to** μ.*

The reader can check that since the Riemann integral is $\mathbb{R}$-linear, so is the integral just defined.

Measures (Optional). The next theorem will show that the integral defined by equation (7.1.4) can be obtained in a unique way from a measure on M. (The reader unfamiliar with measure theory can find the necessary background in Royden [1968]; this result will not be essential for future sections.) The integral we have described can clearly be extended to all continuous functions with compact support. Then we have the following.

7.1.9 Theorem (Riesz Representation Theorem)**.** *Let (M, μ) be a volume manifold. Let β denote the collection of Borel sets of M, the σ-algebra generated by the open* (*or closed, or compact*) *subsets of M. Then there is a unique measure m_μ on β such that for every continuous function of compact support*

$$\int_M f\, dm_\mu = \int_M f\mu. \tag{7.1.5}$$

Proof. Existence of such a measure m_μ is proved in books on measure theory, for example Royden [1968]. For uniqueness, it is enough to consider bounded open sets (by the Hahn extension theorem). Thus, let U be open in M, and let C_U be its characteristic function. We construct a sequence of C^∞ functions of compact support φ_n such that $\varphi_n \downarrow C_U$, pointwise. Hence from the monotone convergence theorem,

$$\int \varphi_n \mu = \int \varphi_n\, dm_\mu \to \int C_U\, dm_\mu = m_\mu(U).$$

Thus, m_μ is unique. ■

The space $L^p(M, \mu)$, $p \in \mathbb{R}$, consists of all measurable functions f such that $|f|^p$ is integrable. For $p \geq 1$, the norm

$$\|f\|_p = \left(\int |f|^p dm_\mu \right)^{1/p}$$

makes $L^p(M, \mu)$ into a Banach space (functions that differ only on a set of measure zero are identified). The use of these spaces in studying objects on M itself is discussed in §7.4. The next propositions give an indication of some of the ideas. If $F : M \to N$ is a measurable mapping and m_M is a measure on M, then $F_* m_M$ is the measure on N defined by $F_* m_M(A) = m_M(F^{-1}(A))$. If F is bijective, we set $F^*(m_N) = (F^{-1})_* m_N$. If $f : M \to \mathbb{R}$ is an integrable function, then fm_M is the measure on M defined by

$$(fm_M)(A) = \int_A f dm_M$$

for every measurable set A in M.

7.1.10 Proposition. *Suppose M and N are orientable n-manifolds with volume forms μ_M and μ_N and corresponding measures m_M and m_N. Let F be an orientation preserving C^1 diffeomorphism of M onto N. Then*

$$F^* m_N = \left(J_{(\mu_M,\mu_N)} F\right) m_M. \tag{7.1.6}$$

Proof. Let f be any C^∞ function with compact support on M. By Theorem 7.1.7,

$$\begin{aligned}\int_N f\, dm_N = \int_N f\, \mu_N = \int_M F^*(f\mu_N) &= \int_M (f \circ F) \left(J_{(\mu_M,\mu_N)} F\right) \mu_M \\ &= \int_M (f \circ F) \left(J_{(\mu_M,\mu_N)} F\right) dm_M.\end{aligned}$$

As in the proof of Theorem 7.1.9, this relation holds for f chosen to be the characteristic function of $F(A)$. That is,

$$m_N(F(A)) = \int_A \left(J_{(\mu_M,\mu_N)} F\right) dm_M.$$

■

Jacobians and Divergence. In preparation for the next result, we notice that on a volume manifold (M, μ), the flow F_t of any vector field X is orientation preserving for each $t \in \mathbb{R}$ (regard this as a statement on the domain of the flow, if the vector field is not complete). Indeed, since F_t is a diffeomorphism, $J_\mu(F_t)$ is nowhere zero; since it is continuous in t and equals one at $t = 0$, it is positive for all t.

7.1.11 Proposition. *Let M be an orientable manifold with volume form μ and corresponding measure m_μ. Let X be a (possibly time-dependent) C^1 vector field on M with flow F_t. The following are equivalent (if the flow of X is not complete, the statements involving it are understood to hold on its domain):*

(i) $\operatorname{div}_\mu X = 0$;

(ii) $J_\mu F_t = 1$ *for all* $t \in \mathbb{R}$;

(iii) $F_{t*} m_\mu = m_\mu$ *for all* $t \in \mathbb{R}$;

(iv) $F_t^* \mu = \mu$ *for all* $t \in \mathbb{R}$;

(v) $\int_M f dm_\mu = \int_M (f \circ F_t) dm_\mu$ *for all* $f \in L^1(M, \mu)$ *and all* $t \in \mathbb{R}$.

Proof. Statement (i) is equivalent to (ii) by Corollary 6.5.19. Statement (ii) is equivalent to (iii) by equation (7.1.6) and to (iv) by definition. We shall prove that (ii) is equivalent to (v). If $J_\mu F_t = 1$ for all $t \in \mathbb{R}$ and f is continuous with compact support, then

$$\int_M (f \circ F_t)\mu = \int_M (f \circ F_t)(F_t^*\mu) = \int_M F_t^*(f\mu) = \int_M f\mu.$$

Hence, by uniqueness in Theorem 7.1.9, we have $\int_M f\, dm_\mu = \int_M (f \circ F_t) dm_\mu$ for all integrable f, and so (ii) implies (v). Conversely, if

$$\int_M (f \circ F_t) dm_\mu = \int_M f\, dm_\mu$$

then taking f to be continuous with compact support, we see that

$$\int_M (f \circ F_t)_\mu = \int_M f\mu = \int_M F_t^*(f\mu) = \int_M (f \circ F_t) F_t^*\mu \qquad = \int_M (f \circ F_t)(J_\mu F_t)\mu.$$

Thus, for every integrable f,

$$\int_M (f \circ F_t) dm_\mu = \int_M (f \circ F_t)(J_\mu F_t) dm_\mu.$$

Hence $J_\mu F_t = 1$, and so (v) implies (ii). ■

Transport Theorem. The following result is central to continuum mechanics (see Example 7.1.13 below and §8.2 for applications).

7.1.12 Theorem (Transport Theorem)**.** *Let (M, μ) be a volume manifold and X a vector field on M with flow F_t. For $f \in \mathcal{F}(M \times \mathbb{R})$ and letting $f_t(m) = f(m,t)$, we have*

$$\frac{d}{dt} \int_{F_t(U)} f_t \mu = \int_{F_t(U)} \left(\frac{\partial f}{\partial t} + \operatorname{div}_\mu(f_t X) \right) \mu \tag{7.1.7}$$

for any open set U in M.

Proof. By the flow characterization of Lie derivatives and Proposition 6.5.17, we have

$$\begin{aligned} \frac{d}{dt} F_t^*(f_t \mu) &= F_t^* \left(\frac{\partial f}{\partial t} \mu \right) + F_t^* \pounds_X (f_t \mu) \\ &= F_t^* \left(\frac{\partial f}{\partial t} \mu \right) + F_t^* [(\pounds_X f_t)\mu + f_t (\operatorname{div}_\mu X)\mu] \\ &= F_t^* \left[\left(\frac{\partial f}{\partial t} + \operatorname{div}_\mu (f_t X) \right) \mu \right]. \end{aligned}$$

Thus, by the change of variables formula,

$$\begin{aligned} \frac{d}{dt} \int_{F_t(U)} f_t \mu = \frac{d}{dt} \int_U F_t^*(f_t \mu) &= \int_U F_t^* \left[\left(\frac{\partial f}{\partial t} + \operatorname{div}_\mu (f_t X) \right) \mu \right] \\ &= \int_{F_t(U)} \left(\frac{\partial f}{\partial t} + \operatorname{div}_\mu (f_t X) \right) \mu. \end{aligned}$$ ■

7.1.13 Example. Let $\rho(x,t)$ be the density of an ideal fluid moving in a compact region of $\mathbb{R}^3$ with smooth boundary. One of the basic assumptions of fluid dynamics is ***conservation of mass***: the mass of the fluid in the open set U remains unchanged during the motion described by a flow F_t. This means that

$$\frac{d}{dt} \int_{F_t(U)} \rho(x,t)\, d^3x = 0 \tag{7.1.8}$$

for all open sets U. By the transport theorem, equation (7.1.8) is equivalent to the ***equation of continuity***

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{7.1.9}$$

here $\mathbf{u}$ represents the velocity of the fluid particles. We shall return to this example in §8.2. ♦

Recurrence. As another application of Proposition 7.1.11, we prove the following.

7.1.14 Theorem (Poincaré Recurrence Theorem)**.** *Let (M, μ) be a volume manifold, m_μ the corresponding measure, and X a time-independent divergence-free vector field with flow F_t. Suppose A is a measurable set in M such that $m_\mu(A) < \infty$, $F_t(x)$ exists for all $t \in \mathbb{R}$ if $x \in A$, and $F_t(A) \subset A$. Then for each measurable subset B of A and $T \geq 0$, there exists $S \geq T$ such that $B \cap F_S(B) \neq \varnothing$. Therefore, a trajectory starting in B returns infinitely often to B.*

Proof. By Proposition 7.1.11, the sets $B, F_T(B), F_{2T}(B), \ldots$ all have the same finite measure. Since $m_\mu(A) < \infty$, they cannot all be disjoint, so there exist integers $k > l > 0$ satisfying

$$F_{kT}(B) \cap F_{lT}(B) \neq \varnothing.$$

Since $F_{kT} = (F_T)^k$ (as X is time-independent), we get

$$F_{(k-l)T}(B) \cap B \neq \varnothing.$$ ■

The Poincaré recurrence theorem is one of the forerunners of ergodic theory, a topic that will be discussed briefly in §7.4. A related result is the following.

7.1.15 Theorem (Schwarzschild Capture Theorem). *Let (M, μ) be a volume manifold, X a time-independent divergence-free vector field with flow F_t, and A a measurable subset of M with finite measure. Assume that for every $x \in A$, the trajectory $t \mapsto F_t(x)$ exists for all $t \in \mathbb{R}$. Then for almost all $x \in A$ (relative to m_μ) the following are equivalent:*

(i) $F_t(x) \in A$ *for all* $t \geq 0$;

(ii) $F_t(x) \in A$ *for all* $t \leq 0$.

Proof. Let $A_1 = \bigcap_{t\geq 0} F_t(A)$ be the set of points in A which have their future trajectory completely in A. Similarly, consider $A_2 = \bigcap_{t\leq 0} F_t(A)$. By Proposition 7.1.11, for any $\tau \geq 0$,

$$\mu(A_1) = \mu(F_{-\tau}(A_1)) = \mu\left(\bigcap_{t\geq -\tau} F_\tau(A)\right)$$

which shows, by letting $\tau \to \infty$, that

$$\mu(A_1) = \mu\left(\bigcap_{t\in\mathbb{R}} F_t(A)\right) = \mu(A_1 \cap A_2).$$

Reasoning similarly for A_2, we get $\mu(A_1) = \mu(A_1 \cap A_2) = \mu(A_2)$, so that

$$\mu(A_1 \backslash (A_1 \cap A_2)) = \mu(A_2 \backslash (A_1 \cap A_2)) = 0.$$

Let

$$S = (A_1 (A_1 \cap A_2)) \cup (A_2 (A_1 \cap A_2));$$

then $m_\mu(S) = 0$ and $S \subset A$. Moreover, we have $A_1 S = A_1 \cap A_2 = A_2 S$, which proves the desired equivalence. ■

So far only integration on orientable manifolds has been discussed. A similar procedure can be carried out to define the integral of a one-density (see §6.5) on any manifold, orientable or not. The only changes needed in the foregoing definitions and propositions are to replace the Jacobians with their absolute values and to use the definition of divergence with respect to a given density as discussed in §6.5. All definitions and propositions go through with these modifications.

Vector Valued Forms. If F is a finite-dimensional vector space, F-valued one-forms and one-densities can also be integrated in the following way. If $\omega = \sum_{i=1}^{l} \omega^i f_i$, where $f_1, \ldots, f_n$ is an ordered basis of F, then we set

$$\int_M \omega = \sum_{i=1}^{l} \left(\int_M \omega^i\right) f_i \in F.$$

It is easy to see that this definition is independent of the chosen basis of F and that all the basic properties of the integral remain unchanged. On the other hand, *the integral of vector-bundle-valued n-forms on M is not defined* unless additional special structures (such as triviality of the bundle) are used. In particular, integration of vector or general tensor fields is not defined.

Exercises

◇ **7.1-1.** Let M be an n-manifold and μ a volume form on M. If X is a vector field on M with flow F_t show that

$$\frac{d}{dt}(J_\mu(F_t)) = J_\mu(F_t)(\operatorname{div}_\mu X \circ F_t).$$

HINT: Compute $(d/dt)F_t^*\mu$ using the Lie derivative formula.

◇ **7.1-2.** Prove the following generalization of the transport theorem

$$\frac{d}{dt}\int_{F_t(V)} \omega_t = \int_{F_t(V)} \left(\frac{\partial \omega_t}{\partial t} + \pounds_X \omega_t \right),$$

where ω_t is a time–dependent k-form on M and V is a k-dimensional submanifold of M.

◇ **7.1-3.** (i) Let $\varphi : S^1 \to S^1$ be the map defined by $\varphi(e^{i\theta}) = e^{2i\theta}$, where $\theta \in [0, 2\pi]$. Let, by abuse of notion, $d\theta$ denote the standard volume of S^1. Show that the following identity holds:

$$\int_{S^1} \varphi^*(d\theta) = 2\int_{S^1} d\theta.$$

(ii) Let $\varphi : M \to N$ be a smooth surjective map. Then φ called a ***k-fold covering map*** if every $n \in N$ has an open neighborhood V such that $\varphi^{-1}(V) = U_1 \cup \cdots \cup U_k$, are disjoint open sets each of which is diffeomorphic by φ to V. Generalize (i) in the following way. If $\omega \in \Omega^n(N)$ is a volume form, show that

$$\int_M \varphi^*\omega = k\int_N \omega.$$

◇ **7.1-4.** Define the integration of Banach space valued n-forms on an n-manifold M. Show that if the Banach space is $\mathbb{R}^l$, you recover the coordinate definition given at the end of this section. If $\mathbf{E}, \mathbf{F}$ are Banach spaces and $A \in L(\mathbf{E}, \mathbf{F})$, define $A_* \in L(\Omega(M, \mathbf{E}), \Omega(M, \mathbf{F}))$ by $(A_*\alpha)(m) = A(\alpha(m))$. Show that

$$\left(\int_M\right) \circ A_* = A \circ \left(\int_M\right)$$

on $\Omega^n(M, \mathbf{E})$.

◇ **7.1-5.** Let M and N be oriented manifolds and endow $M \times N$ with the product orientation. Let $p_M : M \times N \to M$ and $p_N : M \times N \to N$ be the projections. If $\alpha \in \Omega^{\dim M}(M)$ and $\beta \in \Omega^{\dim N}(N)$ have compact support show that

$$\alpha \times \beta := ({p_M}^*\alpha) \wedge ({p_N}^*\beta)$$

has compact support and is a $(\dim M + \dim N)$-form on $M \times N$. Prove ***Fubini's Theorem***

$$\int_{M\times N} \alpha \times \beta = \left(\int_M \alpha\right)\left(\int_N \beta\right).$$

◇ **7.1-6** (Fiber Integral)**.** Let $\varphi : M \to N$ be a surjective submersion, where $\dim M = m$ and $\dim N = n$. The map φ is said to be ***orientable*** if there exists $\eta \in \Omega^p(M)$, where $p = m - n$, such that for each $y \in N$, ${j_y}^*\eta$ is a volume form on $\varphi^{-1}(y)$, where $j_y : \varphi^{-1}(y) \to M$ is the inclusion. An ***orientation*** of φ is an equivalence class of p-forms under the relation: $\eta_1 \sim \eta_2$ iff there exists $f \in \mathcal{F}(M)$, $f > 0$ such that $\eta_2 = f\eta_1$.

(i) If $\varphi : M \to N$ is a vector bundle, show that orientability of φ is equivalent to orientability of the vector bundle as defined in Exercise 6.5-13.

(ii) If φ is oriented by η and N by ω, show that $\varphi^*\omega \wedge \eta$ is a volume form on M. The orientation on M defined by this volume is called the ***local product orientation*** of M (compare with Exercise 6.5-13(vi)).

(iii) Let

$$\Omega^k_\varphi(M) := \{ \alpha \in \Omega^k(M) \mid \varphi^{-1}(K) \cap \operatorname{supp}(\alpha) \text{ is compact, } \\ \text{for any compact set } K \subset N \},$$

the ***fiber-compactly supported*** k-forms on M. Show that $\Omega^k_\varphi(M)$ is an $\mathcal{F}(M)$-submodule of $\Omega^k(M)$, and is invariant under the interior product, exterior differential, and Lie derivative.

(iv) If $\alpha \in \Omega^{k+p}_\varphi(M)$, $k \geq 0$ and $y \in N$, define a p-form α_y on $\varphi^{-1}(y)$, with values in $T^*_yN \wedge \cdots \wedge T^*_yN$ (k times) by

$$[\alpha_y(x)(u_1, \ldots, u_p)](T_x\varphi(v_1), \ldots, T_x\varphi(v_k)) \\ = \alpha(x)(v_1, \ldots, v_k, u_1, \ldots, u_p),$$

where $\varphi(x) = y$, $x \in M$, $v_1, \ldots, v_k \in T_xM$, and $u_1, \ldots, u_p \in \ker(T_x\varphi) = T_x(\varphi^{-1}(y))$. Assume φ is oriented. Define the ***fiber integral***

$$\int_{\text{fib}} : \Omega^{k+p}_\varphi(M) \to \Omega^k(N)$$

by

$$\left(\int_{\text{fib}} \alpha \right)(x) = \int_{\varphi^{-1}(y)} \alpha_y, \quad \text{if} \quad \varphi(x) = y;$$

the right-hand side is understood as the integral of a vector-valued p-form on the oriented p-manifold $\varphi^{-1}(y)$.

(a) Prove that $\int_{\text{fib}} \alpha$ is a smooth k-form on N.
Hint: Use charts in which φ is a projection and apply the theorem of smoothness of the integral with respect to parameters.

(b) Show that if $\varphi : M \to N$ is a locally trivial fiber bundle with N paracompact, $\int_{\text{fib}}$ is surjective.

(v) Let $\beta \in \Omega^l(N)$ have compact support and let $\alpha \in \Omega^{k+p}_\varphi(M)$. Show that $\varphi^*\beta \wedge \alpha \in \Omega^{k+l+p}_\varphi(M)$ and that

$$\int_{\text{fib}} (\varphi^*\beta \wedge \alpha) = \beta \wedge \int_{\text{fib}} \alpha.$$

Hint: Let $E = T^*_yN \wedge \cdots \wedge T^*_yN$ (k times) and let F be the wedge product $l + k$ times. Define $A \in L(E, F)$ by $A(\gamma) = \beta(y) \wedge \gamma$ and show that $(\varphi^*\beta \wedge \alpha)_y = A_*(\alpha_y)$ using the notation of Exercise 7.1-4. Then apply $\int_{\text{fib}}$ to this identity and use Exercise 7.1-4.

(vi) Assume N is paracompact and oriented, φ is oriented, and endow M with the local product orientation. Prove the following ***iterated integration*** (***Fubini-type***) ***formula***

$$\int_M = \int_N \circ \int_{\text{fib}}$$

by following the three steps below.

Step 1: Using a partition of unity, reduce to the case $M = N \times P$ where $\varphi : M \to N$ is the projection and M, N, P are Euclidean spaces.

Step 2: Use (v) and Exercise 7.1-5 to show that for $\beta \in \Omega^n(N)$ and $\gamma \in \Omega^p(P)$ with compact support,

$$\int_N \int_{\text{fib}} (\beta \times \gamma) = \int_M (\beta \times \gamma)$$

Step 3: Since M, N, and P are ranges of coordinate patches, show that any $\omega \in \Omega^m(M)$ with compact support is of the form $\beta \times \gamma$.

(vii) Let $\varphi : M \to N$ and $\varphi' : M' \to N'$ be oriented surjective submersions and let $f : M \to M'$, and $f_0 : N \to N'$ be smooth maps satisfying $f_0 \circ \varphi = \varphi' \circ f$. Show that $\int_{\text{fib}} \circ f^* = f_0^* \circ \int'_{\text{fib}}$, where $\int'_{\text{fib}}$ denotes the fiber integral of φ'.

(viii) Let $\varphi : M \to N$ be an oriented surjective submersion and assume $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are φ-related. Prove that

$$\int_{\text{fib}} \circ \, \mathbf{i}_X = \mathbf{i}_Y \circ \int_{\text{fib}}, \quad \int_{\text{fib}} \circ \, \mathbf{d} = \mathbf{d} \circ \int_{\text{fib}}, \quad \int_{\text{fib}} \circ \, \pounds_X = \pounds_Y \circ \int_{\text{fib}}.$$

(For more information on the fiber integral see Bourbaki [1971] and Greub et al..)

◇ **7.1-7.** Let $\varphi : M \to N$ be a smooth orientation preserving map, where M and N are volume manifolds of dimension m and n respectively. For $\alpha \in \Omega^k(M)$ with compact support, define the linear functional $\varphi_*\alpha : \Omega^{m-k} \to \mathbb{R}$ by

$$(\varphi_*\alpha)(\beta) = \int_M \varphi^*\beta \wedge \alpha$$

for all $\beta \in \Omega^{m-k}(N)$; that is, $\varphi_*\alpha$ is a ***distributional k-form*** on N. If $m < k$, set $\varphi_*\alpha = 0$. If there is a $\gamma \in \Omega^{n-m+k}(N)$ satisfying

$$(\varphi_*\alpha)(\beta) = \int_M \beta \wedge \gamma,$$

identify $\varphi_*\alpha$ with γ and say $\varphi_*\alpha$ is of *form-type*. Prove the following statements.

(i) If φ is a diffeomorphism, then $\varphi_*\alpha$ is the usual push-forward.

(ii) If α is a volume form, this definition corresponds to that for the push-forward of measures.

(iii) If φ is an oriented surjective submersion, show that $\varphi_*\alpha = \int_{\text{fib}} \alpha$, as defined in Exercise 7.1-6(iv).

Hint: Prove the identity

$$\int_M \varphi^*\beta \wedge \alpha = \int_N \left(\beta \wedge \int_{\text{fib}} \alpha \right)$$

using Exercise 7.1-6(v) and (vi).

◇ **7.1-8.** Let (M, μ) be a paracompact n-dimensional volume manifold.

(i) If (N, ν) is another paracompact n-dimensional volume manifold and $f : M \to N$ is an orientation reversing diffeomorphism, show that $\int_N \omega = -\int_M f^*\omega$ for any $\omega \in \Omega^n(N)$ with compact support.

Hint: Use the proof of Theorem 7.1.12.

(ii) If $\eta \in \Omega^n(M)$ has compact support and $-M$ denotes the manifold M endowed with the orientation $[-\mu]$, show that

$$\int_{-M} \eta = -\int_M \eta.$$

HINT: If $\mathcal{A} = \{(U_i, \varphi_i)\}$ is an oriented atlas for $(M, [\mu])$, then

$$-\mathcal{A} = \{(U_i, \varphi_i \circ \psi_i)\}, \quad \psi_i(x^1, \ldots, x^n) = (-x^1, x^2, \ldots, x^n)$$

is an oriented atlas for $(M, [-\mu])$.

◇ **7.1-9.** Let ω_n be the standard volume form on S^n. Show that

$$\int_{S^n} \omega_n = \frac{2^{m+1}\pi^m}{(2m-1)!!}, \quad \text{if } n = 2m,\ m \geq 1$$

and

$$\int_{S^n} \omega_n = \frac{2\pi^{m+1}}{m!}, \quad \text{if } n = 2m+1,\ m \geq 0$$

using the following steps.

(i) Let $M \subset \mathbb{R}^{n+1}$ be the annulus

$$\{\, x \in \mathbb{R}^{n+1} \mid 0 < a < \|x\| < b < \infty \,\}$$

and let $f :]a, b[\times S^n \to A$ be the diffeomorphism $f(t, s) = ts$. Use Exercise 6.5-19(ii) to show that for $x \in \mathbb{R}^{n+1}$,

$$f^*\left(e^{-\|x\|^2}\Omega_{n+1}\right) = t^n e^{-t^2}(dt \times \omega_n)$$

where $\Omega_{n+1} = e_1 \wedge \cdots \wedge e_{n+1}$ for $\{e_1, \ldots, e_{n+1}\}$ the standard basis of $\mathbb{R}^{n+1}$, and where $dt \times \omega_n$ denotes the product volume form on $]a, b[\times S^n$.

(ii) Deduce the equality

$$\int_{\mathbb{R}^{n+1}} e^{-\|x\|^2}\Omega_{n+1} = \int_a^b t^n e^{-t^2} dt \int_{S^n} \omega_n.$$

(iii) Let a $\downarrow 0$ and $b \uparrow \infty$ to deduce the equality

$$\int_0^\infty t^n d^{-t^2} dt \int_{S^n} \omega_n = \left(\int_{-\infty}^{+\infty} e^{-u^2} du\right)^{n+1}.$$

Prove that

$$\int_{-\infty}^\infty e^{-u^2} du = \sqrt{\pi}, \quad \int_0^\infty t^{2m} e^{-t^2} dt = \frac{(2m-1)!!\sqrt{\pi}}{2^{m+1}},$$

and

$$\int_0^\infty t^{2m+1} e^{-t^2} = \frac{m!}{2},$$

to deduce the required formula for $\int_{S^n} \omega_n$.

7.2 Stokes' Theorem

Stokes' theorem states that if α is an $(n-1)$-form on an orientable n-manifold M, then the integral of $\mathbf{d}\alpha$ over M equals the integral of α over ∂M, the boundary of M. As we shall see in the next section, the classical theorems of Gauss, Green, and Stokes are special cases of this result. Before stating Stokes' theorem formally, we need to discuss manifolds with boundary and their orientations.

7.2.1 Definition. *Let* $\mathbf{E}$ *be a Banach space and* $\lambda \in \mathbf{E}^*$. *Let*

$$\mathbf{E}_\lambda = \{\, x \in \mathbf{E} \mid \lambda(x) \geq 0 \,\},$$

called a ***half-space*** *of* $\mathbf{E}$, *and let* $U \subset \mathbf{E}_\lambda$ *be an open set (in the topology induced on* $\mathbf{E}_\lambda$ *from* $\mathbf{E}$*). Call* $\operatorname{Int} U = U \cap \{\, x \in \mathbf{E} \mid \lambda(x) > 0 \,\}$ *the* ***interior*** *of* U *and* $\partial U = U \cap \ker \lambda$ *the* ***boundary*** *of* U. *If* $\mathbf{E} = \mathbb{R}^n$ *and* λ *is the projection on the* j*th factor, then* $\mathbf{E}_\lambda$ *is denoted by* $\mathbb{R}^n_j$ *and is called* ***positive*** j*th* ***half-space***. $\mathbb{R}^n_n$ *is also denoted by* $\mathbb{R}^n_+$.

We have $U = \operatorname{Int} U \cup \partial U$, $\operatorname{Int} U$ is open in U, ∂U is closed in U (*not* in $\mathbf{E}$), and $\partial U \cap \operatorname{Int} U = \varnothing$. The situation is shown in Figure 7.2.1. Note that ∂U is *not* the topological boundary of U in $\mathbf{E}$, but it *is* the topological boundary of U intersected with that of $\mathbf{E}_\lambda$. This inconsistent use of the notation ∂U is temporary.

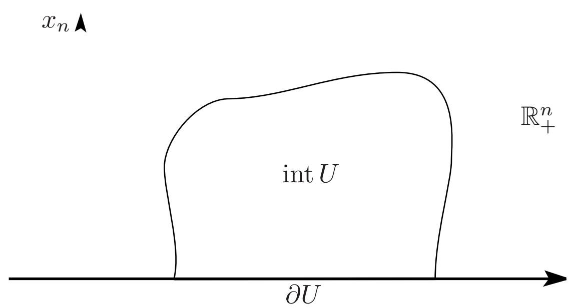


FIGURE 7.2.1. Open sets in a half-space

A manifold with boundary will be obtained by piecing together sets of the type shown in the figure. To carry this out, we need a notion of local smoothness to be used for overlap maps of charts.

7.2.2 Definition. *Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be Banach spaces,* $\lambda \in \mathbf{E}^*$, $\mu \in \mathbf{F}^*$, U *be an open set in* $\mathbf{E}_\lambda$, *and* V *be an open set in* $\mathbf{F}_\mu$. *A map* $f : U \to V$ *is called* ***smooth*** *if for each point* $x \in U$ *there are open neighborhoods* U_1 *of* x *in* $\mathbf{E}$ *and* V_1 *of* $f(x)$ *in* $\mathbf{F}$ *and a smooth map* $f_1 : U_1 \to V_1$ *such that* $f|U \cap U_1 = f_1|U \cap U_1$. *We define* $\mathbf{D}f(x) = \mathbf{D}f_1(x)$. *The map* f *is a* ***diffeomorphism*** *if there is a smooth map* $g : V \to U$ *which is an inverse of* f. *(In this case* $\mathbf{D}f(x)$ *is an isomorphism of* $\mathbf{E}$ *with* $\mathbf{F}$.*)*

We must prove that this definition of $\mathbf{D}f$ is independent of the choice of f_1, that is, we have to show that if $\varphi : W \to \mathbf{E}$ is a smooth map with W open in $\mathbf{E}$ such that $\varphi|(W \cap \mathbf{E}_\lambda) = 0$, then $\mathbf{D}\varphi(x) = 0$ for all $x \in W \cap \mathbf{E}_\lambda$. If $x \in \operatorname{Int}(W \cap \mathbf{E}_\lambda)$, this fact is obvious. If $x \in \partial(W \cap \mathbf{E}_\lambda)$, choose a sequence $x_n \in \operatorname{Int}(W \cap \mathbf{E}_\lambda)$ such that $x_n \to x$; but then $0 = \mathbf{D}\varphi(x_n) \to \mathbf{D}\varphi(x)$ and hence $\mathbf{D}\varphi(x) = 0$, which proves our claim.

7.2.3 Lemma. *Let* $U \subset \mathbf{E}_\lambda$ *be open,* $\varphi : U \to \mathbf{F}_\mu$ *be a smooth map, and assume that for some* $x_0 \in \operatorname{Int} U$, $\varphi(x_0) \in \partial \mathbf{F}_\mu$. *Then*

$$\mathbf{D}\varphi(x_0)(\mathbf{E}) \subset \partial \mathbf{F}_\mu = \ker \mu.$$

Proof. The quotient $\mathbf{F}/\ker \mu$ is isomorphic to $\mathbb{R}$, so that fixing f with $\mu(f) > 0$, the element $[f] \in \mathbf{F}/\ker \mu$ forms a basis. Therefore $[f]$ determines the isomorphism $T_f : \mathbf{F}/\ker \mu \to \mathbb{R}$ given by $T_f([y]) = t$, where $t \in \mathbb{R}$

is the unique number for which $t[f] = [y]$. This isomorphism in turn defines the isomorphism

$$S_f : \ker \mu \oplus \mathbb{R} \to \mathbf{F}$$

given by $S_f(y,t) = y + tf$ which induces diffeomorphisms (in the sense of Definition 7.2.2) of $\ker \mu \times [0, \infty[$ with $\mathbf{F}_\mu$ and of $\ker \mu \times]-\infty, 0]$ with $\{ y \in \mathbf{F} \mid \mu(y) \leq 0 \}$. Denote by $p : \mathbf{F} \to \mathbb{R}$ the linear map given by S_f^{-1} followed by the projection $\ker \mu \oplus \mathbb{R} \to \mathbb{R}$, so that $y \in \mathbf{F}_\mu$ (respectively, $\ker \mu$, $\{ y \in \mathbf{F} \mid \mu(y) \leq 0 \}$) if and only if $p(y) \geq 0$ (respectively, $= 0$, ≤ 0).

Notice that the relation

$$\varphi(x_0 + tx) = \varphi(x_0) + \mathbf{D}\varphi(x_0) \cdot tx + o(tx),$$

where $\lim_{t \to 0} o(tx)/t = 0$, together with the hypothesis $(p \circ \varphi)(x) \geq 0$ for all $x \in U$, implies that

$$0 \leq (p \circ \varphi)(x_0 + tx) = 0 + (p \circ \mathbf{D}\varphi)(x_0) \cdot tx + p(o(tx)),$$

whence for $t > 0$

$$0 \leq (p \circ \mathbf{D}\varphi)(x_0) \cdot x + p\left(\frac{o(tx)}{t}\right).$$

Letting $t \to 0$, we get $(p \circ \mathbf{D}\varphi)(x_0) \cdot x \geq 0$ for all $x \in \mathbf{E}$. Similarly, for $t < 0$ and letting $t \to 0$, we get $(p \circ \mathbf{D}\varphi)(x_0) \cdot x \leq 0$ for all $x \in \mathbf{E}$. The conclusion is

$$(\mathbf{D}\varphi)(x_0)(\mathbf{E}) \subset \ker \mu. \qquad \blacksquare$$

Intuitively, this says that if φ preserves the condition $\lambda(x) \geq 0$ and maps an interior point to the boundary, then the derivative must be zero in the normal direction. The reader may also wish to prove Lemma 7.2.3 from the implicit mapping theorem. Now we carry this idea one step further.

7.2.4 Lemma. *Let U be open in $\mathbf{E}_\lambda$, V be open in $\mathbf{F}_\mu$, and $f : U \to V$ be a diffeomorphism. Then f restricts to diffeomorphisms $\operatorname{Int} f : \operatorname{Int} U \to \operatorname{Int} V$ and $\partial f : \partial U \to \partial V$.*

Proof. Assume first that $\partial U = \varnothing$, that is, that $U \cap \ker \lambda = \varnothing$. We shall show that $\partial V = \varnothing$ and hence we take $\operatorname{Int} f = f$. If $\partial V \neq \varnothing$, there exists $x \in U$ such that $f(x) \in \partial V$ and hence by definition of smoothness there are open neighborhoods $U_1 \subset U$ and $V_1 \subset \mathbf{F}$, such that $x \in U_1$ and $f(x) \in V_1$, and smooth maps $f_1 : U_1 \to V_1$, $g_1 : V_1 \to U_1$ such that

$$f|U_1 = f_1, \quad g_1|V \cap V_1 = f^{-1}|V \cap V_1.$$

Let $x_n \in U_1$, $x_n \to x$, $y_n \in V_1 \backslash \partial V$, and $y_n = f(x_n)$. We have

$$\begin{aligned} \mathbf{D}f(x) \circ \mathbf{D}g_1(f(x)) &= \lim_{y_n \to f(x)} (\mathbf{D}f(g_1(y_n)) \circ \mathbf{D}g_1(y_n)) \\ &= \lim_{y_n \to f(x)} \mathbf{D}(f \circ g_1)(y_n) = \mathrm{Id}_{\mathbf{F}} \end{aligned}$$

and similarly

$$\mathbf{D}g_1(f(x)) \circ \mathbf{D}f(x) = \mathrm{Id}_{\mathbf{E}}$$

so that $\mathbf{D}f(x)^{-1}$ exists and equals $\mathbf{D}g_1(f(x))$. But by Lemma 7.2.3, $\mathbf{D}f(x)(\mathbf{E}) \subset \ker \mu$, which is impossible, $\mathbf{D}f(x)$ being an isomorphism.

Assume that $\partial U \neq \varnothing$. If we assume $\partial V = \varnothing$, then, working with f^{-1} instead of f, the above argument leads to a contradiction. Hence $\partial V \neq \varnothing$. Let $x \in \operatorname{Int} U$ so that x has a neighborhood $U_1 \subset U$, $U_1 \cap \partial U = \varnothing$, and hence $\partial U_1 = \varnothing$. Thus, by the preceding argument, $\partial f(U_1) = \varnothing$, and $f(U_1)$ is open in $V \backslash \partial V$. This shows that $f(\operatorname{Int} U) \subset \operatorname{Int} V$. Similarly, working with f^{-1}, we conclude that $f(\operatorname{Int} U) \supset \operatorname{Int} V$ and hence $f : \operatorname{Int} U \to \operatorname{Int} V$ is a diffeomorphism. But then $f(\partial U) = \partial V$ and $f|\partial U : \partial U \to \partial V$ is a diffeomorphism as well. $\blacksquare$

Now we are ready to define a manifold with boundary.

7.2.5 Definition. *A **manifold with boundary** is a set M together with an **atlas of charts with boundary** on M; **charts with boundary** are pairs (U, φ) where $U \subset M$ and $\varphi(U) \subset E_\lambda$ for some $\lambda \in E^*$ and an **atlas** on M is a family of charts with boundary satisfying* **MA1** *and* **MA2** *of Definition* 3.1.1*, with smoothness of overlap maps φ_{ji} understood in the sense of Definition* 7.2.2*. See Figure* 7.2.2*. If $E = \mathbb{R}^n$, M is called an n-**manifold with boundary**.*

Define

$$\operatorname{Int} M = \bigcup_U \varphi^{-1}(\operatorname{Int}(\varphi(U))) \quad \textit{and} \quad \partial M = \bigcup_U \varphi^{-1}(\partial(\varphi(U)))$$

*called, respectively, the **interior** and **boundary** of M.*

The definition of Int M and ∂M makes sense in view of Lemma 7.2.4. Note that

1. $\operatorname{Int} M$ is a manifold (with atlas obtained from (U, φ) by replacing $\varphi(U) \subset E_\lambda$ by the set $\operatorname{Int} \varphi(U) \subset E$);
2. ∂M is a manifold (with atlas obtained from (U, φ) by replacing $\varphi(U) \subset E_\lambda$ by $\partial\varphi(U) \subset \partial E = \ker \lambda$);
3. ∂M is the topological boundary of $\operatorname{Int} M$ in M (although $\operatorname{Int} M$ is *not* the topological interior of M).

Summarizing, we have proved the following.

7.2.6 Proposition. *If M is a manifold with boundary, then its interior* $\operatorname{Int} M$ *and its boundary* ∂M *are smooth manifolds without boundary. Moreover, if $f : M \to N$ is a diffeomorphism, N being another manifold with boundary, then f induces, by restriction, two diffeomorphisms*

$$\operatorname{Int} f : \operatorname{Int} M \to \operatorname{Int} N \quad \textit{and} \quad \partial f : \partial M \to \partial N.$$

If $n = \dim M$*, then* $\dim(\operatorname{Int} M) = n$ *and* $\dim(\partial M) = n - 1$.

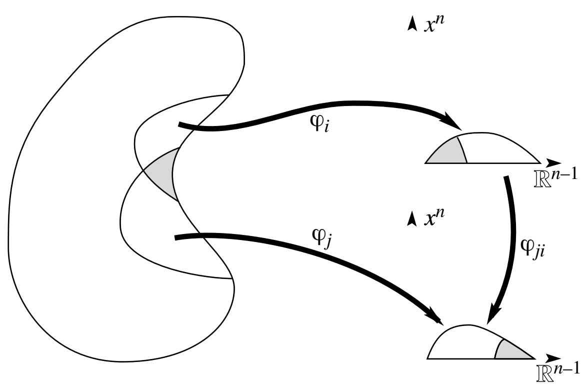


FIGURE 7.2.2. Boundary charts

To integrate a differential n-form over an n-manifold M, M must be oriented. If $\operatorname{Int} M$ is oriented, we want to choose an orientation on ∂M compatible with it. In the classical Stokes theorem for surfaces, it is crucial that the boundary curve be oriented, as in Figure 7.2.3.

The tangent bundle to a manifold with boundary is defined in the same way as for manifolds without boundary. Recall that any tangent vector in T_xM has the form $[dc(t)/dt]|_{t=\tau}$, where $c : [a, b] \to M$ is a C^1 curve, $a < b$, and $\tau \in [a, b]$. If $x \in \partial M$, we consider curves $c : [a, b] \to M$ such that $c(b) = x$. If $\varphi : U \to U' \subset E_\lambda$ is a chart at m, then $[d(\varphi \circ c)(t)/dt]|_{t=b}$ in general points *out* of U', as in Figure 7.2.4.

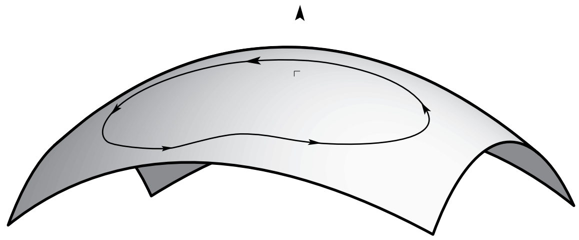


FIGURE 7.2.3. Orientation for surfaces

Therefore, T_xM is isomorphic to the model space E of M even if $x \in \partial M$ (see Figure 7.2.5). It is because of this result that tangent vectors are derivatives of C^1-curves defined on *closed* intervals. Had we defined tangent vectors as derivatives of C^1-curves defined on *open* intervals, T_yM for $y \in \partial M$ would be isomorphic to $\ker \lambda$ and not to E. In Figure 7.2.4, $E = \mathbb{R}^n$, λ is the projection onto the n-th factor, and $c(t)$ is defined on a closed interval whereas the C^1-curve $d(t)$ is defined on an open interval.

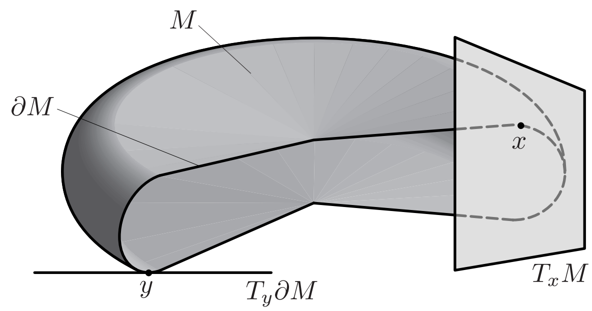


FIGURE 7.2.4. Tangent spaces at the boundary.

Having defined the tangent bundle, all of our previous constructions including tensor fields and exterior forms as well as operations on them such as the Lie derivative, interior product, and exterior derivative carry over directly to manifolds with boundary. One word of caution though: the fundamental relation between Lie derivatives and flows still holds if one is careful to take into account that a vector field on M has integral curves which could run into the boundary in finite time and with finite velocity. (If the vector field is tangent to ∂M, this will not happen.)

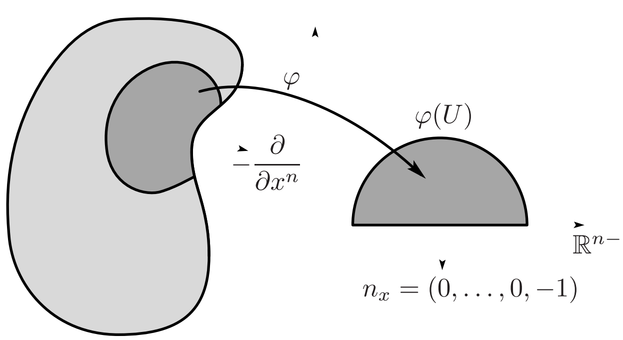


FIGURE 7.2.5. Oriented boundary charts

Next, we turn to the problem of orientation. As for manifolds without boundary a ***volume form*** on an n-manifold with boundary M is a nowhere vanishing n-form on M. Fix an orientation on $\mathbb{R}^n$. Then a chart (U, φ) is called ***positively oriented*** if $T_u\varphi : T_uM \to \mathbb{R}^n$ is orientation preserving for all $u \in U$. If M is paracompact this latter condition is equivalent to orientability of M (the proof is as in Proposition 6.5.2). Therefore, for paracompact manifolds, an orientation on M is just a smooth choice of orientations of all the tangent spaces, "smooth" meaning that for all the charts of a certain atlas, called the ***oriented charts***, the maps $\mathbf{D}(\varphi_j \circ \varphi_i^{-1})(x) : \mathbb{R}^n \to \mathbb{R}^n$ are orientation preserving.

The reader may wonder why for finite dimensional manifolds we did not choose a "standard" half-space, like $x^n \geq 0$ to define the charts at the boundary. Had we done that, the very definition of an oriented chart would be in jeopardy. For example, consider $M = [0, 1]$ and agree that all charts must have range in $\mathbb{R}_+ = \{\, x \in \mathbb{R} \mid x \geq 0 \,\}$. Then an example of an orientation reversing chart at $x = 1$ is $\varphi(x) = 1 - x$; in fact, every chart at $x = 1$ would be orientation reversing. However, if we admit *any* half-space of $\mathbb{R}$, so charts can be also in $\mathbb{R}_- = \{x \in \mathbb{R} \mid x \leq 0\}$, then a positively oriented chart at 1 is $\varphi(x) = x - 1$. See Figure 7.2.6.

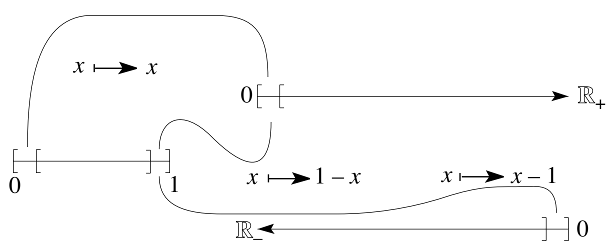


FIGURE 7.2.6. Boundary charts for $[0, 1]$

Once oriented charts and atlases are defined, the theory of integration for oriented paracompact manifolds with boundary proceeds as in §7.1.

Finally we define the ***boundary orientation*** of ∂M. At every $x \in \partial M$, the linear space $T_x(\partial M)$ has codimension one in T_xM so that there are (in a chart on M intersecting ∂M) exactly two kinds of vectors not in $\ker \lambda$: those for which their representatives v satisfy $\lambda(v) > 0$ or $\lambda(v) < 0$, that is, the ***inward*** and ***outward*** pointing vectors. By Lemma 7.2.4, a change of chart does not affect the property of a vector being outward or inward (see Figure 7.2.4). If $\dim M = n$, these considerations enable us to define the induced orientation of ∂M in the following way.

7.2.7 Definition. *Let M be an oriented n-manifold with boundary, $x \in \partial M$ and $\varphi : U \to \mathbb{R}^n_\lambda$ a positively oriented chart, where $\lambda \in (\mathbb{R}^n)^*$. A basis $\{v_1, \ldots, v_{n-1}\}$ of $T_x(\partial M)$ is called **positively oriented** if*

$$\left\{(T_x\varphi)^{-1}(n), v_1, \ldots, v_{n-1}\right\}$$

is positively oriented in the orientation of M, where n is any outward pointing vector to $\mathbb{R}^n_\lambda$ at $\varphi(x)$.

For example, we could choose for n the outward pointing vector to $\mathbb{R}^n_\lambda$ and perpendicular to $\ker \lambda$. If $\lambda : \mathbb{R}^n \to \mathbb{R}$ is the projection on the n-th factor, then $(T_x\varphi)^{-1}(n) = -\partial/\partial x^n$ and the situation is illustrated in Figure 7.2.5.

7.2.8 Theorem (Stokes' Theorem)**.** *Let M be an oriented smooth paracompact n-manifold with boundary and $\alpha \in \Omega^{n-1}(M)$ have compact support. Let $i : \partial M \to M$ be the inclusion map so that $i^*\alpha \in \Omega^{n-1}(\partial M)$. Then*

$$\int_{\partial M} i^*\alpha = \int_M \mathbf{d}\alpha \tag{7.2.1a}$$

or for short,

$$\int_{\partial M} \alpha = \int_M \mathbf{d}\alpha \tag{7.2.1b}$$

If $\partial M = \varnothing$, the left hand side of equation (7.2.1a) *or* (7.2.1b) *is set equal to zero.*

Proof. Since integration was constructed with partitions of unity subordinate to an atlas and both sides of the equation to be proved are linear in α, we may assume without loss of generality that α is a form on $U \subset \mathbb{R}^n_+$ with compact support. Write

$$\alpha = \sum_{i=1}^{n} (-1)^{i-1} \alpha^i dx^1 \wedge \cdots \wedge (dx^i)\hat{} \wedge \cdots \wedge dx^n, \tag{7.2.2}$$

where ˆ above a term means that it is deleted. Then

$$\mathbf{d}\alpha = \sum_{i=1}^{n} \frac{\partial \alpha^i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n, \tag{7.2.3}$$

and thus

$$\int_U \mathbf{d}\alpha = \sum_{i=1}^{n} \int_{\mathbb{R}^n} \frac{\partial \alpha^i}{\partial x^i} dx^1 \cdots dx^n. \tag{7.2.4}$$

There are two cases: $\partial U = \varnothing$ and $\partial U \neq \varnothing$. If $\partial U = \varnothing$, we have $\int_{\partial U} \alpha = 0$. The integration of the ith term in the sum occurring in equation (7.2.4) is

$$\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial \alpha^i}{\partial x^i} dx^i \right) dx^1 \cdots (dx^i)\hat{} \cdots dx^n \quad \text{(no sum)} \tag{7.2.5}$$

and $\int_{-\infty}^{+\infty} (\partial \alpha^i / \partial x^i) dx = 0$ since α^i has a compact support. Thus, the expression in equation (7.2.4) is zero as desired.

If $\partial U \neq \varnothing$, then we can do the same trick for each term except the last, which is, by the fundamental theorem of calculus,

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \left(\int_0^{\infty} \frac{\partial \alpha^n}{\partial x^n} dx^n \right) dx^1 \cdots dx^{n-1} \\ &\qquad = - \int_{\mathbb{R}^{n-1}} \alpha^n (x^1, \ldots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}. \end{aligned} \tag{7.2.6}$$

since α^n has compact support. Thus,

$$\int_U \mathbf{d}\alpha = - \int_{\mathbb{R}^{n-1}} \alpha^n (x^1, \ldots, x^{n-1}, 0) dx^1 \cdots dx^{n-1} \tag{7.2.7}$$

On the other hand,

$$\int_{\partial U} \alpha = \int_{\partial \mathbb{R}^n_+} \alpha = \int_{\partial \mathbb{R}^n_+} (-1)^{n-1} \alpha^n (x^1, \ldots, x^{n-1}, 0) dx^1 \wedge \cdots \wedge dx^{n-1}. \tag{7.2.8}$$

But $\mathbb{R}^{n-1} = \partial \mathbb{R}^n_+$ and the usual orientation on $\mathbb{R}^{n-1}$ is *not* the boundary orientation. The outward unit normal is $-e_n = (0, \ldots, 0, -1)$ and hence the boundary orientation has the sign of the ordered basis $\{-e_n, e_1, \ldots, e_{n-1}\}$, which is $(-1)^n$. Thus equation (7.2.8) becomes

$$\begin{aligned} \int_{\partial U} \alpha &= \int_{\partial \mathbb{R}^n_+} (-1)^{n-1} \alpha^n (x^1, \ldots, x^{n-1}, 0) dx^1 \wedge \cdots \wedge dx^{n-1} \\ &= (-1)^{2n-1} \int_{\mathbb{R}^{n-1}} \alpha^n (x^1, \ldots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}. \end{aligned} \tag{7.2.9}$$

Since $(-1)^{2n-1} = -1$, combining equations (7.2.7) and (7.2.9), we get the desired result. ■

This basic theorem reduces to the usual theorems of Green, Stokes, and Gauss in $\mathbb{R}^2$ and $\mathbb{R}^3$, as we shall see in the next section. For forms with less smoothness or without compact support, the best results are somewhat subtle. See Gaffney [1954], Morrey [1966], Yau [1976], Karp [1981] and the remarks at the end of Supplement 7.2B.

Next we draw some important consequences from Stokes' theorem.

7.2.9 Theorem (Gauss' Theorem)**.** *Let M be an oriented paracompact n-manifold with boundary and X a vector field on M with compact support. Let μ be a volume form on M. Then*

$$\int_M (\operatorname{div} X)\mu = \int_{\partial M} \mathbf{i}_X \mu. \tag{7.2.10}$$

Proof. Recall that

$$(\operatorname{div} X)\mu = \pounds_X \mu = \mathbf{d}\mathbf{i}_X \mu + \mathbf{i}_X \mathbf{d}\mu = \mathbf{d}\mathbf{i}_X \mu.$$

The result is thus a consequence of Stokes' theorem. ■

If M carries a Riemannian metric, there is a unique outward–pointing unit normal $n_{\partial M}$ along ∂M, and M and ∂M carry corresponding uniquely determined volume forms μ_M and $\mu_{\partial M}$. Then Gauss' theorem reads as follows.

7.2.10 Corollary.

$$\int_M (\operatorname{div} X)\, d\mu_M = \int_{\partial M} \langle\langle X, n_{\partial M} \rangle\rangle\, d\mu_{\partial M},$$

where $\langle\langle X, n_{\partial M}\rangle\rangle$ is the inner product of X and $n_{\partial M}$ is the outward unit normal.

Proof. Let $\mu_{\partial M}$ denote the volume element on ∂M induced by the Riemannian volume element $\mu_M \in \Omega^n(M)$; that is, for any positively oriented basis $v_1, \ldots, v_{n-1} \in T_x(\partial M)$, and charts chosen so that $n_{\partial M} = -\partial/\partial x^n$ at the point x,

$$\mu_{\partial M}(x)(v_1, \ldots, v_{n-1}) = \mu_M(x)\left(-\frac{\partial}{\partial x^n}, v_1, \ldots, v_{n-1}\right).$$

Since

$$\begin{aligned}(\mathbf{i}_X \mu_M)(x)(v_1, \ldots, v_{n-1}) &= \mu_M(x)\left(X^i(x) v_i + X^n(x)\frac{\partial}{\partial x^n}, v_1, \ldots, v_{n-1}\right)\\ &= X^n(x)\mu_{\partial M}(x)(v_1, \ldots, v_{n-1})\end{aligned}$$

and $X^n = -\langle\langle X, n_{\partial M}\rangle\rangle$, the corollary follows by Gauss' theorem. ■

7.2.11 Corollary. *If X is divergence-free on a compact boundaryless manifold with a volume element μ, then X as an operator is skew-symmetric; that is, for f and $g \in \mathcal{F}(M)$,*

$$\int_M X[f] g\mu = -\int_M f X[g]\mu.$$

Proof. Since X is divergence free, $\pounds_X(h\mu) = (\pounds_X h)\mu$ for any $h \in \mathcal{F}(M)$. Thus,

$$X[f]g\mu + fX[g]\mu = \pounds_X(fg)\mu = \pounds_X(fg\mu).$$

Integration and the use of Stokes' theorem gives the result. ■

7.2.12 Corollary. *If M is compact without boundary $X \in \mathfrak{X}(M)$, $\alpha \in \Omega^k(M)$, and $\beta \in \Omega^{n-k}(M)$, then*

$$\int_M \pounds_X \alpha \wedge \beta = -\int_M \alpha \wedge \pounds_X \beta.$$

Proof. Since $\alpha \wedge \beta \in \Omega^n(M)$, the formula follows by integrating both sides of the relation $\mathbf{d} \mathbf{i}_X(\alpha \wedge \beta) = \pounds_X(\alpha \wedge \beta) = \pounds_X \alpha \wedge \beta + \alpha \wedge \pounds_X \beta$ and using Stokes' theorem. ■

7.2.13 Corollary. *If M is a compact orientable, boundaryless n-dimensional pseudo-Riemannian manifold with a metric g of index* $\operatorname{Ind}(g)$, *then* $\mathbf{d}$ *and* δ *are adjoints, that is,*

$$\int_M \langle \mathbf{d}\alpha, \beta \rangle \mu = \int_M \mathbf{d}\alpha \wedge *\beta = \int_M \alpha \wedge *\delta\beta = \int_M \langle \alpha, \delta\beta \rangle \mu$$

for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$.

Proof. Recall from Definition 6.5.21 that

$$\delta\beta = (-1)^{nk+1+\operatorname{Ind}(g)} * \mathbf{d} * \beta,$$

so that

$$\begin{aligned} \mathbf{d}\alpha \wedge *\beta - \alpha \wedge *\delta\beta &= \mathbf{d}\alpha \wedge *\beta + (-1)^{nk+\operatorname{Ind}(g)} \alpha \wedge * * \mathbf{d} * \beta \\ &= \mathbf{d}\alpha \wedge *\beta + (-1)^{nk+\operatorname{Ind}(g)+k(n-k)+\operatorname{Ind}(g)} \alpha \wedge \mathbf{d}^* \beta \\ &= \mathbf{d}\alpha \wedge *\beta + (-1)^k \alpha \wedge \mathbf{d} * \beta \\ &= \mathbf{d}(\alpha \wedge *\beta) \end{aligned}$$

since $k^2 + k$ is an even number for any integer k. Integrating both sides of the equation and using Stokes' theorem gives the result. ■

The same identity

$$\iint \langle \mathbf{d}\alpha, \beta \rangle \mu = \int_M \langle \alpha, \delta\beta \rangle \mu$$

holds for noncompact manifolds, possibly with boundary, provided either α or β has compact support in $\operatorname{Int}(M)$.

Supplement 7.2A
Stokes' Theorem for Nonorientable Manifolds

Let M be a nonorientable paracompact n-manifold with a smooth boundary ∂M and inclusion map $i : \partial M \to M$. We would like to give meaning to the formula

$$\int_M \mathbf{d}\rho = \int_{\partial M} i^* \rho.$$

in Stokes' theorem. Clearly, both sides makes sense if $\mathbf{d}\rho$ and $i^*(\rho)$ are defined in such a way that they are densities on M and ∂M, respectively. Here $\mathbf{d}$ should be some operator analogous to the exterior differential, and ρ should be a section of some bundle over M analogous to $\bigwedge^{n-1}(M)$. Denote the as yet unknown bundle analogous to $\bigwedge^k(M)$ by $\bigwedge_t^k(M)$ and its space of sections $\Omega_\tau^k(M)$. Then we desire an operator $\mathbf{d} : \Omega_\tau^k(M) \to \Omega_\tau^{k+1}(M)$, $k = 0, \ldots, n$, and desire $\bigwedge_\tau^n(M)$ to be isomorphic to $|\bigwedge|(M)$.

To guess what $\bigwedge_\tau^k(M)$ might be, let us first discuss $\bigwedge_\tau^n(M)$. The key difference between an n-form ω and a density ρ is their transformation property under a linear map $A : T_mM \to T_mM$ as follows:

$$\omega(m)(A(v_1), \ldots, A(v_n)) = (\det A)\, \omega(m)(v_1, \ldots, v_n)$$

$$\rho(m)(A(v_1), \ldots, A(v_n)) = |\det A|\, \rho(m)(v_1, \ldots, v_n)$$

for $m \in M$ and $v_1, \ldots, v_n \in T_mM$. If $v_1, \ldots, v_n$ is a basis, then $\det(A) > 0$ if A preserves the orientation given by $v_1, \ldots, v_n$ and $\det(A) < 0$ if A reverses this orientation. Thus ρ can be thought of as an object behaving like an n-form at every $m \in M$ once an orientation of T_mM is given; that is, ρ should be thought of as an n-form with values in some line bundle (a bundle with one-dimensional fibers) associated with the concept of orientation. This definition would then generalize to any k; $\bigwedge_\tau^k(M)$ will be line-bundle-valued k-forms on M. We shall now construct this line bundle.

At every point of M there are two orientations. Using them, we construct the oriented double covering $\widetilde{M} \to M$ (see Proposition 6.5.7). Since $\widetilde{M}$ is not a line bundle, some other construction is in order. At every $m \in M$, a line is desired such that the positive half-line should correspond to one orientation of T_mM and the negative half-line to the other. The fact that must be taken into account is that multiplication by a negative number switches these two half-lines. To incorporate this idea, identify $(m, [\mu], a)$ with $(m, [-\mu], -a)$ where $m \in M$, $a \in \mathbb{R}$, and $[\mu]$ is an orientation of T_mM. Thus, define the ***orientation line bundle*** $\sigma(M) = \{\, (m, [\mu], a) \mid m \in M,\ a \in \mathbb{R}, \text{ and } [\mu] \text{ is an orientation of } T_mM)\,\}/\!\sim$ where $\sim$ is the equivalence relation $(m, [\mu], a) \sim (m, [-\mu], -a)$. Denote by $\langle m, [\mu], a\rangle$ the elements of $\sigma(M)$. It can be checked that the map $\pi : \sigma(M) \to M$ defined by $\pi(\langle m, [\mu], a\rangle) = m$ is a line bundle with bundle charts given by

$$\psi : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}, \quad \psi(\langle m, [\mu], a\rangle) = (\varphi(m), \epsilon a),$$

where $\varphi : U \to \mathbb{R}^n$ is a chart for M at m, and $\epsilon = +1$ if $T_m\varphi : (T_mM, [\mu]) \to (\mathbb{R}^n, [\omega])$ is orientation preserving and -1 if it is orientation reversing, $[\omega]$ being a fixed orientation of $\mathbb{R}^n$. The change of chart map of the line bundle $\sigma(M)$ is given by

$$\begin{aligned}&(x, a) \in U' \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}\\ &\qquad \mapsto \left((\varphi_j \circ \varphi_i)^{-1}(x), \operatorname{sign}(\det \mathbf{D}(\varphi_j \circ \varphi_i^{-1})(x))\right) \in U' \times \mathbb{R}.\end{aligned}$$

If M is paracompact, then $\sigma(M)$ is an orientable vector bundle (see Exercise 6.5-14). If in addition M is also connected, then M is orientable if and only if $\sigma(M)$ is trivial line bundle; the proof is similar to that of Proposition 6.5.7.

7.2.14 Definition. *A* ***twisted*** k***-form*** *on M is a $\sigma(M)$-valued k-form on M. The bundle of twisted k-forms is denoted by $\bigwedge_\tau^k(M)$ and sections of this bundle are denoted $\Omega_\tau^k(M)$ or $\Gamma^\infty(\bigwedge_\tau^k(M))$.*

Locally, a section $\rho \in \Omega_\tau^k(M)$ can be written as $\rho = \alpha\xi$ where $\alpha \in \Omega^k(U)$ and ξ is an orientation of U regarded as a locally constant section of $\sigma(M)$ over U. The operators

$$\begin{aligned}&\mathbf{d} : \Omega_\tau^k(M) \to \Omega_\tau^{k+1}(M) \text{ and}\\ &\mathbf{i}_X : \Omega_\tau^k(M) \to \Omega_\tau^{k-1}(M), \text{ where } X \in \mathfrak{X}(M),\end{aligned}$$

are defined to be the unique operators such that if $\rho = \alpha\xi$ in the neighborhood U, then $\mathbf{d}\rho = (\mathbf{d}\alpha)\xi$ and $\mathbf{i}_X\rho = (\mathbf{i}_X\alpha)\xi$. One has $\pounds_X = \mathbf{i}_X \circ \mathbf{d} + \mathbf{d} \circ \mathbf{i}_X$. Note that if M is orientable, $\bigwedge_\tau^k(M)$ coincides with $\bigwedge^k(M)$.

Next we show that the line bundles $|\bigwedge(M)|$ and $\bigwedge_\tau^n(M)$ are isomorphic. If $\lambda \in |\bigwedge(M)|_m$ and $v_1, \ldots, v_n \in T_mM$, define

$$\varphi(\lambda) : T_mM \times \cdots \times T_mM \to \sigma(M)_m$$

by setting

$$\Phi(\lambda)(v_1, \ldots, v_n) = \langle m, [\sigma(v_1, \ldots, v_n)], \lambda(v_1, \ldots, v_n)\rangle,$$

if $\{v_1, \ldots, v_n\}$ is a basis of T_mM, and setting it equal to 0, if $\{v_1, \ldots, v_n\}$ are linearly dependent, where $[\sigma(v_1, \ldots, v_n)]$ denotes the orientation of T_mM given by the ordered basis $\{v_1, \ldots, v_n\}$. $\Phi(\lambda)$ is skew symmetric and homogeneous with respect to scalar multiplication since if $\{v_1, \ldots, v_n\}$ is a basis and $a \in \mathbb{R}$, we have

$$\begin{aligned}\Phi(\lambda)(v_2, v_1, v_3, \ldots, v_n) &= \langle m, [\sigma(v_2, v_1, v_3, \ldots, v_n)], \lambda(v_2, v_1, v_3, \ldots, v_n)\rangle \\ &= \langle m, [-\sigma(v_1, v_2, \ldots, v_n)], \lambda(v_1, \ldots, v_n)\rangle \\ &= \langle m, [\sigma(v_1, \ldots, v_n)], -\lambda(v_1, \ldots, v_n)\rangle \\ &= -\Phi(\lambda)(v_1, \ldots, v_n)\end{aligned}$$

and

$$\begin{aligned}\Phi(\lambda)(av_1, v_2, \ldots, v_n) &= \langle m, [\sigma(av_1, v_2, \ldots, v_n)], \lambda(av_1, \ldots, v_n)\rangle \\ &= \langle m, [(\operatorname{sign} a)\sigma(v_1, \ldots, v_n)], |a|\lambda(v_1, \ldots, v_n)\rangle \\ &= \langle m, [\sigma(v_1, \ldots, v_n)], a\lambda(v_1, \ldots, v_n)\rangle \\ &= a\Phi(\lambda)(v_1, \ldots, v_n).\end{aligned}$$

The proof of additivity is more complicated. Let $v_1, v_1', v_2, \ldots, v_n \in T_mM$. If both $\{v_1, \ldots, v_n\}$ and $\{v_1', v_2, \ldots, v_n\}$ are linearly dependent, then so are $\{v_1+v_1', v_2, \ldots, v_n\}$ and the additivity property of $\Phi(\lambda)$ is trivially verified. So assume that $\{v_1, \ldots, v_n\}$ is a basis of T_mM and write $v_1' = a^1v_1 + \cdots + a^nv_n$. Therefore

$$\begin{aligned}\lambda(v_1', v_2, \ldots, v_n) &= |a^1|\lambda(v_1, \ldots, v_n), \text{ and} \\ \lambda(v_1 + v_1', v_2, \ldots, v_n) &= |1 + a^1|\lambda(v_1, \ldots, v_n).\end{aligned}$$

Moreover, if

(i) $a^1 > 0$, then

$$[\sigma(v_1, \ldots, v_n)] = [\sigma(v_1', v_2, \ldots, v_n)] = [\sigma(v_1 + v_1', v_2, \ldots, v_n)];$$

(ii) $a^1 = 0$, then

$$[\sigma(v_1, \ldots, v_n)] = [\sigma(v_1 + v_1', v_2, \ldots, v_n)]$$

and

$$\Phi(\lambda)(v_1', v_2, \ldots, v_n) = 0;$$

(iii) $-1 < a^1 < 0$, then

$$[\sigma(v_1, \ldots, v_n)] = [-\sigma(v_1', v_2, \ldots, v_n)] = [\sigma(v_1 + v_1', v_2, \ldots, v_n)];$$

(iv) $a^1 = -1$, then

$$[\sigma(v_1, \ldots, v_n)] = [-\sigma(v_1', v_2, \ldots, v_n)]$$

and

$$\Phi(\lambda)(v_1 + v_1', v_2, \ldots, v_n) = 0;$$

(v) $a^1 < -1$, then

$$[\sigma(v_1, \ldots, v_n)] = [-\sigma(v_1', v_2, \ldots, v_n)] = [-\sigma(v_1 + v_1', v_2, \ldots, v_n)].$$

Additivity is now checked in all five cases separately. For example, in case (iii) we have

$$\begin{aligned}\Phi(\lambda)(v_1 + v_1', v_2, \ldots, v_n) &= \langle m, [\sigma(v_1 + v_1', v_2, \ldots, v_n)], \lambda(v_1 + v_1', \ldots, v_n)\rangle \\ &= \langle m, [\sigma(v_1, \ldots, v_n)], (1 + a^1)\lambda(v_1, \ldots, v_n)\rangle \\ &= \langle m, [\sigma(v_1, \ldots, v_n)], \lambda(v_1, \ldots, v_n)\rangle \\ &\quad + \langle m, [-\sigma(v_1', v_2, \ldots, v_n)], -|a^1|\lambda(v_1, \ldots, v_n)] \\ &= \Phi(\lambda)(v_1, \ldots, v_n) + \Phi(\lambda)(v_1', v_2, \ldots, v_n).\end{aligned}$$

Thus Φ has values in $\bigwedge_\tau^n(M)$. The map Φ is clearly linear and injective and thus is an isomorphism of $|\bigwedge(M)|$ with $\bigwedge_\tau^n(M)$. Denote also by Φ the induced isomorphism of $|\Omega(M)|$ with $\Omega_\tau^n(M)$.

The integral of $\rho \in \Omega_\tau^n(M)$ is defined to be the integral of the density $\Phi^{-1}(\rho)$ over M. In local coordinates the expression for Φ is

$$\Phi(a|dx^1 \wedge \cdots \wedge dx^n|) = (a\, dx^1 \wedge \cdots \wedge dx^n)\xi_0^n,$$

where ξ_0^n is the basis element of the space sections of $\sigma(U)$ given by $\xi_0^n(u)(v_1, \ldots, v_n) = \langle u, [\sigma(v_1, \ldots, v_n)], \operatorname{sign}(\det(v_j^i))\rangle$, where (v_j^i) are the components of the vector v_j in the coordinates $(x^1, \ldots, x^n)$ of U. Therefore

$$\Phi^{-1}((a\, dx^1 \wedge \cdots \wedge dx^n)\xi) = \frac{a\xi}{\xi_0^n}|dx^1 \wedge \cdots \wedge dx^n|$$

and

$$\int_U (a\, dx^1 \wedge \cdots \wedge dx^n) b\xi_0^n = \int_U ab|dx^1 \wedge \cdots \wedge dx^n|$$

for any smooth functions $a, b : U \to \mathbb{R}$.

Finally, for the formulation of Stokes' Theorem, if $i : \partial M \to M$ is the inclusion and $\rho \in \Omega_\tau^{n-1}(M)$, the induced twisted $(n-1)$-form $i^*\rho$ on ∂M is defined by setting

$$\begin{aligned}(i^*\rho)(m)(v_1, \ldots, v_{n-1}) = \langle m, [&\operatorname{sign}[\mu_n]\sigma(-\partial/\partial x^n, v_1, \ldots, v_{n-1})], \\ &\rho'(m)(v_1, \ldots, v_{n-1})\rangle,\end{aligned}$$

if $v_1, \ldots, v_{n-1}$ are linearly independent and setting it equal to zero, if $v_1, \ldots, v_{n-1}$ are linearly dependent, where $(x^1, \ldots, x^n)$ is a coordinate system at m with ∂M described by $x^n = 0$ and

$$\rho(m)(v_1, \ldots, v_{n-1}) = \langle m, \operatorname{sign}[\mu_m], \rho'(m)(v_1, \ldots, v_{n-1})\rangle$$

with $\rho'(m)$ skew symmetric; moreover $\operatorname{sign}[\mu_m] = +1$ (respectively, -1) if $[\mu_m]$ and $[\sigma(-\partial/\partial x^n, v_1, \ldots, v_{n-1})]$ define the same (respectively, opposite) orientation of T_mM. If $M = U$, where U is open in $\mathbb{R}_+^n$ and $\rho = \alpha\, a\xi_0 \in \Omega_\tau^{n-1}(U)$, then

$$i^*\rho = (-1)^n i^*(a\alpha)\xi_0^{n-1}.$$

In particular, if

$$\zeta = \sum_{i=1}^n \zeta_i dx^1 \wedge \cdots \wedge (dx^i)\hat{} \wedge \cdots \wedge dx^n,$$

we have

$$\begin{aligned}(i^*\rho)(x^1,\ldots,x^{n-1}) &= (-1)^n a(x^1,\ldots,x^{n-1},0)(-1)^{n-1}\alpha^n(x^1,\ldots,x^{n-1},0)\\ &\quad dx^1\wedge\cdots\wedge(dx^i)\hat{}\wedge\cdots\wedge dx^n\xi_0^{n-1}\\ &= -a(x^1,\ldots,x^{n-1},0)\alpha^n(x^1,\ldots,x^{-1},0)\\ &\quad dx^1\wedge\cdots\wedge(dx^i)\hat{}\wedge\cdots\wedge dx^n\xi_0^{n-1}.\end{aligned}$$

With this observation, the proof of Stokes' theorem 7.2.8 gives the following.

7.2.15 Theorem (Nonorientable Stokes' Theorem)**.** *Let M be a paracompact nonorientable n-manifold with smooth boundary ∂M and $\rho \in \Omega_\tau^{n-1}(M)$, a twisted $(n-1)$-form with compact support. Then*

$$\int_M \mathbf{d}\rho = \int_{\partial M} i^*\rho.$$

The same statement holds for vector-valued twisted $(n-1)$-forms and all corollaries go through replacing everywhere $(n-1)$-forms with twisted $(n-1)$-forms. For example, we have the following.

7.2.16 Theorem (Nonorientable Gauss Theorem)**.** *Let M be a nonorientable Riemannian n-manifold with associated density μ_M. Then for $X \in \mathfrak{X}(M)$ with compact support*

$$\int_M \operatorname{div}(X)\mu_M = \int_{\partial M}(X\cdot n)\mu_{\partial M}$$

where n is the outward unit normal of ∂M, $\mu_{\partial M}$ is the induced Riemannian density of ∂M and $\pounds_X\mu_M = (\operatorname{div} X)\mu_M$.

For a concrete situation in $\mathbb{R}^3$ involving these ideas, see Exercise 7.3-9.

Supplement 7.2B
Stokes' Theorem on Manifolds with Piecewise Smooth Boundary

The statement of Stokes' theorem we have given does not apply when M is, say a cube or a cone, since these sets do not have a smooth boundary. If the singular portion of the boundary (the four vertices and 12 edges in case of the cube, the vertex and the base circle in case of the cone), is of Lebesgue measure zero (within the boundary) it should not contribue to the boundary integral and we can hope that Stokes' theorem still holds. This supplement discusses such a version of Stokes' theorem inspired by Holmann and Rummler [1972]. (See Lang [1972] for an alternative approach.)

First we shall give the definition of a manifold with piecewise smooth boundary. A glance at the definition of a manifold with boundary makes it clear that one could define a manifold with *corners*, by choosing charts that make regions near the boundary diffeomorphic to open subsets of a finite intersection of positive closed half-spaces. Unfortunately, singular points on the boundary—such as the vertex of a cone—need not be of this type. Thus, instead of trying to classify the singular points up to diffeomorphism and then make a formal intrinsic definition, it is simpler to consider manifolds already embedded in a bigger manifold. Then we can impose a condition on the boundary to insure the validity of Stokes' theorem.

7.2.17 Definition. *Definition Let $U \subset \mathbb{R}^{n-1}$ be open and $f : U \to \mathbb{R}$ be continuous. A point p on the graph of f, $\Gamma_f = \{ (x, f(x)) \mid x \in U \}$, is called **regular** if there is an open neighborhood V of p such that $V \cap \Gamma_f$ is an $(n-1)$ dimensional smooth submanifold of V. Let ρ_f denote the set of regular points. Any point in $\sigma_f = \Gamma_f \backslash \rho_f$ is called **singular**. The mapping f is called **piecewise smooth** if ρ_f is Lebesgue measurable, $\pi(\sigma_f)$ has measure zero in U (where $\pi : U \times \mathbb{R} \to U$ is the projection) and $f|\pi(\sigma_f)$ is locally Hölder; that is, for each compact set $K \subset \pi(\sigma_f)$ there are constants $c(K) > 0$, $0 < \alpha(K) \leq 1$ such that*

$$|f(x) - f(y)| \leq c(K)\|x - y\|^{\alpha(K)}$$

for all $x, y \in K$.

Note that ρ_f is open in Γ_f and that $\operatorname{Int}(\Gamma_f^-) \cup \rho_f$, where

$$\Gamma_f^- = \{ (x, y) \in U \times \mathbb{R} \mid y \leq f(x) \},$$

is a manifold with boundary ρ_f. Thus ρ_f has positive orientation induced from the standard orientation of $\mathbb{R}^n$. This will be called the ***positive orientation*** of Γ_f. We are now ready to define manifolds with piecewise smooth boundary.

7.2.18 Definition. *Let M be an n-manifold. A closed subset N of M is said to be a **manifold with piecewise smooth boundary** if for every $p \in N$ there exists a chart (U, φ) of M at p, $\varphi(U) = U' \times U'' \subset \mathbb{R}^{n-1} \times \mathbb{R}$, and a piecewise smooth mapping $f : U' \to \mathbb{R}$ such that*

$$\varphi(\operatorname{bd}(N) \cap U) = \Gamma_f \cap \varphi(U)$$

and $\varphi(N \cap U) = \Gamma_f^- \cap \varphi(U)$. See Figure 7.2.7.

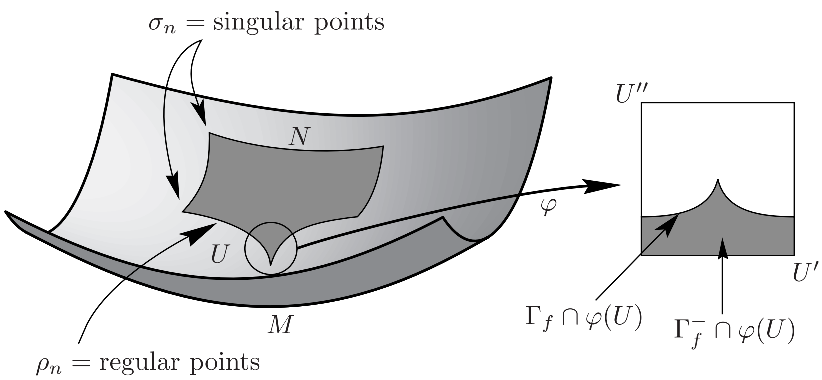


FIGURE 7.2.7. Singular boundary charts

It is readily verified that the condition on N is chart independent, using the fact that the composition of a piecewise smooth map with a diffeomorphism is still piecewise smooth. Thus, regular and singular points of $\operatorname{bd}(N)$ make intrinsic sense and are defined in terms of an arbitrary chart satisfying the conditions of the preceding definition. Let ρ_N and σ_N denote the regular and singular part of the boundary $\operatorname{bd}(N)$ of N in M.

To formulate Stokes' theorem, we define $\int_N \eta$, for η an n-form (respectively, density) on M with compact support. This is done as usual via a partition of unity; ρ_N and σ_N play no role since they have Lebesgue measure zero in every chart: ρ_N because it is an $(n-1)$-manifold and σ_N by definition.

It is not so simple to define $\int_{\operatorname{bd}(N)} \zeta$ for $\zeta \in \Omega^{n-1}(M)$ (respectively, a density). First a lemma is needed.

7.2.19 Lemma. *Let $\zeta \in \Omega^{n-1}(U \times \mathbb{R})$, where U is open in $\mathbb{R}^{n-1}$, $\operatorname{supp}(\zeta)$ is compact and $f : U \to \mathbb{R}$ is a piecewise smooth mapping. Then there is a smooth bounded function $a : \rho_f \to \mathbb{R}$, such that $i^*\zeta = a\lambda$ where $i : \rho_f \to U \times \mathbb{R}$ is the inclusion and $\lambda \in \Omega^{n-1}(\rho_f)$ is the boundary volume form induced by the canonical volume form of $U \times \mathbb{R} \subset \mathbb{R}^n$ on $\operatorname{Int}(\Gamma_f^-) \cup \rho_f$.*

Proof. The existence of the function a on ρ_f follows since $\Omega^{n-1}(\rho_f)$ is one-dimensional with a basis element λ. We prove that a is bounded. Let $p \in \rho_f$ and $v_1, \ldots, v_{n-1} \in T_p(\rho_f)$ be an orthonormal basis with respect to the Riemannian metric on ρ_f induced from the standard metric of $\mathbb{R}^n$, and denote by n the outward unit normal. Then

$$\begin{aligned} a(p) &= a(p)(dx^1 \wedge \cdots \wedge dx^n)(p)(n, v_1, \ldots, v_{n-1}) \\ &= a(p)\lambda(p)(v_1, \ldots, v_{n-1}) = \zeta(p)(v_1, \ldots, v_{n-1}). \end{aligned}$$

Let

$$v_i = v_i^j \left.\frac{\partial}{\partial x^j}\right|_p .$$

Since

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \quad \text{and} \quad n, v_1, \ldots, v_{n-1}$$

are orthonormal bases of $T_p(U \times \mathbb{R})$, we must have $|v_i^j| \le 1$ for all i, j. Hence if

$$\zeta = \sum_{i=1}^n \zeta_i dx^1 \wedge \cdots \wedge \widehat{(dx^i)} \wedge \cdots \wedge dx^n,$$

then

$$\begin{aligned} |a(p)| &= |\zeta(p)(v_1, \ldots, v_{n-1})| \\ &= \left| \sum_{i=1}^n \zeta_i(p) \sum_{\sigma \in S_{n-1}} (-1)^i (\operatorname{sign} \sigma) v_1^{\sigma(1)} \cdots v_{n-1}^{\sigma(n-1)} \right| \\ &= \sum_{i=1}^n |\zeta_i(p)|(n-1)! \end{aligned}$$

which is bounded, since ζ has compact support. ▼

In view of this lemma and the fact that σ_f has measure zero, we can define

$$\int_{\Gamma_f} \zeta = \int_{\rho_f} i^*\zeta = \int_{\rho_f} a\lambda.$$

Now we can define, via a partition of unity, the integral of $\eta \in \Omega^{n-1}(M)$ (or a twisted $(n-1)$-form) by

$$\int_{\operatorname{bd}(N)} \eta = \int_{\rho_N} \eta.$$

7.2.20 Theorem (Piecewise Smooth Stokes Theorem)**.** *Let M be a paracompact n-manifold and N a closed submanifold of M with piecewise smooth boundary. If*

(i) *M is orientable and $\omega \in \Omega^{n-1}(M)$ has compact support, or*

(ii) *M is nonorientable and $\omega \in \Omega_\tau^{n-1}(M)$ is a twisted $(n-1)$-form (see the preceding supplement) which has compact support, then*

$$\int_N \mathbf{d}\omega = \int_{\mathrm{bd}(N)} \omega.$$

The proof of this theorem reduces via a partition of unity to the local case. Thus it suffices to prove that if U is open in $\mathbb{R}^{n-1}$, $\omega \in \Omega^{n-1}(U \times \mathbb{R})$ has compact support, and $f : U \to \mathbb{R}$ is a piecewise smooth mapping, then

$$\int_{\Gamma_f^-} \mathbf{d}\omega = \int_{\Gamma_f} \omega. \tag{7.2.11}$$

The left-hand side of equation (7.2.11) is to be understood as the integral over the compact measurable set $\Gamma_f^- \cap \operatorname{supp}(\omega)$. For the proof of (7.2.11) we use three lemmas.

7.2.21 Lemma. *Equation (7.2.11) holds if ω vanishes in a neighborhood of σ_f in $U \times \mathbb{R}$.*

Proof. Let V be an open neighborhood of σ_f in $U \times \mathbb{R}$ on which ω vanishes and let W be another open neighborhood of σ_f (which is closed in V) such that $\mathrm{cl}(W) \cap (U \times \mathbb{R}) \subset V$. The set $O = (U \times \mathbb{R}) \setminus \mathrm{cl}(W)$ is open and since it is disjoint from σ_f, $\Gamma_f^- \subset O$ is an n-dimensional submanifold of O with $\mathrm{bd}(\Gamma_f^- \cap O) = \Gamma_f \cap O$. Since

$$\operatorname{supp}(\mathbf{d}\omega) \cap \Gamma_f^- \subset \Gamma_f^- \cap O \quad \text{and} \quad \operatorname{supp}(\omega) \cap \Gamma_f \subset \Gamma_f \cap O,$$

by the usual Stokes theorem, we have

$$\int_{\Gamma_f} \mathbf{d}\omega = \int_{\Gamma_f^- \cap O} \mathbf{d}\omega = \int_{\Gamma_{f \cap O}} \omega = \int_{\Gamma_f} \omega. \qquad \blacksquare$$

The purpose of the next two lemmas is to construct approximations to $\mathbf{d}\omega$ and ω if ω does not vanish near σ_f. For this we need translates of bump functions with control on their derivatives.

7.2.22 Lemma. *Let C be a box (rectangular parallelepiped) in $\mathbb{R}^n$ of edge lengths $2l_i$ and let D be the box with the same center as C but of edge lengths $4l_i/3$. There exists a C^∞ function $\varphi : \mathbb{R}^n \to [0,1]$ which is 1 on $\mathbb{R}^n \setminus C$, 0 on D and $|\partial\varphi/\partial x^i| \le A/l_i$, for a constant A independent of l_i.*

Proof. Assume we have found such a function $\varphi : \mathbb{R} \to [0,1]$ for $n = 1$. Then $\psi(x^1, \ldots, x^n) = \varphi(x^1) \ldots \varphi(x^n)$ is the desired function.

The function φ is found in the following way. Let $a = 2l/3$, $\epsilon = l/3$ and choose an integer N such that $2/N < \epsilon$. Let $h : \mathbb{R} \to [0,1]$ be a bump function that is equal to 1 for $|t| < 1/2$ and that vanishes for $|t| > 1$. Then $f : \mathbb{R} \to [0,1]$, defined by $f(t) = 1 - h(t)$ is a C^∞ function vanishing for $|t| < 1/2$ and equal to 1 for $|t| > 1$. Let $f_n(t) = f(nt)$ for all positive integers n and note that

$$|f_n'(t)| = n|f'(nt)| \le Cn.$$

Define the C^∞ function

$$\varphi(t) = \prod f_N\left(t - \frac{z}{2N}\right),$$

where the product is taken over integers z such that $|z| < 2Na + 1$. Note that if $|t| < a + 1/4N$ and $z \in \mathbb{Z}$ is chosen such that $|t - z/2N| < 1/4N$, then

$$f_N\left(t - \frac{z}{2N}\right) = 0 \quad \text{and} \quad |z| \le 2N|t| + \frac{1}{2} < 2Na + 1,$$

so that $\varphi(t) = 0$. Similarly if

$$|t| > a + \frac{2}{N} \quad \text{and} \quad |z| < 2Na + 1,$$

then

$$\left| t - \frac{z}{2N} \right| \geq |t| - \frac{|z|}{2N} > \frac{1}{N}$$

so that $\varphi(t) = 1$.

Finally, let $|t_0 - a| < 2/N$ and let $z_0 \in \mathbb{Z}$ be such that $|t_0 - z_0/2N| < 1/N$. All factors $f_N(t_0 - z/2N)$ are one in a neighborhood of t_0, unless $|t_0 - z/2N| \leq 1/N$. In that case we have the inequality

$$|z - z_0| \leq |z - 2Nt_0| + |2Nt_0 - z_0| \leq 3.$$

Hence at most seven factors in the product are not identically 1 in a neighborhood of t_0. Hence

$$|\varphi'(t_0)| \leq 7CN = \frac{A}{\epsilon}. \qquad \blacksquare$$

7.2.23 Lemma. *Let K be a compact subset of σ_f, the singular set of f. For every $\epsilon > 0$ there is a neighborhood U_ϵ of K in $U \times \mathbb{R}$ and a C^∞ function $\varphi_\epsilon : U \times \mathbb{R} \to [0, 1]$, which vanishes on a neighborhood of K in U_ϵ, is one on the complement of U_ϵ, and is such that*

(i) $\operatorname{vol}(U_\epsilon) \left[\sup_{x \in \mathbb{R}^n} \left| \frac{\partial \varphi_\epsilon(x)}{\partial x^i} \right| \right] \leq \epsilon, \quad i = 1, \ldots, n,$ *and*

(ii) $\operatorname{vol}(U_\epsilon) \leq \epsilon$ *and* $q(U_\epsilon \cap \rho_f) \leq \epsilon$, *where q is the measure on ρ_f associated with the volume form $\lambda \in \Omega^{n-1}(\rho_f)$, and $\operatorname{vol}(U_\epsilon)$ is the Lebesgue measure of U_ϵ in $\mathbb{R}^n$.*

Proof. Partition $\mathbb{R}^{n-1}$ by closed cubes D of edge length $4l/3$, $l \leq 1$. At most 2^n such cubes can meet at a vertex. The set $\pi(K)$, where $\pi : U \times \mathbb{R} \to U$ is the projection, can be covered by finitely many open cubes C of edge length $2l$, each one of these cubes containing a cube D and having the same center as C. Since $\pi(K)$ and K have measure zero, choose l so small that for given $\delta > 0$,

(i) the $(n-1)$-dimensional volume of $\bigcup_{i=1,\ldots,L} C_i$ is smaller than or equal to δ; and

(ii) $q\left(\pi^{-1}\left(\bigcup_{i=1,\ldots,L} C_i\right) \cap \rho_f\right) \leq \delta$.

Since f is locally Hölder and $\pi(K)$ is compact, there exist constants $0 < \alpha \leq 1$ and $k > 0$ such that

$$|f(x) - f(y)| \leq k\|x - y\|^\alpha$$

for $x, y \in \pi(K)$. We can assume $k \geq 1$ without loss of generality. In each of the sets $\pi^{-1}(C_i) = C_i \times \mathbb{R}$, choose a box P_i with base C_i and height $(2kl)^{1/\alpha}$ such that $\pi(K)$ is covered by parallelipeds P_i' with the same center as P_i and edge lengths equal to two-thirds of the edge lengths of P_i.

Let $V = \bigcup_{i=1,\ldots,L} P_i$. Then $\pi(V) = \bigcup_{i=1,\ldots,L} C_i$ and since at most 2^n of the P_i intersect

$$\operatorname{vol}(V) = 2kl2^n \operatorname{vol}(\pi(V)) \leq 2^{n+1}kl\delta \leq 2^{n+1}k\delta$$

and

$$q(V \cap \rho_f) \leq \delta.$$

By the previous lemma, for each P_i there is a C^∞ function $\varphi_i : U \times \mathbb{R} \to [0,1]$ that vanishes on P_i', is equal to 1 on the complement of P_i, and

$$\sup_{x \in \mathbb{R}^n} \left\| \frac{\partial \varphi_i}{\partial x^j} \right\| \leq \frac{A}{l}.$$

Let $\varphi = \Pi_{i=1,\ldots,L}\varphi_i$. Clearly $\varphi : U \times \mathbb{R} \to [0,1]$ is C^∞, vanishes in a neighborhood of K and equals one in the complement of V. But at most 2^n of the P_i can intersect, so that

$$\left| \frac{\partial \varphi}{\partial x^j} \right| = \left| \sum_{i=1}^{L} \frac{\partial \varphi_i}{\partial x^j} \prod_{k \neq i} \varphi_k \right| \leq 2^n \frac{A}{l}, \quad j = 1, \ldots, n.$$

Hence

$$\mathrm{vol}(V) \left[\sup_{x \in \mathbb{R}^n} \left| \frac{\partial \varphi}{\partial x^j} \right| \right] \leq 2^{n+1} k l \delta 2^n \frac{A}{l} = 2^{2n+1} k \delta A.$$

Now let $\delta = \min\{\epsilon, \epsilon/2^{2n+1}kA\}$, $\varphi_\epsilon = \varphi$, and $U_\epsilon = V$. ■

Proof of Equation (7.2.11). Let

$$\omega = \sum_{i=1}^{n} \omega^i dx^1 \wedge \cdots \wedge (dx^1)\hat{} \wedge \cdots \wedge dx^n, \quad \mathbf{d}\omega = b\, dx^1 \wedge \cdots \wedge dx^n,$$

and $i^*\omega = a\lambda$. Then ω^i, b, and a are continuous and bounded on $U \times \mathbb{R}$ and ρ_f respectively; that is, $|\omega^i(x)| \leq M$, $|b(x)| \leq N$ for $x \in U \times \mathbb{R}$ and $|a(y)| \leq N$ for $y \in \rho_f$, where $M, N > 0$ are constants. Let U_ϵ and φ_ϵ be given by the previous lemma applied to $\mathrm{supp}(\omega) \cap \sigma_f$. But $\varphi_\epsilon\omega$ vanishes in a neighborhood of σ_f and Lemma 7.2.21 is applicable; that is

$$\int_{\Gamma_f^-} \mathbf{d}(\varphi_\epsilon \omega) = \int_{\Gamma_f} \varphi_\epsilon \omega. \tag{7.2.12}$$

We have

$$\left| \int_{\Gamma_f} \omega - \int_{\Gamma_f} \varphi_\epsilon \omega \right| \leq \left| \int_{\rho_f} a(1 - \varphi_\epsilon)\lambda \right| \leq N q(U_\epsilon \cap \rho_f) \leq N\epsilon$$

and

$$\begin{aligned}
\left| \int_{\Gamma_f^-} \mathbf{d}\omega - \int_{\Gamma_f^-} \mathbf{d}(\varphi_\epsilon \omega) \right| &\leq \left| \int_{\Gamma_f^-} (\mathbf{d}\omega - \varphi_\epsilon d\omega) \right| + \left| \int_{\Gamma_f^-} \mathbf{d}\varphi_\epsilon \wedge \omega \right| \\
&\leq \left| \int_{\Gamma_f^-} b(1 - \varphi_\epsilon) dx^1 \wedge \cdots \wedge dx^n \right| + \sum_{i=1}^{n} \int_{\Gamma_f^-} |\omega^i| \left| \frac{\partial \varphi_\epsilon}{\partial x^i} \right| dx^1 \wedge \cdots \wedge dx^n \\
&\leq N\, \mathrm{vol}(U_\epsilon) + M \left[\sum_{i=1}^{n} \sup_{x \in \mathbb{R}} \left| \frac{\partial \varphi_\epsilon(x)}{\partial x^i} \right| \right] \mathrm{vol}(U_\epsilon) \leq N\epsilon + Mn\epsilon.
\end{aligned} \tag{7.2.13}$$

From equations (7.2.12) and (7.2.13) we get

$$\left| \int_{\Gamma_f^-} \mathbf{d}\omega - \int_{\Gamma_f} \omega \right| \leq (2N + nM)\epsilon$$

for all $\epsilon > 0$, which proves the equality. ■

In analysis it can be useful to have hypotheses on the smoothness of ω as well as on the boundary that are as weak as possible. Our proofs show that ω need only be C^1. An effective strategy for sharper results is to approximate ω by smooth forms ω_k so that both sides of Stokes' theorem converge as $k \to \infty$. A useful class of forms for which this works are those in Sobolev spaces, function spaces encountered in the study of partial differential equations. The Hölder nature of the boundary of N in Stokes' theorem is exactly what is needed to make this approximation process work. The key ingredients are approximation properties in M (which are obtained from those in $\mathbb{R}^n$) and the ***Calderón extension theorem*** to reduce approximations in N to those in $\mathbb{R}^n$. (Proofs of these facts may be found in Stein [1970], Marsden [1973], and Adams [1975].)

Supplement 7.2C
Stokes' Theorem on Chains

In algebraic topology it is of interest to integrate forms over images of simplexes. This box adapts Stokes' theorem to this case. The result could be obtained as a corollary of the piecewise smooth Stokes Theorem, but we shall give a self-contained and independent proof.

7.2.24 Definition. *The **standard** p-**simplex** is the closed set*

$$\Delta_p = \left\{ x \in \mathbb{R}^p \;\middle|\; 0 \le x^i \le 1,\ \sum_{i=1}^{p} x^i \le 1 \right\}.$$

*The **vertices** of Δ_p are the $p+1$ points*

$$v_0 = (0, \dots, 0),\ v_1 = (1, 0, \dots, 0), \dots,\ v_p = (0, \dots, 0, 1).$$

*Opposite to each v_i there is the ith **face** $\Phi_{p-1,i} : \Delta_{p-1} \to \Delta_p$ given by (see Figure 7.2.8):*

$$\Phi_{p-1,0}(y^1, \dots, y^{p-1}) = \left(1 - \sum_{i=1}^{p-1} y^i, y^1, \dots, y^{p-1}\right), \quad \text{if } i = 0$$

and

$$\Phi_{p-1,i}(y^1, \dots, y^{p-1}) = (y^1, \dots, y^{i-1}, 0, y^i, \dots, y^{p-1}), \quad \text{if } i \ne 0.$$

*A C^k-**singular** p-**simplex** on a C^r-manifold M, $1 \le k \le r$, is a C^k-map $s : U \to M$, where U is an open neighborhood of Δ_p in $\mathbb{R}^p$. The points $s(v_0), \dots, s(v_p)$ are the **vertices** of the singular p-simplex s and the map $s \circ \Phi_{p-1,i} : V \to M$, for V an open neighborhood of Δ_{p-1} in $\mathbb{R}^{p-1}$ and $\Phi_{p-1,i}$ extended by the same formula from Δ_{p-1} to V, is called the ith **face** of the singular p-simplex s. A C^k-**singular** p-**chain** on M is a finite formal linear combination with real coefficients of C^k-singular p-simplexes. The **boundary** of a singular p-simplex s is the singular $(p-1)$-chain ∂s defined by*

$$\partial s = \sum_{i=0}^{p} (-1)^i s \circ \Phi_{p-1,i}$$

and that of a singular p-chain is obtained by extending ∂ from the simplexes by linearity to chains. It is straightforward to verify that $\partial \circ \partial = 0$ using the relation

$$\Phi_{p-1,j} \circ \Phi_{p-2,j} = \Phi_{p-2,i-1} \circ \Phi_{p-2,i-1}$$

for $j < i$.

If $s : U \to M, \Delta_p \subset U$, *is a singular* p*-simplex,* $\omega \in \Omega^p(M)$, *and*

$$s^*\omega = a dx^1 \wedge \cdots \wedge dx^p \in \Omega^p(U),$$

the ***integral*** *of* ω ***over*** s *is defined by*

$$\int_s \omega \int_{\Delta_p} a\, dx^1 \cdots dx^p,$$

where the integral on the right is the usual integral in $\mathbb{R}^p$. *The* ***integral of*** ω ***over a*** p***-chain*** *is obtained by linear extension.*

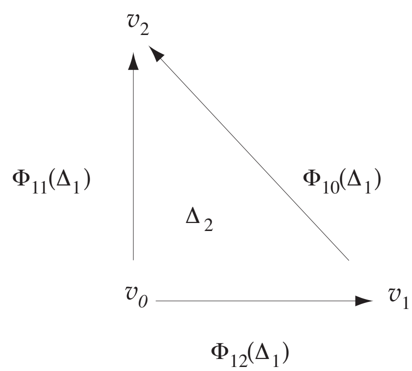


FIGURE 7.2.8. Integrating over chains

7.2.25 Theorem (Stokes' Theorem on Chains)**.** *If* c *is any singular* p*-chain and* $\omega \in \Omega^{p-1}(M)$, *then*

$$\int_c \mathbf{d}\omega = \int_{\partial c} \omega.$$

Proof. By linearity it suffices to prove the formula if $c = s$, a singular p-simplex. If

$$s^*\omega = \sum_{j=1}^{p} (-1)^{j-1} \omega^j dx^1 \wedge \cdots \wedge \widehat{(dx^j)} \wedge \cdots \wedge dx^p,$$

then

$$\mathbf{d}(s^*\omega) = \sum_{j=1}^{p-1} \frac{\partial \omega^j}{\partial x^j} dx^1 \wedge \cdots \wedge dx^p$$

and denoting the coordinates in a neighborhood V of Δ_{p-1} by $(y^1, \ldots, y^{p-1}) = y$, we get

$$\Phi^*_{p-1,0} s^*\omega(y) = \sum_{j=1}^{p} \omega^j \left(1 - \sum_{i=1}^{p-1} y^i, y^1, \ldots, y^{p-1}\right) dy^1 \wedge \cdots \wedge dy^{p-1},$$

if $i = 0$ and

$$\Phi^*_{p-1,i} s^*\omega(y) = (-1)^{i-1} \omega^i (y^1, \ldots, y^{i-1}, 0, y^i, \ldots, y^{p-1})\, dy^1 \wedge \cdots \wedge dy^{p-1},$$

if $i \neq 0$. Thus, the formula in the statement becomes

$$\sum_{j=1}^{p} \int_{\Delta_p} \frac{\partial \omega^j(x)}{\partial x^j} dx^1 \cdots dx^p$$
$$= \sum_{j=1}^{p} \int_{\Delta_{p-1}} \left[\omega^j \left(1 - \sum_{i=1}^{p-1} y^i, y^1, \dots y^{p-1} \right) \right. \tag{7.2.14}$$
$$\left. - \omega^j(y^1, \dots, y^{j-1}, 0, y^j, \dots, y^{p-1}) \right] dy^1 \cdots dy^{p-1}.$$

By Fubini's theorem, each summand on the left hand side of equation (7.2.14) equals

$$\int_{\Delta_p} \frac{\partial \omega^j(x)}{\partial x^j} dx^1 \cdots dx^p$$
$$= \int_{\Delta_{p-1}} \left(\int_0^{1-\sum_{k\neq j} x^k} \frac{\partial \omega^j}{\partial x^j} dx^j \right) dx^1 \cdots (dx^j)^\wedge \cdots dx^p$$
$$= \int_{\Delta_{p-1}} \left[\omega^j \left(x^1, \dots, x^{j-1}, 1 - \sum_{k \neq j} x^k, x^{j+1}, \dots, x^p \right) \right.$$
$$\left. - \omega^j(x^1, \dots, x^{j-1}, 0, x^{j+1}, \dots, x^p) \right] dx^1 \cdots (dx^j)^\wedge \cdots dx^p.$$

Break up this integral as a difference of two terms. In the first term perform the change of variables

$$(y^1, \dots, y^{p-1}) \mapsto \left(x^2, \dots, x^{j-1}, 1 - \sum_{k \neq j} x^k, x^{j+1}, \dots, x^p \right)$$

which has Jacobian equal to $(-1)^j$, use the change of variables formula from calculus in the multiple integral involving the absolute value of the Jacobian, and note that $x^1 = 1 - \Sigma_{i=1,\dots,p-1} y^i$. In the second term perform the change of variables

$$(y^1, \dots, y^{p-1}) \mapsto (x^1, \dots, x^{j-1}, x^{j+1}, \dots, x^p)$$

which has Jacobian equal to one. Then we get

$$\int_{\Delta_p} \frac{\partial \omega^j(x)}{\partial x^j} dx^1 \dots dx^p$$
$$= \int_{\Delta_{p-1}} \left[\omega^j \left(1 - \sum_{i=1}^{p-1} y^i, y^1, \dots, y^{p-1} \right) \right.$$
$$\left. - \omega^j(y^1, \dots, y^{j-1}, 0, y^j, \dots, y^{p-1}) \right] dy^1 \dots dy^{p-1}$$

and formula (7.2.14) is thus proved for each corresponding summand. ■

Instead of singular p-chains one can consider ***infinite singular p-chains*** defined as infinite formal sums with real coefficients $\sum_{i \in I} a_i S_i$ such that for each $i \in I$ the family of sets $\{ s_i(\Delta_p) \mid a_i \neq 0 \}$ is locally finite, that is, each $m \in M$ has a neighborhood intersecting only finitely many (or no) sets of this family. On compact manifolds only finitely many coefficients in an infinite singular p-chain are non-zero and thus infinite singular p-chains are singular p-chains. The statement and the proof of Stokes' theorem on chains remain unchanged if c is an infinite p-chain and $\omega \in \Omega^{\pi-1}(M)$ has compact support.

Exercises

◇ **7.2-1.** Let M and N be oriented n-manifolds with boundary and $f : M \to N$ an orientation-preserving diffeomorphism. Show that the change of variables formula and Stokes' theorem imply that $f^* \circ \mathbf{d} = \mathbf{d} \circ f^*$.

◇ **7.2-2.** Let M be a compact orientable boundaryless n-manifold and $\alpha \in \Omega^{n-1}(M)$. Show that $\mathbf{d}\alpha$ vanishes at some point.

◇ **7.2-3.** Let M be a compact $(n+1)$-dimensional manifold with boundary, $f : \partial M \to N$ a smooth map and $\omega \in \Omega^n(N)$ where $\mathbf{d}\omega = 0$. Show that if f extends to M, then $\int_{\partial M} f^*\omega = 0$.

◇ **7.2-4.** Let (M, μ) be a volume manifold with $\partial M = \varnothing$.

(i) Show that the divergence of a vector field X is uniquely determined by the condition

$$\int_M f(\operatorname{div} X)\mu = -\int_M (\pounds_X f)\mu$$

for any f with compact support.

(ii) What does the equation in (i) become if M is compact with boundary?

(iii) $X(x, y, z) = (y, -x, 0)$ defines a vector field on S^2. Calculate $\operatorname{div} X$.

◇ **7.2-5.** Let M be a paracompact manifold with boundary. Show that there is a positive smooth function $f : M \to [0, \infty[$ with 0 a regular value, such that $\partial M = f^{-1}(0)$.
Hint: First do it locally and then patch the local functions together with a partition of unity.

◇ **7.2-6.** Let M be a boundaryless manifold and $f : M \to \mathbb{R}$ a C^∞ mapping having a regular value a. Show that $f^{-1}([a, \infty[)$ is a manifold with boundary $f^{-1}(a)$.

◇ **7.2-7.** Let $f : M \to N$ be a C^∞ mapping, $\partial M \neq \varnothing$, $\partial N \neq \varnothing$, and let $P \subset N$ be a submanifold of N. Assume that $f \pitchfork P$, $(f|\partial M) \pitchfork P$ and that in addition one of the following conditions hold.

(i) P is boundaryless and $P \subset \operatorname{Int} N$; or

(ii) $\partial P \neq \varnothing$ and $\partial P \subset \partial N$; or

(iii) $\partial P \neq \varnothing$, $f \pitchfork \partial P$, and $(f|\partial M) \pitchfork \partial P$.

Show that $f^{-1}(P)$ is a submanifold of M whose boundary equals

$$\partial f^{-1}(P) = f^{-1}(P) \cap \partial M,$$

in case (i), and $\partial f^{-1}(P) = f^{-1}(\partial P)$ in cases (ii) and (iii). If all manifolds are finite dimensional, show that

$$\dim M - \dim f^{-1}(P) = \dim N - \dim P.$$

Formulate and prove the statement replacing this equality between dimensions for infinite dimensional manifolds.
Hint: At the boundary, work with a boundary chart using the technique in the proof of Theorem 3.5.12.

◇ **7.2-8.** Without some kind of transversality conditions on $f|\partial M$ for $f : M \to N$ a smooth map, even if $f \pitchfork P$, where P is a submanifold of N (like the ones in the previous exercise), $f^{-1}(P)$ is in general *not* a submanifold. For example, let $M = \mathbb{R}^2_+$, $N = \mathbb{R}$, $P = \{0\}$ and $f(x, y) = y + \chi(x)$ for a smooth function $\chi : \mathbb{R} \to \mathbb{R}$. Show that f is a smooth surjective submersion. Find the conditions under which $f|\partial M$ has 0 as a regular value. Construct a smooth function χ for which these conditions are violated and $f^{-1}(0)$ is not a manifold.
HINT: Take for χ a smooth function which has infinitely many zeros converging to zero.

◇ **7.2-9.** (i) Show that if M is a boundaryless manifold, there is a connected manifold N with $M = \partial N$.
HINT: Think of semi-infinite cylinders.

(ii) Construct an example for (i) in which M is compact but N cannot be chosen to be compact.
HINT: Assume $\dim M = 0$.

◇ **7.2-10.** Let M be a manifold, X a smooth vector field on M with flow F_t and $\alpha \in \Omega^k(M)$. We call α an ***invariant k-form*** of X when $\pounds_X \alpha = 0$. Prove the following.

Poincaré–Cartan Theorem. *α is an invariant k-form of X iff for all oriented compact k-manifolds with boundary $(V, \partial V)$ and C^∞ mappings $\varphi : V \to M$, such that the domain of F_t contains $\varphi(V)$, $0 \le t \le T$, we have*

$$\int_V (F_t \circ \varphi)^* \alpha = \int_V \varphi^* \alpha.$$

HINT: For the converse show that the equality between integrals implies $(F_t \circ \varphi)^* \alpha = \varphi^* \alpha$; then differentiate relative to t.

◇ **7.2-11.** Let X be a vector field on a manifold M and α, β invariant forms of X. (See Exercise 7.2-10.) Prove the following.

(i) $\mathbf{i}_X \alpha$ is an invariant form of X.

(ii) $\mathbf{d}\alpha$ is an invariant form of X.

(iii) $\pounds_X \gamma$ is closed iff $\mathbf{d}\gamma$ is an invariant form, for any $\gamma \in \Omega^k(M)$.

(iv) $\alpha \wedge \beta$ is an invariant form of X.

(v) Let $\mathcal{A}_X$ denote the invariant forms of X. Then $\mathcal{A}_X$ is a $\wedge$ subalgebra of $\Omega(M)$, which is closed under $\mathbf{d}$ and $\mathbf{i}_X$.

◇ **7.2-12.** Let X be a vector field on a manifold M with flow F_t and $\alpha \in \Omega^k(M)$. Then α is called a ***relatively invariant k-form of*** X if $\pounds_X \alpha$ is closed. Prove the following

Poincaré–Cartan Theorem. *α is a relatively invariant $(k-1)$-form of X iff for all oriented compact k-manifolds with boundary $(V, \partial V)$ and C^∞ maps $\varphi : V \to M$ such that the domain of F_t contains $\varphi(V)$ for $0 \le t \le T$, we have*

$$\int_{\partial V} (F_t \circ \varphi \circ i)^* \alpha = \int_{\partial V} (\varphi \circ i)^* \alpha.$$

where $i : \partial V \to V$ is the inclusion map.

◇ **7.2-13.** If $X \in \mathfrak{X}(M)$, let $\mathcal{A}_X$ be the set of all invariant forms of $X, \mathcal{R}_X$ the set of all relatively invariant forms of $X, \mathcal{C}$ the set of all closed forms in $\Omega(M)$, and $\mathcal{E}$ the set of all exact forms in $\Omega(M)$. Show that

(i) $\mathcal{A}_X \subset \mathcal{R}_X$, $\mathcal{E} \subset \mathcal{C} \subset \mathcal{R}_X$, $\mathcal{A}_X$ is a differential subalgebra of $\Omega(M)$, but $\mathcal{R}_X$ is only a real vector subspace.

(ii) $0 \to \mathcal{A}_X \xrightarrow{i} \Omega(M) \xrightarrow{\pounds_X} \Omega(M) \xrightarrow{\pi} \Omega(M)/\operatorname{Im}(\pounds_X) \to 0$ is exact.

(iii) $0 \to \mathcal{C} \xrightarrow{i} \mathcal{R}_X \xrightarrow{\mathbf{d}} \mathcal{A}_X \xrightarrow{\pi} \mathcal{A}_X/\mathcal{E} \cap \mathcal{A}_X \to 0$ is exact.

(iv) $\mathbf{d}(\mathcal{A}_X) \subset \mathcal{A}_X$ and $\mathbf{i}_X(\mathcal{A}_X) \subset \mathcal{A}_X$.

◇ **7.2-14** (Smale–Sard Theorem for manifolds with boundary)**.** Let M and N be C^k manifolds, where M is Lindelöf, having a boundary ∂M, and N is boundaryless. Let $f : M \to N$ be a C^k Fredholm map and let $\partial f = f|\partial M$. If $k > \max(0, \operatorname{index}(T_x f))$ for every $x \in M$, show that $\mathcal{R}_f \cap \mathcal{R}_{\partial f}$ is residual in N.

◇ **7.2-15** (The Boundaryless Double)**.** Let M be a manifold with boundary. Show that the topological space obtained by identifying the points of ∂M in the disjoint union of M with itself is a boundaryless manifold in which M embeds, called the ***boundaryless double*** of M.
HINT: Glue together the two boundary charts.

◇ **7.2-16.** Let M be a manifold with $\partial M \neq \varnothing$. Assume M admits partitions of unity. Show that ∂M is orientable and hence by Exercise 6.5-17, the algebraic normal bundle

$$\nu(\partial M) = (TM|\partial M)/T(\partial M)$$

is trivial.
HINT: Use proposition 6.5.8. Locally $n(m) = \partial/\partial x^n$ for $m \in \partial M$; glue these together.

◇ **7.2-17** (Collars)**.** Let M be a manifold with boundary. A ***collar*** for M is a diffeomorphism of $\partial M \times [0,1[$ onto an open neighborhood of ∂M in M that is the identity on ∂M.

(i) Show that a manifold with boundary and admitting partitions of unity has a collar.

HINT: Via a partition of unity, construct a vector field on M that points inward when restricted to ∂M. Then look at the integral curves starting on ∂M to define the collar.

(ii) Let $\varphi_1 : \partial M \times [0,1[\to M$, $i = 0,1$ be two collars. Show that φ_0 and φ_1 are ***isotopic***, that is, there is a smooth map $H :]-\epsilon, 1+\epsilon[\times \partial M \times [0,1[\to M$ such that

$$H(0,m,t) = \varphi_0(m,t), \quad H(1,m,t) = \varphi_1(m,t)$$

for all $(m,t) \in \partial M \times [0,1[$ and that $H(s,\cdot,\cdot)$ is an embedding for all $s \in]-\epsilon, 1+\epsilon[$.[1]

HINT: Let U_i be the image of φ_i, an open set in M containing ∂M. Let $X_i = \varphi_i^*(0, \partial/\partial t)$ and look at the flow of $(1-s)X_0 + sX_1$ on $U_0 \cap U_1$.

(iii) Let N be a submanifold of M such that $\partial N = N \cap \partial M$ and $T_n N$ is not a subset of $T_n(\partial M)$ for all $n \in \partial N$. Show that ∂M has a collar which restricts to a collar of ∂N in N.

◇ **7.2-18.** Let M and N be manifolds with boundary and let $\varphi : \partial M \to \partial N$ be a diffeomorphism. Form the topological space $M \cup_\varphi N$ which is the quotient of the disjoint union of M with N by the equivalence relation which identifies m with $\varphi(m)$. Let V be the image of ∂M and ∂N in $M \cup_\varphi N$.

(i) Use collars to construct a homeomorphism of a neighborhood U of V with the space $]-1,1[\times V$ which maps V pointwise to $V \times \{0\}$ and which maps $V \cap M$ and $V \cap N$ diffeomorphically onto $V \times]0,1[$ and $V \times]-1,0[$, respectively. Construct a differentiable structure out of those on M, N, and U.

The "uniqueness theorem of glueing" states that the differentiable structures on the space $M \cup_\varphi N$ obtained in (i) by making various choices are all diffeomorphic. The rest of this exercise uses this fact.

[1] It can be shown that φ_0 and φ_1 are diffeotopic using Thom's theorem of embedding of isotopies into diffeotopies. This then provides the basis of glueing manifolds together along their boundaries; see Hirsch [1976, Chapter 8], for proofs and the preamble to the next exercise for a discussion.

(ii) Two compact boundaryless manifolds M_1, M_2 are called ***cobordant*** if there is a compact manifold with boundary N, called the ***cobordism***, such that ∂N equals the disjoint union of M_1 with M_2. Show that "cobordism" is an equivalence relation.

HINT: For transitivity, glue the manifolds along one common component of their boundaries.

(iii) Show that the operation of disjoint union induces the structure of an abelian group on the set of cobordism classes in which each element has order two.

HINT: The zero element is the class of any compact manifold which is the boundary of another compact manifold.

(iv) Show that the operation of taking products of manifolds induces a multiplicative law on the set of cobordism classes, thus making this set $\mathfrak{N}$ a ring.

(v) Repeat parts (ii) and (iii) for oriented manifolds obtaining a graded ring, that is,

$$[M, \mu] \cdot [N, \nu] = (-1)^{\dim M + \dim N} [N, \nu] \cdot [M, \mu],$$

the graded ring of oriented cobordism classes $\mathfrak{O}$. Are the elements of $\mathfrak{O}$ still of order two relative to addition?

(vi) Denote by $\mathfrak{N}^n, \mathfrak{O}^n$, the cobordism classes of a given dimension. Show that $\mathfrak{N}^0 = \mathbb{Z}/2\mathbb{Z}$, $\mathfrak{O}^0 = \mathbb{Z}$, $\mathfrak{N}^1 = \mathfrak{O}^1 = 0$.

(vii) Assume M and N are boundaryless manifolds, M compact, and P a boundaryless submanifold of N. Assume $f, g : M \to N$ are smoothly homotopic maps such that $f \pitchfork P$ and $g \pitchfork P$. Show that $f^{-1}(P)$ and $g^{-1}(P)$ are cobordant.

HINT: Choose a smooth homotopy H transverse to P. What is $\partial H^{-1}(P)$?

◇ **7.2-19.** (i) Let χ be a vector field density on a finite dimensional manifold M, that is, $\chi = X \otimes \rho$ for $X \in \mathfrak{X}(M)$ and $\rho \in |\Omega(M)|$. Recall from Exercise 6.5-16 that the density $\operatorname{div}\chi$, defined to be $(\operatorname{div}_\rho X)\rho$, is independent of the representation of χ as $X \otimes \rho$. Show that for any $f \in \mathcal{F}(M)$, $Y \in \mathfrak{X}(M)$, and any diffeomorphism $\varphi : M \to M$ we have

$$\varphi^*(\operatorname{div}\chi) = \operatorname{div}(^*\chi), \quad \pounds_Y(\operatorname{div}\chi) = \operatorname{div}(\pounds_Y \chi),$$

and

$$\operatorname{div}(f\chi) = \mathbf{d}f \cdot \chi + f \operatorname{div}\chi.$$

(ii) If M is paracompact and Riemannian, phrase Gauss' theorem for vector field densities.

(iii) Let $\alpha \in \Omega^{k-1}(M)$ and τ be a tensor density of type $(k, 0)$ which is completely antisymmetric. Let $\alpha \cdot \tau$ denote the contraction of α with the first $k-1$ indices of the tensor part of τ producing a vector field density. Define the ***contravariant exterior derivative*** $\partial\tau$ by requiring the following relation for all $\alpha \in \Omega^{k-1}(M)$:

$$\operatorname{div}(\alpha \cdot \tau) = \mathbf{d}\alpha \cdot \tau + \alpha \cdot \partial\tau,$$

where $\mathbf{d}\alpha \cdot \tau$ and $\alpha \cdot \partial\tau$ means contraction on all indices. Show that if $\tau = t \otimes \rho$, where locally

$$t = t^{i_1 \ldots i_k} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}}, \quad \text{and} \quad \rho = |dx^1 \wedge \cdots \wedge dx^n|,$$

then the local expression of ∂t is

$$\partial\tau = \frac{\partial}{\partial x^j}(t^{i_1 \ldots i_{k-1} j}) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_{k-1}}} \otimes |dx^1 \wedge \cdots \wedge dx^n|.$$

Show that $\partial^2 = 0$.

(iv) Prove the following properties of ∂:

$$\partial(\tau \wedge \sigma) = \partial\tau \wedge \sigma + (-1)^k \tau \wedge \partial\sigma,$$

and

$$\pounds_X \partial\tau = \partial \pounds_X \tau, \quad \varphi^* \partial\tau = \partial\varphi^* \tau$$

where τ is a $(k,0)$-tensor density, σ is a $(l,0)$-tensor density, $X \in \mathfrak{X}(M)$, and $\varphi : M \to M$ is a diffeomorphism. Show that if χ is a vector field density, $\partial\chi = \operatorname{div}\chi$.

(v) Let $\mathbf{j}_X \tau = X \wedge \tau$ for $X \in \mathfrak{X}(M)$. Show that $\mathbf{i}_X \alpha \cdot \tau = \alpha \cdot \mathbf{j}_X \tau$ for any $\alpha \in \Omega^{k+1}(M)$, $X \in \mathfrak{X}(M)$ and τ is a completely antisymmetric $(k,0)$-tensor density. Prove the analog of **Cartan's formula**: $\pounds_X = \mathbf{j}_X \circ \partial + \partial \circ \mathbf{j}_X$.

HINT: Integrate the defining relation in (iii) for a any form with support in $\operatorname{Int} M$ and extend the formula by continuity to ∂M.

(vi) Formulate and prove a global formula for $\partial\tau(\alpha_1, \ldots, \alpha_{k-1})$, τ a $(k,0)$-tensor density, analogous to Palais' formula (Proposition 6.4.11(ii)).

◇ **7.2-20** (Prüfer Manifold). Let $\mathbb{R}_d$ denote the set $\mathbb{R}$ with the discrete topology; it is thus a zero dimensional manifold. Let $P = (\mathbb{R}^2_+ \times \mathbb{R}_d)/R$, where $\mathbb{R}^2_+ = \{ (x,y) \mid y \geq 0 \}$ and R is the equivalence relation: (x,y,a) $R(x',y',a')$ iff $[(y = y' > 0$ and $a + xy = a' + x'y')$ or $(y = y' = 0$ and $a = a', x = x')]$. Show that P is a Hausdorff two-dimensional manifold, $\partial P \neq \varnothing$, and ∂P is a disjoint union of uncountably many copies of $\mathbb{R}$. Show that P is not paracompact.

◇ **7.2-21.** In the notation of Supplement 7.2C, verify that $\partial \circ \partial = 0$.

7.3 The Classical Theorems of Green, Gauss, and Stokes

This section obtains these three classical theorems as a consequence of Stokes' theorem for differential forms. We begin with Green's theorem, which relates a line integral along a closed piecewise smooth curve C in the plane $\mathbb{R}^2$ to a double integral over the region D enclosed by C. (Piecewise smooth means that the curve C has only finitely many corners.) Recall from advanced calculus that the ***line integral*** of a one-form $\omega = P\,dx + Q\,dy$ along a curve C parameterized by $\gamma : [a,b] \to \mathbb{R}^2$ is defined by

$$\int_C \omega = \int_b^a \{P(\gamma_1(t), \gamma_2(t))\gamma_1'(t) + Q(\gamma_1(t), \gamma_2(t))\gamma_2'(t)\} dt;$$

that is

$$\int_C \omega = \int_a^b \gamma^* \omega.$$

7.3.1 Theorem (Green's Theorem). *Let D be a closed bounded region in $\mathbb{R}^2$ bounded by a closed positively oriented piecewise smooth curve C. (Positively oriented means the region D is on your left as you traverse the curve in the positive direction.) Suppose $P : D \to \mathbb{R}$ and $Q : D \to \mathbb{R}$ are C^1. Then*

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

Proof. We assume the boundary $C = \partial D$ is smooth. (The piecewise smooth case follows from the generalization of Stokes' theorem outlined in Supplement 7.2B).

Let

$$\omega = P(x, y)dx + Q(x, y)dy \in \Omega^1(D).$$

Since

$$\mathbf{d}\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

and the measure associated with the volume $dx \wedge dy$ on $\mathbb{R}^2$ is the usual Lebesgue measure $dx\, dy$, the formula of the theorem is a restatement of Stokes' theorem for this case. ■

This theorem may be phrased in terms of the divergence and the outward unit normal. If C is given parametrically by $t \mapsto (x(t), y(t))$, then the outward unit normal is

$$\mathbf{n} = \frac{(y'(t), -x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}} \tag{7.3.1}$$

and the infinitesimal arc-length (the volume element of C) is

$$ds = \sqrt{x'(t)^2 + y'(t)^2} d.$$

(See Figure 7.3.1.) If

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} \in \mathfrak{X}(D),$$

recall that

$$\operatorname{div} X = *\mathbf{d} * \mathbf{X}^\flat = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

7.3.2 Corollary. *Let D be a region in $\mathbb{R}^2$ bounded by a closed piecewise smooth curve C. If $\mathbf{X} \in \mathfrak{X}(D)$, then*

$$\int_C (\mathbf{X} \cdot \mathbf{n}) ds = \iint_D (\operatorname{div} X) dx\, dy.$$

where $\int_C f\, ds$ denotes the line integral of the function f over the positively oriented curve C and $\mathbf{X} \cdot \mathbf{n}$ is the dot product.

Proof. Using formula (7.3.1) for $\mathbf{n}$, we have

$$\begin{aligned} \int_C (\mathbf{X} \cdot \mathbf{n}) ds &= \int_a^b [P(x(t)y(t))y'(t) - Q(x(t), y(t))x'(t)]dt \\ &= \int_C P\, dy - Q\, dx \end{aligned} \tag{7.3.2}$$

by the definition of the line integral. By Theorem 7.3.1, equation (7.3.2) equals

$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\, dy = \iint_D (\operatorname{div} X) dx\, dy.$$ ■

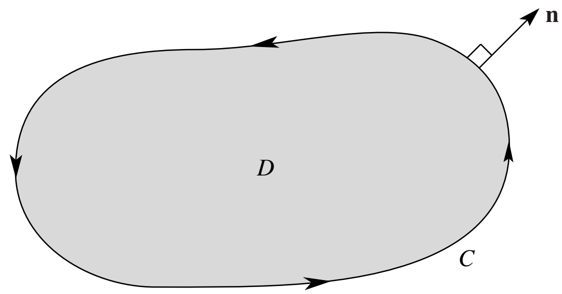


FIGURE 7.3.1. Green's Theorem

Taking $P(x, y) = x$ and $Q(x, y) = y$ in Green's theorem, we get the following.

7.3.3 Corollary. *Let D be a region in $\mathbb{R}^2$ bounded by a closed piecewise smooth curve C. The area of D is given by*

$$\operatorname{area}(D) = \frac{1}{2}\int_C x\,dy - y\,dx.$$

The classical Stokes theorem for surfaces relates the line integral of a vector field around a simple closed curve C in $\mathbb{R}^3$ to an integral over a surface S for which $C = \partial S$. Recall from advanced calculus that the ***line integral*** of a vector field $\mathbf{X}$ in $\mathbb{R}^3$ over the curve $\sigma : [a, b] \to \mathbb{R}^3$ is defined by

$$\int_\sigma \mathbf{X}\cdot d\mathbf{s} = \int_a^b \mathbf{X}(\sigma(t))\cdot\sigma'(t)dt. \tag{7.3.3}$$

The surface integral of a compactly supported two-form ω in $\mathbb{R}^3$ is defined to be the integral of the pull-back of ω to the oriented surface. If S is an oriented surface, $\mathbf{n}$ is called the ***outward unit normal*** at $x \in S$ if $\mathbf{n}$ is perpendicular to T_xS and $\{\mathbf{n}, \mathbf{e}_1, \mathbf{e}_2\}$ is a positively oriented basis of $\mathbb{R}^3$ whenever $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a positively oriented basis of T_xS. Thus S is orientable iff the normal bundle to S, which has one-dimensional fiber, is trivial. Also, the area element ν of S is given by Proposition 6.5.8. That is,

$$\nu(x)(\mathbf{v}_1, \mathbf{v}_2) = \mu(x)(\mathbf{n}, \mathbf{v}_1, \mathbf{v}_2), \tag{7.3.4}$$

where $\mathbf{v}_1, \mathbf{v}_2 \in T_xS$, and $\mu = dx \wedge dy \wedge dz$. We want to express $\int_S \omega$ in a form familiar from vector calculus. Let $\omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$ so that $\omega = *\mathbf{X}^\flat$, where

$$\mathbf{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}.$$

Recall that $\alpha \wedge *\beta = \langle\alpha, \beta\rangle\mu$ so that letting $\alpha = \mathbf{n}^\flat$, and $\beta = \mathbf{X}^\flat$, we get

$$\mathbf{n}^\flat \wedge *\mathbf{X}^\flat = (\mathbf{X}\cdot\mathbf{n})\mu.$$

Applying both sides to $(\mathbf{n}, \mathbf{v}_1, \mathbf{v}_2)$ and using equation (7.3.2) gives

$$(\mathbf{n}^\flat \wedge *\mathbf{X}^\flat)(\mathbf{n}, \mathbf{v}_1, \mathbf{v}_2) = (\mathbf{X}\cdot\mathbf{n})\nu(\mathbf{v}_1, \mathbf{v}_2) \tag{7.3.5}$$

(the base point x is suppressed). The left side of (7.3.5) is $*\mathbf{X}^\flat(\mathbf{v}_1, \mathbf{v}_2)$ since $\mathbf{n}^\flat$ is one on $\mathbf{n}$ and zero on $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus (7.3.5) becomes

$$*\mathbf{X}^\flat = (\mathbf{X}\cdot\mathbf{n})\nu. \tag{7.3.6}$$

Therefore,

$$\int_S \omega = \int_S (\mathbf{X}\cdot\mathbf{n})dS = \int_S \mathbf{X}\cdot\mathbf{dS},$$

where dS, the measure on S defined by ν, is identified with a surface integral familiar from vector calculus.

A physical interpretation of $\int_S(\mathbf{X}\cdot\mathbf{n})dS$ may be useful. Think of $\mathbf{X}$ as the velocity field of a fluid, so $\mathbf{X}$ is pointing in the direction in which the fluid is moving across the surface S and $\mathbf{X}\cdot\mathbf{n}$ measures the volume of fluid passing through a unit square of the tangent plane to S in unit time. Hence the integral $\int_S(\mathbf{X}\cdot\mathbf{n})dS$ *is the net quantity of fluid flowing across the surface per unit time, that is, the rate of fluid flow.* Accordingly, this integral is also called the ***flux*** of $\mathbf{X}$ across the surface.

7.3.4 Theorem (Classical Stokes Theorem)**.** *Let S be an oriented compact surface in $\mathbb{R}^3$ and $\mathbf{X}$ a C^1 vector field defined on S and its boundary. Then*

$$\int_S (\operatorname{curl}\mathbf{X})\cdot\mathbf{n}\,dS = \int_{\partial S}\mathbf{X}\cdot\mathbf{ds}.$$

where $\mathbf{n}$ *is the outward unit normal to* S (*Figure* 7.3.2).

Proof. First extend $\mathbf{X}$ via a bump function to all of $\mathbb{R}^3$ so that the extended $\mathbf{X}$ still has compact support. By definition, $\int_{\partial S}\mathbf{X}\cdot\mathbf{d}s = \int_{\partial S}\mathbf{X}^\flat$ where $^\flat$ denotes the index lowering action defined by the standard metric in $\mathbb{R}^3$. But $\mathbf{d}\mathbf{X}^\flat = *(\operatorname{curl}\mathbf{X})^\flat$ (see Example 6.4.3C) so that by equations (7.3.3), (7.3.6), and Stokes' theorem,

$$\int_{\partial S}\mathbf{X}\cdot\mathbf{ds} = \int_S \mathbf{d}\mathbf{X}^\flat = \int_S *(\operatorname{curl}\mathbf{X})^\flat = \int_S(\operatorname{curl}\mathbf{X}\cdot\mathbf{n})dS. \qquad \blacksquare$$

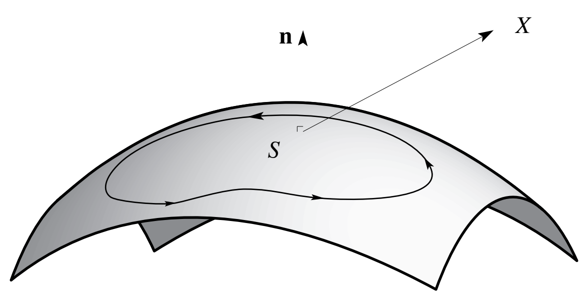


FIGURE 7.3.2. Stokes' Theorem

7.3.5 Examples.

A. The historical origins of Stokes' formula, are connected with Faraday's law, which is discussed in Chapter 8 and example B below. In fluid dynamics, Stokes' formula is useful in the development of Kelvin's circulation theorem, to be discussed in §8.2. Here we concentrate on a physical interpretation of the curl operator. Suppose $\mathbf{X}$ represents the velocity vector field of a fluid. Let us apply Stokes' theorem to a disk D_r of radius r at a point $P\in\mathbb{R}^3$ (Figure 7.3.3). We get

$$\int_{\partial D_r}\mathbf{X}\cdot\mathbf{ds} = \int_{\partial D_r}(\operatorname{curl}\mathbf{X})\cdot\mathbf{nds} = (\operatorname{curl}\mathbf{X}\cdot\mathbf{n})(Q)\pi r^2,$$

the last equality coming from the mean value theorem for integrals; here $Q \in D_r$ is some point given by the mean value theorem and πr^2 is the area of D_r. Thus

$$((\operatorname{curl} \mathbf{X}) \cdot \mathbf{n})(P) = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{\partial D_r} \mathbf{X} \cdot \mathbf{ds}. \tag{7.3.7}$$

The number $\int_C \mathbf{X} \cdot \mathbf{ds}$ is called the ***circulation*** of $\mathbf{X}$ around the closed curve C. It represents the net amount of turning of the fluid in a counterclockwise direction around C.

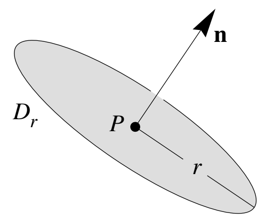


FIGURE 7.3.3. Curl is the circulation per unit area

Formula (7.3.7) gives the following physical interpretation for curl $\mathbf{X}$, namely: $(\operatorname{curl} \mathbf{X}) \cdot \mathbf{n}$ *is the circulation of* $\mathbf{X}$ *per unit area on a surface perpendicular* to $\mathbf{n}$. The magnitude of $(\operatorname{curl} \mathbf{X}) \cdot \mathbf{n}$ is clearly maximized when $\mathbf{n} = (\operatorname{curl} \mathbf{X})/\|\operatorname{curl} \mathbf{X}\|$. The vector curl $\mathbf{X}$ is called the ***vorticity vector***.

B. One of Maxwell's equations of electromagnetic theory states that if $\mathbf{E}(x, y, z, t)$ and $\mathbf{H}(x, y, z)$ represent the electric and magnetic fields at time t, then

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t},$$

where $\nabla \times \mathbf{E}$ is computed by holding t fixed, and $\partial \mathbf{H}/\partial t$ is computed by holding x, y, and z constant. Let us use Stokes' theorem to determine what this means physically. Assume S is a surface to which Stokes' theorem applies. Then

$$\int_{\partial S} \mathbf{E} \cdot \mathbf{ds} = \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{dS} = -\int_S \frac{d\mathbf{H}}{\partial t} \cdot \mathbf{dS} = -\frac{\partial}{\partial t} \int_S \mathbf{H} \cdot \mathbf{dS}.$$

(The last equality may be justified if $\mathbf{H}$ is C^1.) Thus we obtain

$$\int_{\partial S} \mathbf{E} \cdot \mathbf{ds} = -\frac{\partial}{\partial t} \int_S \mathbf{H} \cdot \mathbf{dS}. \tag{7.3.8}$$

Equality (7.3.8) is known as ***Faraday's law***. The quantity $\int_{\partial S} \mathbf{E} \cdot \mathbf{ds}$ represents the "voltage" around ∂S, and if ∂S were a wire, a current would flow in proportion to this voltage. Also $\int_S \mathbf{H} \cdot \mathbf{dS}$ is called the ***flux*** of $\mathbf{H}$, or the magnetic flux. Thus, Faraday's law says that *the voltage around a loop equals the negative of the rate of change of magnetic flux through the loop.*

C. Let $\mathbf{X} \in \mathfrak{X}(\mathbb{R}^3)$. Since $\mathbb{R}^3$ is contractible, the Poincaré lemma shows that curl $\mathbf{X} = 0$ iff $\mathbf{X} = \operatorname{grad} f$ for some function $f \in \mathcal{F}(\mathbb{R}^3)$. This in turn is equivalent (by Stokes' theorem) to either of the following: (i) for any oriented simple closed curve C, $\int_C \mathbf{X} \cdot \mathbf{ds} = 0$, or (ii) for any oriented simple curves C_1, C_2 with the same end points,

$$\int_{C_1} \mathbf{X} \cdot \mathbf{ds} = \int_{C_2} \mathbf{X} \cdot \mathbf{ds}.$$

The function f can be found in the following way:

$$f(x,y,z) = \int_0^x X^1(t,0,0)dt + \int_0^y X^2(x,t,0)dt + \int_0^z X^3(x,y,t)dt. \tag{7.3.9}$$

Thus, for example, if

$$\mathbf{X} = y\frac{\partial}{\partial x} + (z\cos(yz) + x)\frac{\partial}{\partial y} + y\cos(yz)\frac{\partial}{\partial z},$$

then $\operatorname{curl}\mathbf{X} = 0$ and so $\mathbf{X} = \operatorname{grad} f$, for some f. Using the formula (7.3.9), one finds

$$f(x,y,z) = xy + \sin yz. \tag{7.3.10}$$

D. The same arguments apply in $\mathbb{R}^2$ using Green's theorem in place of Stokes' theorem. Namely, if

$$\mathbf{X} = X^1\frac{\partial}{\partial x} + X^2\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2) \quad \text{and} \quad \frac{\partial X^2}{\partial x} = \frac{\partial X^1}{\partial y},$$

then $\mathbf{X} = \operatorname{grad} f$, for some $f \in \mathcal{F}(\mathbb{R}^2)$ and conversely.

E. The following statement is again a reformulation of the Poincaré lemma: let $\mathbf{X} \in \mathfrak{X}(\mathbb{R}^3)$, then $\operatorname{div}\mathbf{X} = 0$ iff $\mathbf{X} = \operatorname{curl}\mathbf{Y}$ for some $\mathbf{Y} \in \mathfrak{X}(\mathbb{R}^3)$. ♦

7.3.6 Theorem (Classical Gauss Theorem). *Let Ω be a compact set with nonempty interior in $\mathbb{R}^3$ bounded by a surface S that is piecewise smooth. If $\mathbf{X}$ is a C^1 vector field on $\Omega \cup S$, then*

$$\int_\Omega (\operatorname{div}\mathbf{X})dV = \int_S (\mathbf{X}\cdot\mathbf{n})dS, \tag{7.3.11}$$

where dV denotes the standard volume element (Lebesgue measure) in $\mathbb{R}^3$ (Figure 7.3.4).

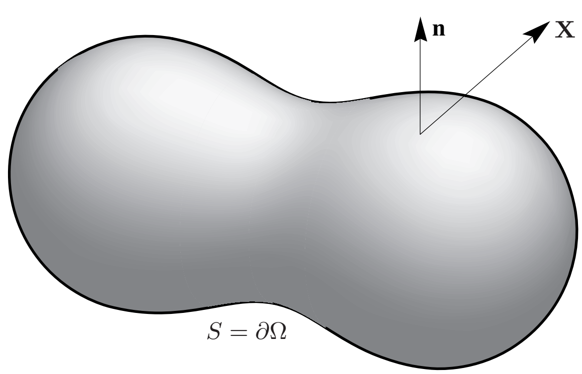


FIGURE 7.3.4. Gauss' Theorem

Proof. Either use Corollary 7.2.10 or argue as in Theorem 7.3.4. By equation (7.3.6),

$$\int_S (\mathbf{X}\cdot\mathbf{n})dS = \int_S *\mathbf{X}^\flat.$$

By Stokes' theorem, this equals

$$\int_\Omega \mathbf{d} * \mathbf{X}^\flat = \int_\Omega (\operatorname{div} \mathbf{X}) dV.$$

since $\mathbf{d} * \mathbf{X}^\flat = (\operatorname{div} \mathbf{X}) dx \wedge dy \wedge dz$. ■

7.3.7 Example. We shall use the preceding theorem to prove ***Gauss' law***

$$\int_{\partial\Omega} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} ds = \begin{cases} 4\pi, & \text{if } 0 \in \Omega; \\ 0, & \text{if } 0 \notin \Omega, \end{cases} \tag{7.3.12}$$

where Ω is a compact set in $\mathbb{R}^3$ with nonempty interior, $\partial\Omega$ is the surface bounding Ω, which is assumed to be piecewise smooth, $\mathbf{n}$ is the outward unit normal, $\mathbf{0} \notin \partial\Omega$, and where

$$r = (x^2 + y^2 + z^2)^{1/2}, \quad \mathbf{r} = (x, y, z).$$

If $\mathbf{0} \notin \Omega$, apply Theorem 7.3.6 and the fact that $\operatorname{div}(\mathbf{r}/r^3) = 0$ to get the result. If $\mathbf{0} \in \Omega$, surround $\mathbf{0}$ inside Ω by a ball D_ϵ of radius ϵ (Figure 7.3.5). Since the orientation of ∂D_ϵ induced from $\Omega \backslash D_\epsilon$ is the opposite of that induced from D_ϵ (namely it is given by the *inward* unit normal), Gauss' theorem gives

$$\int_{\partial\Omega} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS + \int_{\partial D_\epsilon} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \int_{\partial(\Omega \backslash D_\epsilon)} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = 0 \tag{7.3.13}$$

since $\mathbf{0} \notin \partial\Omega \backslash D_\epsilon$ and thus on $\Omega \backslash D_\epsilon$, we have $\operatorname{div}(\mathbf{r}/r^3) = 0$. But on ∂D_ϵ, $r = \epsilon$ and $\mathbf{n} = -\mathbf{r}/\epsilon$, so that

$$\int_{\partial D_\epsilon} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = -\int_{\partial D_\epsilon} \frac{\epsilon^2}{\epsilon^4} dS = -\frac{1}{\epsilon^2} 4\pi\epsilon^2 = -4\pi,$$

since

$$\int_{\partial D_\epsilon} dS = 4\pi\epsilon^2,$$

the area of the sphere of radius ϵ. ♦

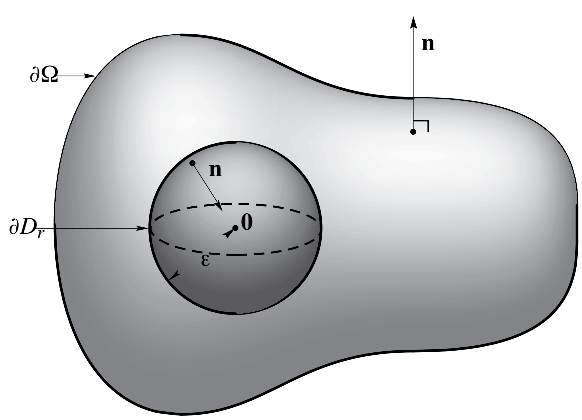


FIGURE 7.3.5. ??????????????

In electrostatics Gauss' law (7.3.12) is used in the following way. The potential due to a point charge q at $\mathbf{0} \in \mathbb{R}^3$ is given by $q/(4\pi r)$, where $r = (x^2 + y^2 + z^2)^{1/2}$. The corresponding electric field is defined to be minus the gradient of this potential; that is,

$$\mathbf{E} = \frac{q\mathbf{r}}{4\pi r^3}.$$

Thus Gauss' law states that *the total electric flux* $\int_{\partial\Omega} \mathbf{E} \cdot \mathbf{n}\, dS$ *equals* q if $\mathbf{0} \in \Omega$ *and equals zero*, if $\mathbf{0} \notin \Omega$.

A continuous charge distribution in Ω described by a ***charge density*** ρ is related to $\mathbf{E}$ by $\rho = \operatorname{div} \mathbf{E}$. By Gauss' theorem the electric flux

$$\int_{\partial\Omega} \mathbf{E} \cdot \mathbf{n}\, dS = \int_{\Omega} \rho\, dV,$$

which represents the total charge inside Ω. Thus, the relationship $\operatorname{div} \mathbf{E} = \rho$ may be phrased as follows. *The flux out of a surface of an electric field equals the total charge inside the surface.*

Exercises

◇ **7.3-1.** Use Green's theorem to show that

(i) The area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is πab.

(ii) The area of the hypocycloid $x = a\cos^3\theta$, $y = b\sin^3\theta$ is $3\pi a^2/8$.

(iii) The area of one loop of the four-leaved rose $r = 3\sin 2\theta$ is $9\pi/8$.

◇ **7.3-2.** Why does Green's theorem fail in the unit disk for

$$-y\frac{dx}{(x^2+y^2)} + \frac{xdy}{(x^2+y^2)}?$$

◇ **7.3-3.** For an oriented surface S and a fixed vector $\mathbf{a}$, show that

$$2\int_S \mathbf{a} \cdot \mathbf{n}\, dS = \int_{\partial S} (\mathbf{a} \times \mathbf{r}) \cdot \mathbf{dS}.$$

◇ **7.3-4.** Let the components of the vector field $\mathbf{X} \in \mathfrak{X}(\mathbb{R}^3)$ be ***homogeneous of degree one***; that is, $\mathbf{X}$ satisfies $X^i(tx, ty, tz) = tX^i(x, y, z)$, $i = 1, 2, 3$. Show that if $\operatorname{curl} \mathbf{X} = \mathbf{0}$, then $\mathbf{X} = \operatorname{grad} f$, where $f = (xX^1 + yX^2 + zX^3)/2$.

◇ **7.3-5.** Let S be the surface of a region Ω in $\mathbb{R}^3$. Show that

$$\text{volume}(\Omega) = \frac{1}{3}\int_S \mathbf{r} \cdot \mathbf{n} dS.$$

Give an intuitive argument why this should be so.
Hint: Think of cones.

◇ **7.3-6.** Let S be a closed (i.e., compact boundaryless) oriented surface in $\mathbb{R}^3$.

(i) Show in two ways that $\int_S (\operatorname{curl} \mathbf{X}) \cdot \mathbf{n}\, dS = 0$.

(ii) Let $\mathbf{X} = \mathfrak{X}(S)$ and $f \in \mathcal{F}(S)$. Make sense of and show that

$$\int_S (\operatorname{grad} f)\mathbf{X} \cdot \mathbf{dS} = -\int_S (f \operatorname{curl} \mathbf{X}) dS$$

where grad, curl, and div are taken in $\mathbb{R}^3$.

◇ **7.3-7.** Let $\mathbf{X}$ and $\mathbf{Y}$ be smooth vector fields on an open set $D \subset \mathbb{R}^3$, with ∂D smooth, and $\mathrm{cl}(D)$ compact. Show that

$$\int_D \mathbf{Y} \cdot \operatorname{curl} \mathbf{X} \, dV = \int_D \mathbf{X} \cdot \operatorname{curl} \mathbf{Y} \, dV + \int_{\partial D} (\mathbf{X} \times \mathbf{Y}) \cdot \mathbf{n} \, dS.$$

where $\mathbf{n}$ is the outward unit normal to ∂D and dS the induced surface measure on ∂D.
Hint: Show that

$$\mathbf{Y} \cdot \operatorname{curl} \mathbf{X} - \mathbf{X} \cdot \operatorname{curl} \mathbf{Y} = \operatorname{div}(\mathbf{X} \times \mathbf{Y}).$$

◇ **7.3-8.** If C is a closed curve bounding a surface S show that

$$\int_C f(\operatorname{grad} g) \cdot \mathbf{ds} = \int_S (\operatorname{grad} f \times \operatorname{grad} g) \cdot \mathbf{n} \, dS = -\int_C g(\operatorname{grad} f) \cdot \mathbf{ds}$$

where f and g are C^2 functions.

◇ **7.3-9** (A. Lenard)**.** Faraday's law relates the line integral of the electric field around a loop C to the surface integral of the rate of change of the magnetic field over a surface S with boundary C. Regarding the equation $\nabla \times \mathbf{E} = -\partial \mathbf{H}/\partial t$ as the basic equation, Faraday's law is a consequence of Stokes' theorem, as we have seen in Example 7.3.5B. Suppose we are given electric and magnetic fields in space that satisfy the equation $\nabla \times \mathbf{E} = -\partial \mathbf{H}/\partial t$. Suppose C is the boundary of the Möbius band shown in Figure 6.5.1. Since the Möbius band cannot be oriented, Stokes' theorem does not apply. What becomes of Faraday's law? Resolve the issue in two ways: (i) by using the results of Supplement 7.2A or a direct reformulation of Stokes' theorem for nonorientable surfaces, and (ii) realizing C as the boundary of an orientable surface. If $\partial \mathbf{H}/\partial t$ is arbitrary, in general does a current flow around C or not?

7.4 Induced Flows on Function Spaces and Ergodicity

This section requires some results from functional analysis. Specifically *we shall require a knowledge of Stone's theorem and self-adjoint operators*. The required results may be found in Supplements 7.4A and 7.4B at the end of this section.

Flows on manifolds induce flows on tangent bundles, tensor bundles, and spaces of tensor fields by means of push-forward. In this section we shall be concerned mainly with the induced flow on the space of functions. This induced flow is sometimes called the ***Liouville flow***.

Let M be a manifold and μ a volume element on M; that is, (M, μ) is a volume manifold. If F_t is a (volume-preserving) flow on M, then F_t induces a *linear* one-parameter group (of isometries) on the Hilbert space $H = L^2(M\mu)$ by

$$U_t(f) = f \circ F_{-t}.$$

The association of U_t with F_t replaces a *nonlinear finite-dimensional* problem with a *linear infinite-dimensional* one.

There have been several theorems that relate properties of F_t and U_t. The best known of these is the result of Koopman [1931], which shows that U_t has one as a simple eigenvalue for all t if and only if F_t is ergodic. (If there are no other eigenvalues, then F_t is called *weakly mixing*.) A few basic results on ergodic theory are given below. We refer the reader to the excellent texts of Halmos [1956], Arnol'd and Avez [1967], and Bowen [1975] for more information.

We shall first present a result of Povzner [1966], which relates the completeness of the flow of a divergence-free vector field X to the skew-adjointness of X as an operator. (The hypothesis of divergence free is removed in Exercises 7.4-1–7.4-3.) We begin with a lemma due to Ed Nelson.

7.4.1 Lemma. *Let A be an (unbounded) self-adjoint operator on a complex Hilbert space $\mathcal{H}$. Let $D_0 \subset D(A)$ (the domain of A) be a dense linear subspace of $\mathcal{H}$ and suppose $U_t = e^{itA}$ (the unitary one-parameter group generated by A) leaves D_0 invariant. Then $A_0 := (A$ restricted to $D_0)$ is essentially self-adjoint; that is, the closure of A_0 is A.*

Proof. Let $\mathbf{A}$ denote the closure of A_0. Since A is closed and extends A_0, A extends $\mathbf{A}$. We need to prove that $\mathbf{A}$ extends A.

For $\lambda > 0$, $\lambda - iA$ is surjective with a bounded inverse. First of all, we prove that $\lambda - iA_0$ has dense range. If not, there is a $v \in \mathcal{H}$ such that

$$\langle v, \lambda x - iA_0 x\rangle = 0 \quad \text{for all } x \in D_0.$$

In particular, since D_0 is U_t-invariant,

$$\frac{d}{dt}\langle v, U_t x\rangle = \langle v, iAU_t x\rangle = \lambda\langle v, U_t x\rangle$$

so

$$\langle v, U_t x\rangle = e^{\lambda t}\langle v, x\rangle.$$

Since D_0 is dense, this holds for all $x \in \mathcal{H}$. Thus, $\|U_t\| = 1$ and $\lambda > 0$ imply $v = 0$. Therefore $(\lambda - iA_0)^{-1}$ makes sense and $(\lambda - iA)^{-1}$ is its closure. It follows from Supplement 7.4A that A is the closure of A_0 (see Corollary 7.4.15 and the remarks following it). ■

7.4.2 Proposition. *Let X be a C^∞ divergence-free vector field on (M, μ) with a complete flow F_t. Then iX is an essentially self-adjoint operator on $C_c^\infty = C^\infty$ functions with compact support in the complex Hilbert space $L^2(M, \mu)$.*

Proof. Let $U_t f = f \circ F_{-t}$ be the unitary one-parameter group induced from F_t. A straightforward convergence argument shows that $U_t f$ is continuous in t in $L^2(M, \mu)$. In Lemma 7.4.1, choose $D_0 = C^\infty$ functions with compact support. This is clearly invariant under U_t. If $f \in D_0$, then

$$\left.\frac{d}{dt}U_t f\right|_{t=0} = \left.\frac{d}{dt} f \circ F_{-t}\right|_{t=0} = -\mathbf{d}f \cdot X,$$

so the generator of U_t is an extension of $-X$ (as a differential operator) on D_0. The corresponding essentially self–adjoint operator is therefore iX. ■

Now we prove the converse of Proposition 7.4.2. That is, if iX is essentially self-adjoint, then X has an almost everywhere complete flow. This then gives a functional-analytic characterization of completeness.

7.4.3 Theorem. *Let M be a manifold with a volume element μ and X be a C^∞ divergence-free vector field on M. Suppose that, as an operator on $L^2(M, \mu)$, iX is essentially self-adjoint on the C^∞ functions with compact support. Then, except possibly for a set of points x of measure zero, the flow $F_t(x)$ of X is defined for all $t \in \mathbb{R}$.*

We shall actually prove that, if the defect index of iX is zero in the upper half-plane (i.e., if $(iX + i)(C_c^\infty)$ is dense in L^2), then the flow is defined, except for a set of measure zero, for all $t > 0$. Similarly, if the defect index of iX is zero in the lower half-plane, the flow is essentially complete for $t < 0$. The converses of these more general results can be established along the lines of the proof of Lemma 7.4.1.

Proof (E. Nelson—private communication)**.** Suppose that there is a set E of finite positive measure such that if $x \in E$, $F_t(x)$ fails to be defined for t sufficiently large. Let E_T be the set of $x \in E$ for which $F_t(x)$ is undefined for $t \geq T$. Since $E = \bigcup_{T \geq 1} E_T$, some E_T has positive measure. Replacing E by E_T, we may assume that all points of E "move to infinity" in a time $\leq T$.

If f is any function on M, we adopt the convention that $f(F_t(x)) = 0$ if $F_t(x)$ is undefined. For any $x \in M$, and $t < -T$, $F_t(x)$ must be either in the complement of E or undefined; otherwise it would be a point of E that did not move to infinity in time T. Hence $\chi_E(F_t(x)) = 0$ for $t < -T$, where χ_E is the characteristic function of E. We now define a function on M by

$$g(x) = \int_{-\infty}^{\infty} e^{-\tau} \chi_E(F_\tau(x)) d\tau.$$

Note that the integral converges because the integrand vanishes for $t < -T$. In fact, we have

$$0 \leq g(x) \leq \int_{-T}^{\infty} e^{-\tau} d\tau = e^T.$$

Moreover, g is in L^2. Indeed, because F_t is measure-preserving, where defined, denoting by $\|\cdot\|_2$ the L^2 norm, we have $\|\chi_E \circ F_\tau\|_2 \leq \|\chi_E\|_2$, so that

$$\|g\|_2 \leq \int_{-T}^{\infty} e^{-\tau} \|\chi_E \circ F_\tau\|_2 d\tau \leq \|\chi_E\|_2 e^T.$$

The function g is nonzero because E has positive measure.

Fix a point $x \in M$. Then $F_t(x)$ is defined for t sufficiently small. It is easy to see that in this case $F_\tau(F_t(x))$ and $F_{\tau+t}(x)$ are defined or undefined together, and in the former case they are equal. Hence we have $\chi_E(F_\tau(F_t(x)) = \chi_E(F_{\tau+t}(x))$ for t sufficiently small. Therefore, for

$$g(F_t(x)) = \int_{-\infty}^{\infty} e^{-\tau} \chi_E(F_{\tau+t}(x)) d\tau = \int_{-\infty}^{\infty} e^{t-\tau} \chi_E(F_\tau(x)) d\tau = e^t g(x).$$

Now if φ is C^∞ with compact support, we have

$$\begin{aligned}
\int g(x) X[\varphi](x) d\mu &= \lim_{t \to 0} \int g(x) \frac{\varphi(F_t(x)) - \varphi(x)}{t} d\mu \\
&= \lim_{t \to 0} \int \frac{g(F_{-t}(x)) - g(x)}{t} \varphi(x) d\mu \\
&= \lim_{t \to 0} \int \frac{e^{-t} - 1}{t} g(x) \varphi(x) d\mu = -\int g(x) \varphi(x) d\mu.
\end{aligned}$$

These equalities are justified because on the support of φ the flow F_t exists for sufficiently small t and is measure-preserving. Thus g is orthogonal to the range of $X + 1$, and therefore the defect index of iX in the upper half-plane is nonzero.

The case of completeness for $t < 0$ is similar. ■

Methods of functional analysis applied to $L^2(M, \mu)$ can, as we have seen, be used to obtain theorems relevant to flows on M. Related to this is a measure-theoretic analogue of the fact that any automorphism of the algebra $\mathcal{F}(M)$ is induced by a diffeomorphism of M (see Supplement 4.2C). This result, due to Mackey [1962], states that if U_t is a linear isometry on $L^2(M, \mu)$, which is multiplicative (i.e., $U_t(fg) = (U_t f)(U_t g)$, where defined), then U_t is induced by some measure preserving flow F_t on M. This may be used to give another proof of Theorem 7.4.3.

An important notion for statistical mechanics is that of ergodicity; this is intended to capture the idea that a flow may be random or chaotic. In dealing with the motion of molecules, the founders of statistical mechanics, particularly Boltzmann and Gibbs, made such hypotheses at the outset. One of the earliest precise definitions of randomness of a dynamical system was ***minimality***: the orbit of almost every point is dense. In order to prove useful theorems, von Neumann and Birkhoff in the early 1930s required the strong assumption of ergodicity, defined as follows.

7.4.4 Definition. *Let S be a measure space and F_t a (measurable) flow on S. We call F_t **ergodic** if the only invariant measurable sets are $\varnothing$ and all of S.*

Here, invariant means $F_t(A) = A$ for all $t \in \mathbb{R}$ and we agree to write $A = B$ if A and B differ by a set of measure zero. (It is not difficult to see that ergodicity implies minimality if we are on a second countable Borel space.)

A function $f : S \to \mathbb{R}$ will be called a ***constant of the motion*** if $f \circ F_t = f$ a.e. (almost everywhere) for each $t \in \mathbb{R}$.

7.4.5 Proposition. *A flow F_t on S is ergodic iff the only constants of the motion are constant a.e.*

Proof. If F_t is ergodic and f is a constant of the motion, the two sets

$$\{ x \in S \mid f(x) \geq a \}$$

and $\{ x \in S \mid f(x) \leq a \}$ are invariant, so f must be constant a.e. The converse follows by taking f to be a characteristic function. ■

The first major step in ergodic theory was taken by von Neumann [1932], who proved the mean ergodic theorem which remains as one of the most important basic theorems. The setting is in Hilbert space, but we shall see how it applies to flows of vector fields in Corollary 7.4.7.

7.4.6 Theorem (Mean Ergodic Theorem)**.** *Let $\mathbf{H}$ be a real or complex Hilbert space and $U_t : \mathbf{H} \to \mathbf{H}$ a strongly continuous one-parameter unitary group (i.e., U_t is unitary for each t, is a flow on $\mathbf{H}$ and for each $x \in \mathbf{H}$, the map $t \mapsto U_t x$ is continuous).*

Let the closed subspace $\mathbf{H}_0$ be defined by

$$\mathbf{H}_0 = \{ x \in \mathbf{H} \mid U_t x = x \text{ for all } t \in \mathbb{R} \}$$

and let $\mathbb{P}$ be the orthogonal projection onto $\mathbf{H}_0$. Then for any $x \in \mathbf{H}$,

$$\lim_{t\to\infty} \left\| \frac{1}{t} \int_0^t U_s x \, ds - \mathbb{P}x \right\| = 0.$$

*The point $\operatorname{av}(x) = \mathbb{P}x$ so defined is called the **time average** of x.*

Proof (Riesz [1944])**.** We must show that

$$\lim_{t\to\infty} \left\| \frac{1}{t} \int_0^t U_s x \, ds - \mathbb{P}x \right\| = 0.$$

If $\mathbb{P}x = x$, this means $x \in \mathbf{H}_0$, so $U_s(x) = x$; the result is clearly true in this case. We can therefore suppose that $\mathbb{P}x = 0$ by considering the decomposition $x = \mathbb{P}x + (x - \mathbb{P}x)$. Note that

$$\{ U_t y - y \mid y \in \mathbf{H},\, t \in \mathbb{R} \}^{\perp} = \mathbf{H}_0$$

where $\perp$ denotes the orthogonal complement. This is an easy verification using unitarity of U_t and $U_t^{-1} = U_{-t}$. It follows that $\ker \mathbb{P}$ is the closure of the space spanned by elements of the form $U_s y - y$. Indeed $\ker \mathbb{P} = \mathbf{H}_0^{\perp}$, and if A is any set in $\mathbf{H}$, and $B = A^{\perp}$, then $B^{\perp}$ is the closure of the span of A. Therefore, for any $\epsilon > 0$, there exists $t_1, \ldots, t_n$ and $x_1, \ldots, x_n$ such that

$$\left\| x - \sum_{j=1}^{n} (U_{t_j} x_j - x_j) \right\| < \epsilon.$$

It follows from this, again using unitarity of U_t, that it is enough to prove our assertion for x of the form $U_\tau y - y$. Thus we must establish

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t U_s(U_\tau y - y)ds = 0.$$

For $t > \tau$ we may estimate this integral as follows:

$$\begin{aligned}\left\|\frac{1}{t}\int_0^t (U_sU_\tau y - U_s y)ds\right\| &= \left\|-\frac{1}{t}\int_0^\tau U_s(y)ds + \frac{1}{t}\int_0^{t+\tau} U_s(y)ds\right\| \\ &\le \frac{1}{t}\int_0^\tau \|y\|ds + \frac{1}{t}\int_0^{t+\tau}\|y\|ds \\ &= \frac{2\tau\|y\|}{t} \to 0 \text{ as } t\to\infty.\end{aligned}$$

■

To apply Theorem 7.4.6 to a measure-preserving flow F_t on S, we consider the unitary one-parameter group $U_t(f) = f \circ F_t$ on $L^2(S,\mu)$. We only require a minimal amount of continuity on F_t here, namely, we assume that if $s \to t$, $F_s(x) \in F_t(x)$ for a.e. $x \in S$. We shall also assume $\mu(S) < \infty$ for convenience. Under these hypotheses, U_t is a strongly continuous unitary one–parameter group. The verification can be done with the aid of the dominated convergence theorem.

7.4.7 Corollary. *In the hypotheses above F_t is ergodic if and only if for each $f \in L^2(S)$ its time average*

$$\operatorname{av}(f)(x) = \lim_{t\to\pm\infty}\frac{1}{t}\int_0^t (f\circ F_s)(x)ds$$

(*the limit being in the L^2-mean*) *is constant a.e. In this case the time average* av(f) *necessarily equals the* ***space average*** $\int_S f d\mu/\mu(S)$ *a.e.*

Proof. Ergodicity of F_t is equivalent, by Proposition 7.4.5, to $\dim \mathbf{H}_0 = 1$, where $\mathbf{H}_0$ is the closed subspace of $L^2(S)$ given in Theorem 7.4.6. If $\dim \mathbf{H}_0 = 1$,

$$\mathbb{P}(f) = \int_S \frac{f d\mu}{\mu(S)}$$

so the equality of av(f) with $\mathbb{P}(f)$ a.e. is a consequence of Theorem 7.4.6. Conversely if any $f \in L^2(S)$ has a.e. constant time average av(f) then taking f to be a constant of motion, it follows that $f = \operatorname{av}(f)$ is constant a.e. Therefore, $\dim \mathbf{H}_0 = 1$. ■

Thus, if F_t is ergodic, the time average of a function is constant a.e. and equals its space average. A refinement of this is the *individual ergodic theorem* of Birkhoff [1931], in which one obtains convergence almost everywhere. Also, if $\mu(S) = \infty$ but $f \in L^1(S) \cap L^2(S)$, one still concludes a.e. convergence of the time average. (If f is only L^2, mean convergence to zero is still assured by Proposition 7.4.5.)

Modern work in dynamical systems, following the ideas in §4.3, has shown that for many interesting flows arising in the physical sciences, the motion can be "chaotic" on large regions of phase space without being ergodic. Much current research is focused on trying to prove analogues of the ergodic theorems for such cases. (See, for instance, Guckenheimer and Holmes [1983], Eckmann and Ruelle [1985], and references therein.)

A particularly important example of an ergodic flow is the irrational flow on the torus.

7.4.8 Definition. *The flow $F_t : \mathbb{T}^n \to \mathbb{T}^n$ given by $F_t([\varphi]) = [\varphi + \nu t]$, for $\nu \in \mathbb{R}^n$ is called the* ***quasiperiodic*** *or* ***linear flow*** *on $\mathbb{T}^n$ determined by ν. The quasiperiodic flow is called* ***irrational*** *if the components $(\nu^1,\ldots,\nu^n)$ of ν are linearly independent over $\mathbb{Z}$ (or, equivalently, over $\mathbb{Q}$), that is, $\mathbf{k}\cdot\nu = 0$ for $\mathbf{k} \in \mathbb{Z}^n$ implies $\mathbf{k} = 0$.*

7.4.9 Proposition. *The linear flow F_t on $\mathbb{T}^n$ determined by $\nu \in \mathbb{R}^n$ is ergodic if and only if it is irrational.*

Proof. Assume the flow is irrational and let $f \in L^2(\mathbb{T}^n)$ be a constant of the motion. Expand $f([\varphi])$ and $(f \circ F_t)([\varphi])$ in Fourier series:

$$f([\varphi]) = \sum_{\mathbf{k}\in\mathbb{Z}^n} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\varphi} \quad \text{and} \quad (f \circ F_t)([\varphi]) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{b}_k(t) e^{i\mathbf{k}\cdot\varphi}$$

where the convergence is in L^2 and the Fourier coefficients are given by

$$\begin{aligned} a_{\mathbf{k}} &= \int_{T^n} e^{i\mathbf{k}\cdot\varphi} f([\varphi])d\varphi \\ b_{\mathbf{k}}(t) &= \int_{T^n} e^{i\mathbf{k}\cdot\varphi} f([\varphi + \nu t])d\varphi \\ &= \int_{T^n} e^{i\mathbf{k}\cdot(\varphi-\nu t)} f([\varphi])d\varphi = e^{i\mathbf{k}\cdot\nu t} a_{\mathbf{k}}. \end{aligned}$$

(The measure $d\varphi$ is chosen such that the total volume of $\mathbb{T}^n$ equals one.) Since f is a constant of the motion, $a_{\mathbf{k}} = b_{\mathbf{k}}(t)$ for all $\mathbf{k} \in \mathbb{Z}^n$ and all $t \in \mathbb{R}$ which implies that $e^{\mathbf{k}\cdot\nu t} = 1$ for all $\mathbf{k} \in \mathbb{Z}^n$, $t \in \mathbb{R}$. Thus $\mathbf{k}\cdot\nu = 0$ which by hypothesis forces $\mathbf{k} = 0$. Consequently all $a_{\mathbf{k}} = 0$ with the exception of a_0 and thus $f = a_0$ a.e.

Conversely, assume F_t is ergodic and that $\mathbf{k}\cdot\nu = 0$ for some $\mathbf{k} \in \mathbb{Z}^n\backslash\{\mathbf{0}\}$. Then the set $A = \{\, [\psi] \in \mathbb{T}^n \mid \mathbf{k}\cdot\psi = 0 \,\}$ is closed and hence measurable and invariant under F_t. But clearly $A \neq \varnothing$ and $A \neq \mathbb{T}^n$ which shows that F_t is not ergodic. ■

7.4.10 Corollary. *Let F_t be an irrational flow on $\mathbb{T}^n$ determined by ν. Then every trajectory of F_t is **uniformly distributed** on $\mathbb{T}^n$, that is, for any measurable set A in $\mathbb{T}^n$,*

$$\lim_{t\to\pm\infty} \frac{\text{measure}\, A(t)}{t} = \text{measure}\, A$$

where

$$A(t) = \{\, s \in [0, t] \mid F_s([\psi]) \in A \,\}$$

and the measure of $\mathbb{T}^n$ is assumed to be equal to one.

Proof. Let χ_A be the characteristic function of A. Then

$$\begin{aligned} \text{av}(\chi_A)([\psi]) &= \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \chi_A(F_s([\psi]))ds \\ &= \lim_{t\to\pm\infty} \frac{1}{t}(\text{measure}\, A(t)) \\ &= \int_{T^n} (\chi_A([\psi]))d\varphi \\ &= \text{measure}\, A \end{aligned}$$

by Corollary 7.4.7 and Proposition 7.4.9. ■

7.4.11 Corollary. *Every trajectory of a quasiperiodic flow F_t on $\mathbb{T}^n$ is dense if and only if the flow is irrational.*

Proof. By translation of the initial condition it is easily seen that every trajectory is dense on $\mathbb{T}^n$ if and only if the trajectory through $[\mathbf{0}]$ is dense. Assume first that the flow is irrational. If $\{\, F_t([\mathbf{0}]) \mid t \in \mathbb{R} \,\}$ is not dense in $\mathbb{T}^n$ then there is an open set U in $\mathbb{T}^n$ not containing any point of this trajectory. Thus, in the notation of Corollary 7.4.10, $U(t) = \varnothing$. This contradicts 7.4.10 since the measure of U is strictly positive.

Conversely, assume that the trajectory through $[\mathbf{0}]$ is dense and let f be a continuous constant of the motion for F_t. This implies that f is a constant. Since continuous functions are dense in L^2, this in turn implies that any L^2-constant of the motion is constant a.e. By Proposition 7.4.5, F_t is ergodic and by Proposition 7.4.9, F_t is irrational. ■

Supplement 7.4A
Unbounded and Self Adjoint Operators.[2]

In many applications involving differential equations, the operators one meets are not defined on the whole Banach space E and are not continuous. Thus we are led to consider a linear transformation $A : D_A \subset E \to E$ where D_A is a linear subspace of E (the domain of A). If D_A is dense in E, we say A is ***densely defined***. We speak of A as an ***operator*** and this shall mean ***linear*** operator unless otherwise specified.

Even though A is not usually continuous, it might have the important property of being closed. We say A is ***closed*** if its graph Γ_A

$$\Gamma_A = \{ (x, Ax) \in E \times E \mid x \in D_A \}$$

is a closed subset of $E \times E$. This is equivalent to

$$(x_n \in D_A,\ x_n \to x \in E \text{ and } Ax_n \to y \in E)$$
$$\text{implies } (x \in D_A \text{ and } Ax = y).$$

An operator A (with domain D_A) is called ***closable*** if $\mathrm{cl}(\Gamma_A)$, the closure of the graph of A, is the graph of an operator, say, $\mathbf{A}$. We call $\mathbf{A}$ the ***closure*** of A. It is easy to see that A is closable iff $\{(x_n \in D_A,\ x_n \to 0$ and $Ax_n \to y)$ implies $y = 0\}$. Clearly $\mathbf{A}$ is a closed operator that is an ***extension*** of A; that is, $D_{\mathbf{A}} \supset D_A$ and $\mathbf{A} = A$ on D_A. One writes this as $\mathbf{A} \supset A$.

The closed graph theorem from §2.2 asserts that an everywhere defined closed operator is bounded. However, if an operator is only densely defined, "closed" is weaker than "bounded." If A is a closed operator, the map $x \mapsto (x, Ax)$ is an isomorphism between D_A and the closed subspace Γ_A. Hence if we set

$$|||x|||^2 = \|x\|^2 + \|Ax\|^2,$$

D_A becomes a Banach space. We call the norm $|||\cdot|||$ on D_A the ***graph norm***.

Let A be an operator on a real or complex Hilbert space $\mathbf{H}$ with dense domain D_A. The ***adjoint*** of A is the operator A^* with domain D_{A^*}, defined as follows:

$$D_{A^*} = \{ y \in \mathbf{H} \mid \text{ there is a } z \in \mathbf{H} \text{ such that}$$
$$\langle Ax, y \rangle = \langle x, z \rangle \text{ for all } x \in D_A \}$$

and

$$A^* : D_{A^*} \to \mathbf{H}, \quad y \mapsto z.$$

From the fact that D_A is dense, we see that A^* is indeed well defined (there is at most one such z for any $y \in \mathbf{H}$). It is easy to see that if $A \supset B$ then $B^* \supset A^*$.

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[2]This supplement was written in collaboration with P. Chernoff.

If A is everywhere defined and bounded, it follows from the Riesz representation theorem (Supplement 2.2A) that A^* is everywhere defined; moreover it is not hard to see that, in this case, $\|A^*\| = \|A\|$.

An operator A is ***symmetric*** (***Hermitian*** in the complex case) if $A^* \supset A$; that is, $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in D_A$. If $A^* = A$ (this includes the condition $D_{A^*} = D_A$), then A is called ***self-adjoint***. An everywhere defined symmetric operator is bounded (from the closed graph theorem) and so is self-adjoint. It is also easy to see that a self-adjoint operator is closed.

One must be aware that, for technical reasons, it is the notion of self-adjoint rather than symmetric, which is important in applications. Correspondingly, verifying self-adjointness is often difficult while verifying symmetry is usually trivial.

Sometimes it is useful to have another concept at hand, that of essential self-adjointness. First, it is easy to check that any symmetric operator A is closable. The closure $\mathbf{A}$ is easily seen to be symmetric. One says that A is ***essentially self-adjoint*** when its closure $\mathbf{A}$ is self-adjoint.

Let A be a self-adjoint operator. A dense subspace $C \subset \mathbf{H}$ is said to be a ***core*** of A if $C \subset D_A$ and the closure of A restricted to C is again A. Thus if C is a core of A one can recover A just be knowing A on C.

We now give a number of propositions concerning the foregoing concepts, which are useful in applications. Most of this is classical work of von Neumann. We begin with the following.

7.4.12 Proposition. *Let A be a closed symmetric operator of a complex Hilbert space $\mathbf{H}$. If A is self-adjoint then $A + \lambda I$ is surjective for every complex number λ with $\operatorname{Im}\lambda \neq 0$ (I is the identity operator).*

Conversely, if A is symmetric and $A - iI$ and $A + iI$ are both surjective then A is self-adjoint.

Proof. Let A be self-adjoint and $\lambda = \alpha + i\beta$, $\beta \neq 0$. For $x \in D_A$ we have

$$\begin{aligned}\|(A+\lambda)x\|^2 &= \|(A+\alpha)x\|^2 + i\beta\langle x, Ax\rangle - i\beta\langle Ax, x\rangle + \beta^2\|x\|^2 \\ &= \|(A+\alpha)x\|^2 + \beta^2\|x\|^2 \geq \beta^2\|x\|^2,\end{aligned}$$

where $A + \lambda$ means $A + \lambda I$. Thus we have the inequality

$$\|(A+\lambda)x\| \geq |\operatorname{Im}\lambda|\,\|x\| \tag{7.4.1}$$

Since A is closed, it follows from equation (7.4.1) that the range of $A+\lambda$ is a closed set for $\operatorname{Im}\lambda \neq 0$. Indeed, let $y_n = (A+\lambda)x_n \to y$. By the inequality (7.4.1),

$$\|x_n - x_m\| \leq \frac{\|y_n - y_m\|}{|\operatorname{Im}\lambda|}$$

so x_n converges to, say x. Also Ax_n converges to $y - \lambda x$; thus $x \in D_A$ and $y - \lambda x = Ax$ as A is closed.

Now suppose y is orthogonal to the range of $A + \lambda I$. Thus

$$\langle Ax + \lambda x, y\rangle = 0 \text{ for all } x \in D_A, \quad \text{or} \quad \langle Ax, y\rangle = -\langle x, \lambda y\rangle.$$

By definition, $y \in D_{A^*}$ and $A^*y = -\bar{\lambda}y$; since $A = A^*$, $y \in D_A$, and $Ay = -\bar{\lambda}y$, we obtain $(A+\lambda I)y = 0$. Thus the range of $A + \lambda I$ is all of $\mathbf{H}$.

Conversely, suppose $A + i$ and $A - i$ are onto. Let $y \in D_{A^*}$. Thus for all $x \in D_A$,

$$\langle (A+i)x, y\rangle = \langle x, (A^* - i)y\rangle = \langle x, (A-i)z\rangle$$

for some $z \in D_A$ since $A - i$ is onto. Thus,

$$\langle (A+i)x, y\rangle = \langle (A+i)x, z\rangle$$

and it follows that $y = z$. This proves that $D_{A^*} \subset D_A$ and so $D_A = D_{A^*}$. The result follows. ■

If A is self-adjoint then for $\operatorname{Im}\lambda \neq 0$, $\lambda I - A$ is onto and from equation (7.4.1) is one-to-one. Thus $(\lambda I - A)^{-1} : \mathbf{H} \to \mathbf{H}$ exists, is bounded, and we have

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{|\operatorname{Im}\lambda|} \tag{7.4.2}$$

This operator $(\lambda I - A)^{-1}$ is called the ***resolvent*** of A. Notice that even though A is an unbounded operator, the resolvent is bounded. The same argument used to prove Proposition 7.4.12 shows the following.

7.4.13 Proposition. *A symmetric operator A is essentially self-adjoint iff the ranges of $A+iI$ and $A-iI$ are dense.*

If A is a (closed) symmetric operator then the ranges of $A + iI$ and $A - iI$ are (closed) subspaces. The dimensions of their orthogonal complements are called the ***deficiency indices*** of A. Thus, Propositions (7.4.12) and (7.4.13) can be restated as: *a closed symmetric operator* (*resp., a symmetric operator*) *is self-adjoint* (*resp., essentially self-adjoint*) *iff it has deficiency indices* $(0, 0)$.

If A is a closed symmetric operator then from equation (7.4.1), $A + iI$ is one-to-one and we can consider the inverse $(A + iI)^{-1}$, defined on the range of $A + iI$. One calls

$$(A - iI)(A + iI)^{-1}$$

the ***Cayley transform*** of A. It is always isometric, as is easy to check. Thus A *is self–adjoint iff its Cayley transform is unitary.*

Let us return to the graph of an operator A for a moment. The adjoint can be described entirely in terms of its graph and this is often convenient. Define an isometry $J : \mathbf{H} \oplus \mathbf{H} \to \mathbf{H} \oplus \mathbf{H}$ by $J(x, y) = (-y, x)$; note that $J^2 = -I$.

7.4.14 Proposition. *Let A be densely defined. Then $(\Gamma_A)^\perp = J(\Gamma_{A^*})$ and $-\Gamma_{A^*} = J(\Gamma_A)^\perp$. In particular, A^* is closed, and if A is closed, then*

$$\mathbf{H} \oplus \mathbf{H} = \Gamma_A \oplus J(\Gamma_{A^*}),$$

where $\mathbf{H} \oplus \mathbf{H}$ carries the usual inner product:

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle .$$

Proof. Let $(z, y) \in J(\Gamma_{A^*})$, so $y \in D_{A^*}$ and $z = -A^*y$. Let $x \in D_A$. We have

$$\langle (x, Ax), (-A^*y, y) \rangle = \langle x, -A^*y \rangle + \langle Ax, y \rangle = 0,$$

and so $J(\Gamma_{A^*}) \subset \Gamma_A^\perp$.

Conversely if $(z, y) \in (\Gamma_A)^\perp$, then $\langle x, z \rangle + \langle Ax, y \rangle = 0$ for all $x \in D_A$. Thus by definition, $y \in D_{A^*}$ and $z = -A^*y$. This proves the opposite inclusion. ■

Thus if A is a closed operator, the statement $\mathbf{H} \oplus \mathbf{H} = \Gamma_A \oplus J(\Gamma_{A^*})$ means that given $e, f \in \mathbf{H}$, the equations

$$x - A^*y = e \quad \text{and} \quad Ax + y = f$$

have exactly one solution (x, y). If A is densely defined and symmetric, then its closure $\mathbf{A}$ satisfies $\mathbf{A} \subset A^*$ since A^* is closed. There are other important consequences of Proposition 7.4.14 as well.

7.4.15 Corollary. *For A densely defined and closeable, we have*

(i) $\mathbf{A} = A^{**}$, *and*

(ii) $\mathbf{A}^* = A^*$.

Proof. (i) Note that

$$\Gamma_{A^{**}} = -J\{(\Gamma_{A^*})^{\perp}\} = -(J(\Gamma_{A^*}))^{\perp}$$

since J is an isometry. But

$$-(J(\Gamma_{A^*}))^{\perp} = -(J^2\Gamma_A^{\perp})^{\perp} = \Gamma_A^{\perp\perp} = \mathrm{cl}(\Gamma_A) = \Gamma_{\mathbf{A}}.$$

(ii) follows since $\Gamma_{\mathbf{A}}^{\perp} = \mathrm{cl}(\Gamma_A^{\perp})$. ■

Suppose $A : D_A \subset \mathbf{H} \to \mathbf{H}$ is one-to-one. Then we get an operator A^{-1} defined on the range of A. In terms of graphs:

$$\Gamma_{A^{-1}} = K(\Gamma_A),$$

where $K(x, y) = (y, x)$; note that $K^2 = I$, K is an isometry and $KJ = -JK$. It follows for example that *if A is self-adjoint, so is A^{-1},* since

$$\Gamma_{(A^{-1})^*} = -J(\Gamma_{A^{-1}}^{\perp}) = -J(K\Gamma_A^{\perp}) = KJ\Gamma_A^{\perp} = K\Gamma_{(A^*)} = \Gamma_{A^{*-1}}.$$

Next we consider possible self-adjoint extensions of a symmetric operator.

7.4.16 Proposition. *Let A be a symmetric densely defined operator on $\mathbf{H}$ and $\mathbf{A}$ its closure. The following are equivalent*:

(i) *A is essentially self-adjoint.*

(ii) *A^* is self-adjoint.*

(iii) *$A^{**} \supset A^*$.*

(iv) *A has exactly one self-adjoint extension.*

(v) *$\mathbf{A} = A^*$.*

Proof. By definition, (i) means $\mathbf{A}^* = \mathbf{A}$. But we know $\mathbf{A}^* = A^*$ and $\mathbf{A} = A^{**}$ by Corollary 7.4.15. Thus (i), (ii), (v) are equivalent. These imply (iii). Also (iii) implies (ii) since $A \subset \mathbf{A} \subset A^* \subset A^{**} = \mathbf{A}$ and so $A^* = A^{**}$. To prove (iv) is implied let Y be any self-adjoint extension of A. Since Y is closed, $Y \supset \mathbf{A}$. But $\mathbf{A} = A^*$ so Y extends the self-adjoint operator A^*; that is, $Y \supset A$. Taking adjoints, $A^* = A \supset Y^* = Y$ so $Y = A$.

To prove that (iv) implies the others is a bit more complicated. We shall in fact give a more general result in Proposition 7.4.18 below. First we need some notation. Let

$$D_+ = \mathrm{range}(A + iI)^{\perp} \subset \mathbf{H} \quad \text{and} \quad D_- = \mathrm{range}(A - iI)^{\perp} \subset \mathbf{H}$$

called the ***positive*** and ***negative defect spaces***. Using the argument in Proposition 7.4.12 it is easy to check that

$$D_+ = \{\, x \in D_{A^*} \mid A^*x = ix \,\} \quad \text{and} \quad D_- = \{\, x \in D_{A^*} \mid A^*x = -ix \,\}.$$ ■

7.4.17 Lemma. *Using the graph norm on D_{A^*}, we have the orthogonal direct sum*

$$D_{A^*} = D_{\mathbf{A}} \oplus D_+ \oplus D_-.$$

Proof. Since D_+, D_- are closed in $\mathbf{H}$ they are closed in D_{A^*}. Also $D_{\mathbf{A}} \subset D_{A^*}$ is closed since A^* is an extension of A and hence of $\mathbf{A}$. It is easy to see that the indicated spaces are orthogonal. For example let $x \in D_{\mathbf{A}}$ and $y \in D_-$. Then using the inner product

$$\langle\!\langle x, y \rangle\!\rangle = \langle x, y \rangle + \langle A^*x, A^*y \rangle$$

gives

$$\langle\!\langle x, y \rangle\!\rangle = \langle x, y \rangle + \langle A^*x, -iy \rangle = \langle x, y \rangle - i\langle A^*x, y \rangle.$$

Since $x \in D_{\mathbf{A}} = D_{A^*}$, by Proposition 7.4.16(v), we get

$$\langle\!\langle x, y \rangle\!\rangle = \langle x, y \rangle - i\langle x, A^*y \rangle = \langle x, y \rangle - \langle x, y \rangle = 0.$$

To see that $D_{A^*} = D_{\mathbf{A}} \oplus D_+ \oplus D_-$ it suffices to show that the orthogonal complement of $D_{\mathbf{A}} \oplus D_+ \oplus D_-$ is zero. Let $u \in (D_{\mathbf{A}} \oplus D_+ \oplus D_-)^\perp$, so

$$\langle\!\langle u, x \rangle\!\rangle = \langle\!\langle u, y \rangle\!\rangle = \langle\!\langle u, z \rangle\!\rangle = 0$$

for all $x \in D_{\mathbf{A}}$, $y \in D_+$, $z \in D_-$. From $\langle\!\langle u, x \rangle\!\rangle = 0$ we get

$$\langle u, x \rangle + \langle A^*u, A^*x \rangle = 0$$

that is, or $A^*u \in D_{A^*}$ and $A^*A^*u = -u$. It follows that $(I - iA^*)u \in D_+$. But from $\langle\!\langle u, y \rangle\!\rangle = 0$ we have $\langle (I - iA^*)u, y \rangle = 0$ and so $(I - iA^*)u = 0$. Hence $u \in D_-$. Taking $z = u$ gives $u = 0$. ■

7.4.18 Proposition. *The self-adjoint extensions of a symmetric densely defined operator A (if any) are obtained as follows. Let $T : D_+ \to D_-$ be an isometry mapping D_+ onto D_- and let $\Gamma_T \subset D_+ \oplus D_-$ be its graph. Then the restriction of A^* to $D_{\mathbf{A}} \oplus \Gamma_T$ is a self-adjoint extension of A.*

Thus, A has self-adjoint extensions iff its defect indices $(\dim D_+, \dim D_-)$ are equal and these extensions are in one-to-one correspondence with all isometries of D_+ onto D_-. Assuming this result for a moment, we give the following.

Completion of Proof of Proposition 7.4.16. If there is only one self-adjoint extension it follows from proposition 7.4.18 that $D_+ = D_- = \{0\}$ so by proposition 7.4.13, A is essentially self-adjoint. ■

Proof of Proposition 7.4.18. Let B be a self-adjoint extension of $\mathbf{A}$. Then $\mathbf{A}^* = A^* \supset B$ so B is the restriction of A^* to some subspace containing $D_{\mathbf{A}}$. We want to show that these subspaces are of the form $D_{\mathbf{A}} \oplus \Gamma_T$ as stated.

Suppose first that $T : D_+ \to D_-$ is an isometry onto and let $\mathcal{A}$ be the restriction of A^* to $D_{\mathbf{A}} \oplus \Gamma_T$. First of all, one proves that $\mathcal{A}$ is symmetric: that is, for $u, x \in D_{\mathbf{A}}$ and $v, y \in D_+$ that

$$\langle Ax + A^*y + A^*Ty, u + v + Tv \rangle = \langle x + y + Ty, Au + A^*v + A^*Tv \rangle.$$

This is a straightforward computation using the definitions.

To show that $\mathcal{A}$ is self-adjoint, we show that $D_{\mathcal{A}^*} \subset D_{\mathcal{A}}$. If this does not hold there exists a nonzero $z \in D_{\mathcal{A}^*}$ such that either $\mathcal{A}^*z = iz$ or $\mathcal{A}^*z = -iz$. This follows from Lemma 7.4.17 applied to the operator $\mathcal{A}$. (Observe that $\mathcal{A}$ is a closed operator—this easily follows.) Now $\mathcal{A} \supset A$ so $A^* \supset \mathcal{A}^*$. Thus $z \in D_+$ or $z \in D_-$. Suppose $z \in D_+$. Then $z + Tz \in D_{\mathcal{A}}$ so as $\langle\!\langle D_{\mathcal{A}}, z \rangle\!\rangle = 0$, where $\langle\!\langle\, , \rangle\!\rangle$ denotes the inner product relative to $\mathcal{A}$,

$$0 = \langle\!\langle z + Tz, z \rangle\!\rangle = \langle\!\langle z, z \rangle\!\rangle + \langle\!\langle Tz, z \rangle\!\rangle = 2\langle z, z \rangle,$$

since $Tz \in D_-$. Hence $z = 0$. In a similar way one sees that if $z \in D_-$ then $z = 0$. Hence $\mathcal{A}$ is self-adjoint.

We will leave the details of the converse to the reader (they are similar to the foregoing). The idea is this: if $\mathcal{A}$ is restriction of A^* to a subspace $D_{\mathbf{A}} \oplus V$ for $V \subset D_+ \oplus D_-$ and $\mathcal{A}$ is symmetric, then V is the graph of a map $T : W \subset D_+ \to D_-$ and $\langle Tu, Tv \rangle = \langle u, v \rangle$, for a subspace $W \subset D_+$. Then self-adjointness of $\mathcal{A}$ implies that in fact $W = D_+$ and T is onto. ■

A convenient test for establishing the equality of the deficiency indices is to show that T commutes with a conjugation U; that is, an antilinear isometry $U : \mathbf{H} \to \mathbf{H}$ satisfying $U^2 = I$; antilinear means that

$$U(\alpha x) = \bar{\alpha} U x$$

for complex scalars α and

$$U(x + y) = Ux + Uy$$

for $x, y \in \mathbf{H}$. It is easy to see that U is the isometry required from D_+ to D_- (use $D_+ = \mathrm{range}(A + iI)^\perp$).

As a corollary, we obtain an important classical result of von Neumann: *Let* $\mathbf{H}$ *be* L^2 *of a measure space and let* A *be a (closed) symmetric operator that is real in the sense that it commutes with complex conjugation. Then* A *admits self-adjoint extensions.* (Another sufficient condition of a different nature, due to Friedrichs, is given below.) This result applies to many quantum mechanical operators. However, one is also interested in essential self-adjointness, so that the self-adjoint extension will be unique. Methods for proving this for specific operators in quantum mechanics are given in Kato [1951, 1976] and Reed and Simon [1974]. For corresponding questions in elasticity, see Marsden and Hughes [1983]. We now give some additional results that illustrate methods for handling self-adjoint operators.

7.4.19 Proposition. *Let* A *be a self-adjoint and* B *a bounded self-adjoint operator. Then* $A + B$ *(with domain* D_A*) is self-adjoint. If* A *is essentially self-adjoint on* D_A *then so is* $A + B$*.*

Proof. $A + B$ is certainly symmetric on D_A. Let $y \in D_{(A+B)^*}$ so that for all $x \in D_A$,

$$\langle (A + B)x, y \rangle = \langle x, (A + B)^* y \rangle.$$

The left side is

$$\langle Ax, y \rangle + \langle Bx, y \rangle = \langle Ax, y \rangle + \langle x, By \rangle$$

since B is everywhere defined. Thus

$$\langle Ax, y \rangle = \langle x, (A + B)^* y - By \rangle.$$

Hence $y \in D_{A^*} = D_A$ and

$$Ay = A^* y = (A + B)^* y - By.$$

Hence $y \in D_{A+B} = D_A$.

Let $\mathbf{A}$ be the closure of A. For the second part, it suffices to show that the closure of $A + B$ equals $\mathbf{A} + B$. But if $x \in D_{\mathbf{A}}$ there is a sequence $x_n \in D_A$ such that $x_n \to x$, and $Ax_n \to \mathbf{A}x$. Then $Bx_n \to Bx$ as B is bounded so x belongs to the domain of the closure of $A + B$. ■

In general, the sum of two self-adjoint operators need not be self-adjoint. (See Nelson [1959] and Chernoff [1974] for this and related examples.)

7.4.20 Proposition. *Let* A *be a symmetric operator. If the range of* A *is all of* $\mathbf{H}$ *then* A *is self-adjoint.*

Proof. We first observe that A is one-to-one. Indeed let $Ax = 0$. Then for any $y \in D_A$,

$$0 = \langle Ax, y \rangle = \langle x, Ay \rangle.$$

But A is onto and so $x = 0$. Thus A admits an everywhere defined inverse A^{-1}, which is therefore self-adjoint. Hence A is self-adjoint (we proved earlier than the inverse of a self-adjoint operator is self-adjoint). ■

We shall use these results to prove a theorem that typifies the kind of techniques one uses.

7.4.21 Proposition. *Let A be a symmetric operator on $\mathbf{H}$ and suppose $A \leq 0$; that is $\langle Ax, x\rangle \leq 0$ for $x \in D_A$. Suppose $I - A$ has dense range. Then A is essentially self-adjoint.*

Proof. Note that

$$\langle (I - A)u, u\rangle = \langle u, u\rangle - \langle Au, u\rangle \geq \|u\|^2$$

and so by the Schwarz inequality we have

$$\|(I - A)u\| \geq \|u\|.$$

It follows that the closure of $I - A$ which equals $I - \mathbf{A}$ has closed range, which by hypothesis must be all of $\mathbf{H}$. By Proposition 7.4.19 $I - A$ is self-djoint, so by Proposition 7.4.20 $\mathbf{A}$ is self-adjoint. ■

7.4.22 Corollary. *If A is self-adjoint and $A \leq 0$, then for any $\lambda > 0$, $\lambda - A$ is onto, $(\lambda - A)^{-1}$ exists and*

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda}. \tag{7.4.3}$$

Proof. As before, we have

$$\langle (\lambda - A)u, u\rangle \geq \lambda \|u\|^2,$$

which yields

$$\|(\lambda - A)u\| \geq \lambda \|u\|.$$

As A is closed, this implies that the range of $\lambda - A$ is closed. If we can show it is dense, the result will follow. Suppose y is orthogonal to the range

$$\langle (\lambda - A)u, y\rangle = 0 \quad \text{for all } u \in D_A.$$

This means that $(\lambda - A)^* y = 0$, or since A is self-adjoint, $y \in D_A$. Making the choice $u = y$ gives

$$0 = \langle (\lambda - A)y, y\rangle \geq \lambda \|y\|^2,$$

so $y = 0$. ■

Note that *an operator A has dense range iff A^* is one-to-one*; that is, $A^* w = 0$ implies $w = 0$.

For a given symmetric operator A, we considered the general problem of self-adjoint extensions of A and classified these in terms of the defect spaces. Now, under different hypotheses, we construct a special self-adjoint extension (even though A need not be essentially self-adjoint). This result is useful in many applications, including quantum mechanics.

A symmetric operator A on $\mathbf{H}$ is called ***lower semi-bounded*** if there is a constant $c \in \mathbb{R}$ such that $\langle Ax, x\rangle \geq c\|x\|^2$ for all $x \in D_A$. ***Upper semi-bounded*** is defined similarly. If A is either upper or lower ***semi-bounded*** then A is called semi-bounded. Observe that if A is positive or negative then A is semi-bounded.

As an example, let $A = -\nabla^2 + V$ where ∇^2 is the Laplacian and let V be a real valued continuous function and bounded below, say $V(x) \geq \alpha$. Let $\mathbf{H} = L^2(\mathbb{R}^n, \mathbb{C})$ and D_A the C^∞ functions with compact support. Then $-\nabla^2$ is positive, so

$$\langle Af, f\rangle = \langle -\nabla^2 f, f\rangle + \langle Vf, f\rangle \geq \alpha \langle f, f\rangle,$$

and thus A is semi-bounded.

We already know that this operator is real so has self-adjoint extensions by von Neumann's theorem. However, the self-adjoint extension constructed below (called the ***Friedrichs extension***) is "natural." Thus the actual construction is as important as the statement.

7.4.23 Theorem. *A semi-bounded symmetric (densely defined) operator admits a self-adjoint extension.*

Proof. After multiplying by -1 if necessary and replacing A by $A+(1-\alpha)I$ we can suppose $\langle Ax, x\rangle \geq \|x\|^2$. Consider the inner product on D_A given by $\langle\langle x, y\rangle\rangle = \langle Ax, y\rangle$. (Using symmetry of A and the preceding inequality one easily checks that this is an inner product.)

Let $\mathbf{H}^1$ be the completion of D_A in this inner product. Since the $\mathbf{H}^1$-norm is stronger than the $\mathbf{H}$-norm, we have $\mathbf{H}^1 \subset \mathbf{H}$ (i.e., the injection $D_A \subset \mathbf{H}$ extends uniquely to the completion).

Now let $\mathbf{H}^{-1}$ be the dual of $\mathbf{H}^1$. We have an injection of $\mathbf{H}$ into $\mathbf{H}^{-1}$ defined as follows: if y is fixed and $x \mapsto \langle x, y\rangle$ is a linear functional on $\mathbf{H}$, it is also continuous on $\mathbf{H}^1$ since

$$|\langle x, y\rangle| \leq \|x\|\,\|y\| \leq |||x|||\,||y||,$$

where $|||\cdot|||$ is the norm of $\mathbf{H}^1$. Thus $\mathbf{H}^1 \subset \mathbf{H} \subset \mathbf{H}^{-1}$.

Now the inner product on $\mathbf{H}^1$ defines an isomorphism $B : \mathbf{H}^1 \to \mathbf{H}^{-1}$. Let C be the operator with domain $D_C = \{\, x \in \mathbf{H}^1 \mid B(x) \in \mathbf{H}\,\}$, and $Cx = Bx$ for $x \in D_C$. Thus C is an extension of A. This will be the extension we sought. We shall prove that C is self-adjoint. By definition, C is surjective; in fact $C : D_C \to \mathbf{H}$ is a linear isomorphism. Thus by Proposition 7.4.20 it suffices to show that C is symmetric. Indeed for $x, y \in D_C$ we have, by definition,

$$\overline{\langle Cx, y\rangle} = \overline{\langle\langle x, y\rangle\rangle} = \langle\langle y, x\rangle\rangle = \langle Cy, x\rangle = \langle x, Cy\rangle. \qquad \blacksquare$$

The self-adjoint extension C can be alternatively described as follows. Let $\mathbf{H}^1$ be as before and let C be the restriction of A^* to $D_{A^*} \cap \mathbf{H}^1$. We leave the verification as an exercise.

Supplement 7.4B
Stone's Theorem[3]

Here we give a self-contained proof of Stone's theorem for unbounded self-adjoint operators A on a complex Hilbert space $\mathbf{H}$. This guarantees that the one-parameter group e^{itA} of unitary operators exists. In fact, there is a one-to-one correspondence between self-adjoint operators and continuous one-parameter unitary groups. A ***continuous one-parameter unitary group*** is a homomorphism $t \mapsto U_t$ from $\mathbb{R}$ to the group of unitary operators on $\mathbf{H}$, such that for each $x \in \mathbf{H}$ the map $t \mapsto U_t x$ is continuous. The ***infinitesimal generator*** A of U_t is defined by

$$iAx = \frac{d}{dt}U_t x\bigg|_{t=0} = \lim_{h\to 0}\frac{U_h(x) - x}{h},$$

its domain D consisting of those x for which the indicated limit exists. We insert the factor i for convenience; iA is often called the ***generator***.

7.4.24 Theorem (Stone's Theorem)**.** *Let U_t be a continuous one-parameter unitary group. Then the generator A of U_t is self-adjoint.* (*In particular, by Supplement* 7.4A, *it is closed and densely defined.*) *Conversely, let A be a given self-adjoint operator. Then there exists a unique one-parameter unitary group U_t whose generator is A.*

[3]This supplement was written in collaboration with P. Chernoff.

Before we begin the proof, let us note that if A is a *bounded* self-adjoint operator then one can form the series

$$U_t = e^{itA} = I + itA + \frac{1}{2!}(itA)^2 + \frac{1}{3!}(itA)^3 + \dots$$

which converges in the operator norm. It is straightforward to verify that U_t is a continuous one-parameter unitary group and that A is its generator. Because of this, one often writes e^{itA} for the unitary group whose generator is A even if A is unbounded. (In the context of the so-called "operational calculus" for self-adjoint operators, one can show that e^{itA} really *is* the result of applying the function $e^{it(\cdot)}$ to A; however, we shall not go into these matters here.)

Proof of Stone's Theorem (first half)**.** Let U_t be a given continuous unitary group. In a series of lemmas, we shall show that the generator A of U_t is self-adjoint. ■

7.4.25 Lemma. *The domain D of A is invariant under each U_t, and moreover $AU_tx = U_tAx$ for each $x \in D$.*

Proof. Suppose $x \in D$. Then

$$\frac{1}{h}(U_hU_tx - U_tx) = U_t\left(\frac{1}{h}(U_hx - h)\right).$$

which converges to $U_t(iAx) = iU_tAx$ as $h \to 0$. The lemma follows by the definition of A. ■

7.4.26 Corollary. *The operator A is closed.*

Proof. If $x \in D$ then, by Lemma 7.4.25

$$\frac{d}{dt}U_tx = iAU_tx = iU_tAx.$$

Hence,

$$U_tx = x + i\int_0^t U_\tau Ax\, d\tau. \tag{7.4.4}$$

Now suppose that $x_n \in D, x_n \to x$, and $Ax_n \to y$. Then we have, by equation (7.4.4),

$$U_tx = \lim_{n\to\infty} U_tx_n = \lim_{n\to\infty}\left\{x_n + i\int_0^t U_\tau Ax_n d\tau\right\}.$$

Thus,

$$U_tx = x + i\int_0^t U_\tau y\, d\tau. \tag{7.4.5}$$

(Here we have taken the limit under the integral sign because the convergence is uniform; indeed

$$\|U_\tau Ax_n - U_\tau y\| = \|Ax_n - y\| \to 0$$

independent of $\tau \in [0, t]$.) Then, by equation (7.4.5),

$$\left.\frac{d}{dt}U_tx\right|_{t=0} = iy.$$

Hence $x \in D$ and $y = Ax$. Thus A is closed. ■

7.4.27 Lemma. *A is densely defined.*

Proof. Let $x \in H$, and let φ be a C^∞ function with compact support on $\mathbb{R}$. Define

$$x_\varphi = \int_{-\infty}^{\infty} \varphi(t) U_t x \, dt.$$

We shall show that x_φ is in D, and that $x = \lim_{n\to\infty} x_{\varphi_n}$ for a suitable sequence $\{\varphi_n\}$. To take the latter point first, let $\varphi_n(t)$ be nonnegative, zero outside the interval $[0, 1/n]$, and such that $\varphi_n(t)$ has integral 1. By continuity, if $\epsilon > 0$ is given, one can find N so large that $\|U_t x - x\| < \epsilon$ if $|t| < 1/N$. Suppose that $n > N$. Then

$$\begin{aligned} \|x_{\varphi_n} - x\| &= \left\| \int_{-\infty}^{\infty} \varphi_n(t)(U_t x - x) dt \right\| = \left\| \int_0^{1/n} \varphi_n(t)(U_t x - x) dt \right\| \\ &\le \int_0^{1/n} \varphi_n(t) \|U_t x - x\| dt \le \epsilon \int_0^{1/n} \varphi_n(t) dt = \epsilon. \end{aligned}$$

Finally, we show that $x_\varphi \in D$; moreover, we shall show that $iAx_\varphi = -x_{\varphi'}$. Indeed,

$$\begin{aligned} -\int_0^t U_\tau x_{\varphi'} d\tau &= \int_0^t U_\tau d\tau \int_{-\infty}^{\infty} \varphi'(\sigma) U_\sigma d\sigma \\ &= -\int_{-\infty}^{\infty} d\sigma \cdot \varphi'(\sigma) \cdot \int_0^t U_{\tau+\sigma} x \, d\tau \\ &= -\int_{-\infty}^{\infty} d\sigma \cdot \varphi'(\sigma) \cdot \int_\sigma^{\sigma+t} U_\tau x \, d\tau. \end{aligned}$$

Integrating by parts and using the fact that φ has compact support, we get

$$-\int_0^t U_\tau x_{\varphi'} d\tau = \int_{-\infty}^{\infty} (U_{\sigma+t} x - U_\sigma x)\varphi(\sigma) d\sigma = (U_t - I) \int_{-\infty}^{\infty} U_\sigma x_\varphi(\sigma) d\sigma.$$

That is,

$$-\int_0^t U_\tau x_{\varphi'} d\tau = U_t x_\varphi - x_\varphi.$$

from which the assertion follows. ■

Thus far we have made no significant use of the unitarity of U_t. We shall now do so.

7.4.28 Lemma. *A is symmetric.*

Proof. Take $x, y \in D$. Then we have

$$\begin{aligned} \langle Ax, y \rangle &= \frac{1}{i} \frac{d}{dt} \langle U_t x, y \rangle \Big|_{t=0} = \frac{1}{i} \frac{d}{dt} \langle x, U_t^* y \rangle \Big|_{t=0} \\ &= \frac{1}{i} \frac{d}{dt} \langle x, U_{-t} y \rangle \Big|_{t=0} = -\frac{1}{i} \frac{d}{dt} \langle x, U_t y \rangle \Big|_{t=0} \\ &= -\frac{1}{i} \langle x, iAy \rangle = \langle x, Ay \rangle. \end{aligned}$$

■

To complete the proof that A is self-adjoint, let $y \in D^*$ and $x \in D$. By Lemmas 7.4.25, and 7.4.28,

$$\begin{aligned}\langle U_t y, x\rangle = \langle y, U_{-t}x\rangle = \langle y, x\rangle + \left\langle y, i\int_0^{-t} U_\tau Ax\, d\tau \right\rangle \\ = \langle y, x\rangle - i\int_0^{-t}\langle y, U_\tau Ax\rangle d\tau = \langle y, x\rangle - i\int_0^{-t}\langle y, AU_\tau x\rangle d\tau \\ = \langle y, x\rangle - i\int_0^{-t}\langle U_{-\tau}A^* y, x\rangle d\tau = \langle y, x\rangle + i\int_0^{-t}\langle U_\tau A^* y, x\rangle d\tau \\ = \left\langle y + i\int_0^{t} U_\tau A^* y\, d\tau, x\right\rangle.\end{aligned}$$

Because D is dense, it follows that

$$U_t y = y + i\int_0^t U_\tau A^* y\, d\tau.$$

Hence, differentiating, we see that $y \in D$ and $A^* y = Ay$. Thus $A = A^*$.

Proof of Stone's theorem (second half)**.** We are now given a self-adjoint operator A. We shall construct a continuous unitary group U_t whose generator is A.

7.4.29 Lemma. *If $\lambda > 0$, then $I + \lambda A^2 : D_{A^2} \to H$ is bijective,*

$$(I + \lambda A^2)^{-1} : H \to D_{A^2}$$

is bounded by 1, and D_{A^2}, the domain of A^2, is dense.

Proof. If A is self-adjoint, so is $\sqrt{\lambda}A$. It is therefore enough to establish the lemma for $\lambda = 1$. First we establish surjectivity.

By Proposition 7.4.14 and Lemma 7.4.26, if $z \in H$ is given there exists a unique solution (x, y) to the equations

$$x - Ay = 0, \quad Ax + y = z.$$

From the first equation, $x = Ay$. The second equation then yields $A^2 y + y = z$, so $I + A^2$ is surjective.

For $x \in D_{A^2}$, note that

$$\langle (I + A^2)x, x\rangle \geq \|x\|^2, \quad \text{so} \quad \|(I + A^2)x\| \geq \|x\|.$$

Thus $I + A^2$ is one-to-one and $\|(I + A^2)^{-1}\| \leq 1$. Now suppose that u is orthogonal to D_{A^2}. We can find a v such that $u = v + A^2 v$. Then

$$0 = \langle u, v\rangle = \langle v + A^2 v, v\rangle = \|v\|^2 + \|Av\|^2,$$

whence $v = 0$ and therefore $u = 0$. Consequently D_{A^2} is dense in H. ■

For $\lambda > 0$, define an operator A_λ by $A_\lambda = A(I + \lambda A^2)^{-1}$. Note that A_λ is defined on all of H because if $x \in H$ then $(I + \lambda A^2)^{-1}x \in D_{A^2} \subset D$, so $A(I + \lambda A^2)^{-1}x$ makes sense.

7.4.30 Lemma. *A_λ is a bounded self-adjoint operator. Also A_λ and A_μ commute for all $\lambda, \mu > 0$.*

Proof. Pick $x \in H$. Then by Lemma 7.4.29

$$\begin{aligned}\lambda\|A_\lambda x\|^2 &= \langle \lambda A(I+\lambda A^2)^{-1}x, A(I+\lambda A^2)^{-1}x\rangle \\ &= \langle \lambda A^2(I+\lambda A^2)^{-1}x, (I+\lambda A^2)^{-1}x\rangle \\ &\le \langle (I+\lambda A^2)(I+\lambda A^2)^{-1}x, (I+\lambda A^2)^{-1}x\rangle \\ &\le \|(I+\lambda A^2)^{-1}x\|^2 \\ &\le \|x\|^2,\end{aligned}$$

so $\|A_\lambda\| \le 1/\sqrt{\lambda}$, and thus A_λ is bounded.

We now show that A_λ is self-adjoint. First we shall show that if $x \in D$, then

$$A_\lambda x = (I+\lambda A^2)^{-1}Ax.$$

Indeed, if $x \in D$ we have $A_\lambda x \in D_{A^2}$ by Lemma 7.4.29, so

$$\begin{aligned}(I+\lambda A^2)A_\lambda x &= (I+\lambda A^2)A(I+\lambda A^2)^{-1}x \\ &= A(I+\lambda A^2)(I+\lambda A^2)^{-1}x \\ &= Ax.\end{aligned}$$

Now suppose $x \in D$ and y is arbitrary. Then

$$\begin{aligned}\langle A_\lambda x, y\rangle &= \langle (I+\lambda A^2)^{-1}Ax, y\rangle \\ &= \langle (I+\lambda A^2)^{-1}Ax, (I+\lambda A^2)(I+\lambda A^2)^{-1}y\rangle \\ &= \langle Ax, (I+\lambda A^2)^{-1}y\rangle = \langle x, A_\lambda y\rangle.\end{aligned}$$

Because D is dense and A_λ bounded, this relation must hold for all $x \in H$. Hence A_λ is self-adjoint. The proof that $A_\lambda A_\mu = A_\mu A_\lambda$ is a calculation that we leave to the reader. ■

Since A_λ is bounded, we can form the continuous one-parameter unitary groups $U_t^\lambda = e^{itA_\lambda}$, $\lambda > 0$ using power series or the results of §4.1. Since A_λ and A_μ commute, it follows that U_s^λ and U_t^μ commute for every s and t.

7.4.31 Lemma. *If $x \in D$ then $\lim_{\lambda\to 0} A_\lambda x = Ax$.*

Proof. If $x \in D$ we have

$$A_\lambda x - Ax = (I+\lambda A^2)^{-1}Ax - Ax = -\lambda A^2(I+\lambda A^2)^{-1}Ax.$$

It is therefore enough to show that for every $y \in H$, $\lambda A^2(I+\lambda A^2)^{-1}y \to 0$. From the inequality

$$\|(I+\lambda A^2)y\|^2 \ge \|\lambda A^2 y\|^2,$$

valid for $\lambda \ge 0$, we see that $\|\lambda A^2(I+\lambda A^2)^{-1}\| \le 1$. Thus it is even enough to show the preceding equality for all y in some dense subspace of H. Suppose $y \in D_{A^2}$, which is dense by Lemma 7.4.29. Then

$$\|\lambda A^2(I+\lambda A^2)^{-1}y\| = \lambda\|(I+\lambda A^2)^{-1}A^2y\| \le \lambda\|A^2y\|,$$

which indeed goes to zero with λ. ■

7.4.32 Lemma. *For each $x \in H$, $\lim_{\lambda\to 0} U_t^\lambda x$ exists. If we call the limit $U_t x$, then $\{U_t\}$ is a continuous one-parameter unitary group.*

Proof. We have

$$\begin{aligned} U_t^\lambda x - U_t^\mu x &= \int_0^t \frac{d}{d\tau}(U_\tau^\lambda U_{t-\tau}^\mu)x\,d\tau \\ &= \int_0^t \left[iA_\lambda U_\tau^\lambda U_{t-\tau}^\mu x - U_\tau^\lambda iA_u U_{t-\tau}^\mu x\right] d\tau \\ &= i\int_0^t U_\tau^\lambda U_{t-\tau}^\mu (A_\lambda x - A_\mu x) d\tau, \end{aligned}$$

whence

$$\|U_t^\lambda x - U_t^\mu x\| \le |t|\,\|A_\lambda x - A_\mu x\|. \tag{7.4.6}$$

Now suppose that $x \in D$. Then by Lemma 7.4.31, $A_\lambda x \to Ax$, so that

$$\|A_\lambda x - A_\mu x\| \to 0 \quad \text{as} \quad \lambda, \mu \to 0.$$

Because of equation (7.4.6) it follows that $\{U_t^\lambda \xi\}_{\lambda>0}$ is uniformly Cauchy as $\lambda \to 0$ on every compact t-interval. It follows that $\lim_{\lambda\to0} U_t^\lambda x = U_t x$ exists and is a continuous function of t. Moreover, since D is dense and all of the U_t^λ have norm 1, an easy approximation argument shows that the preceding conclusion holds even if $x \notin D$.

It is obvious that each U_t is a linear operator. Furthermore,

$$\langle U_t x, U_t y\rangle = \lim_{\lambda\to0}\langle U_t^\lambda x, U_t^\lambda y\rangle = \lim_{\lambda\to0}\langle x, y\rangle = \langle x, y\rangle$$

so U_t is isometric. Trivially, $U_0 = I$. Finally,

$$\begin{aligned} \langle U_s U_t, x, y\rangle &= \lim_{\lambda\to0}\langle U_s^\lambda U_t, x, y\rangle = \lim_{\lambda\to0}\langle U_t x, U_{-s}^\lambda y\rangle \\ &= \lim_{\lambda\to0}\langle U_t^\lambda x, U_{-s}^\lambda y\rangle = \lim_{\lambda\to0}\langle U_{s+t}^\lambda x, y\rangle = \langle U_{s+t} x, y\rangle, \end{aligned}$$

so $U_s U_t = U_{s+t}$.

Thus, U_s has an inverse, namely U_{-s}, and so U_s is unitary. ■

7.4.33 Lemma. *If $x \in D$, then*

$$\lim_{t\to0}\frac{U_t x - x}{t} = iAx.$$

Proof. We have

$$U_t^\lambda x - x = i\int_0^t U_\tau^\lambda A_\lambda x\,d\tau. \tag{7.4.7}$$

Now

$$U_\tau^\lambda A_\lambda x - U_\tau Ax = U_\tau^\lambda(A_\lambda x - Ax) + U_\tau^\lambda Ax - U_\tau Ax \to 0$$

uniformly for $\tau \in [0, t]$ as $\lambda \to 0$. Thus letting $\lambda \to 0$ in equation (**??**), we get

$$U_t x - x = i\int_0^t U_\tau Ax\,d\tau \tag{7.4.8}$$

for all $x \in D$. The lemma follows directly from equation (7.4.8). ■

7.4.34 Lemma. *If*

$$\lim_{t\to 0} \frac{U_t x - x}{t} = iw$$

exists, then $x \in D$.

Proof. It suffices to show that $x \in D^*$, the domain of A^*, since $D = D^*$. Let $y \in D^*$. Then by Lemma 7.4.33,

$$\langle x, iAy\rangle = \lim_{t\to 0} \left\langle x, \frac{U_{-t}y - y}{-t}\right\rangle = -\lim_{t\to 0}\left\langle \frac{U_t x - x}{t}, y\right\rangle = -\langle iw, y\rangle.$$

Therefore, $\langle x, Ay\rangle = \langle w, y\rangle$. Thus $x \in D^*$ and so as A is self-adjoint, $x \in D$. ■

Let us finally prove uniqueness. Let $c(t)$ be a differentiable curve in H such that $c(t) \in D$ and $c'(t) = iA(c(t))$. We claim that $c(t) = U_t c(0)$. Indeed consider, $h(t) = U_{-t}c(t)$. Then

$$\begin{aligned}\|h(t+\tau) - h(t)\| &= \|U_{-t-\tau}c(t+\tau) - U_{-t-\tau}U_t c(t)\| \\ &= \|c(t+\tau) - U_\tau c(t)\| \\ &= \|(c(t+\tau) - c(t)) - (U_\tau c(t) - c(t))\|.\end{aligned}$$

Hence

$$\frac{h(t+\tau) - h(t)}{\tau} \to 0$$

as $\tau \to 0$, so h is constant. But $h(t) = h(0)$ means $c(t) = U_t c(0)$. ■

From the proof of Stone's theorem, one can deduce the following Laplace transform expression for the resolvent, which we give for the sake of completeness.

7.4.35 Corollary. *Let* $\operatorname{Re}\lambda > 0$. *Then for all* $x \in H$,

$$(\lambda - iA)^{-1}x = \int_0^\infty e^{-\lambda t} U_t x dt.$$

Proof. The foregoing is formally an identity if one thinks of U_t as e^{itA}. Indeed, if A is *bounded* then it follows just by manipulation of the power series: One has $e^{-\lambda t}e^{itA} = e^{-t(\lambda - iA)}$, as one can see by expanding both sides. Next note that

$$\int_0^R e^{-t(\lambda - iA)} x dt = (\lambda - iA)^{-1}\left[x - e^{-R(\lambda - iA)}x\right],$$

as is seen by integrating the series term by term. Letting $R \to \infty$, one gets the result.

Now for arbitrary A we know that $U_t x = \lim_{\mu\to 0} U_t^\mu x$, uniformly on bounded intervals. It follows that

$$\int_0^\infty e^{-\lambda t} U_t x\, dt = \lim_{\mu\to 0}\int_0^\infty e^{-\lambda t} U_t^\mu x\, dt = \lim_{\mu\to 0}(\lambda - iA_\mu)^{-1}x.$$

It remains to show that this limit is $(\lambda - iA)^{-1}x$. Now

$$(\lambda - iA)^{-1}x - (\lambda - iA_\mu)^{-1}x = (\lambda - iA_\mu)^{-1}[(\lambda - iA_\mu)(\lambda - iA)^{-1}x - x].$$

But $(\lambda - iA)^{-1}x \in D$ (see Proposition 7.4.12), so by Lemma 7.4.31,

$$(\lambda - iA_\mu)(\lambda - iA)^{-1}x \to (\lambda - iA)(\lambda - iA)^{-1}x = x \quad \text{as } \mu \to 0.$$

Because $\|(\lambda - iA_\mu)^{-1}\| \leq |\operatorname{Re}\lambda|^{-1}$ it follows that

$$\|(\lambda - iA)^{-1}x - (\lambda - iA_\mu)^{-1}x\| \to 0. \qquad \blacksquare$$

In closing, we mention that many of the results proved have generalizations to continuous one-parameter groups or semi-groups of linear operators in Banach spaces (or on locally convex spaces). The central result, due to Hille and Yosida, characterizes generators of semi-groups. Our proof of Stone's theorem is based on methods that can be used in the more general context. Expositions of this more general context are found in, for example, Kato [1976] and Marsden and Hughes [1983, Chapter 6].

Exercises

(Exercises 7.4-1–7.4-3 form a unit.)

◇ **7.4-1.** Given a manifold M, show that the space of half-densities on M carries a natural inner product. Let its completion be denoted $\mathfrak{H}(M)$, which is called the ***intrinsic Hilbert space*** of M. If μ is a density on M, define a bijection of $L^2(M, \mu)$ with $\mathfrak{H}(M)$ by $f \mapsto f\sqrt{\mu}$. Show that it is an isometry.

◇ **7.4-2.** If F_t is the (local) flow of a smooth vector field X, show that F_t induces a flow of isometries on $\mathfrak{H}(M)$. (Make no assumption that X is divergence-free.) Show that the generator $iX' = \pounds_X$ of the induced flow on $\mathfrak{H}(M)$ is

$$iX'(f\sqrt{\mu}) = \left(X[f] + \frac{1}{2}(\operatorname{div}_\mu X)f\right)\sqrt{\mu}$$

and check directly that X' is a symmetric operator on the space of half-densities with compact support.

◇ **7.4-3.** Prove that F_t is complete a.e. if and only if X' is essentially self-adjoint.

◇ **7.4-4.** Consider the flow in $\mathbb{R}^2$ associated with a reflecting particle: for $t > 0$, set

$$F_t(q, p) = q + tp \quad \text{if } q > 0,\ q + tp > 0$$

and

$$F_t(q, p) = -q - tp \quad \text{if } q > 0,\ q + tp < 0$$

and set

$$F_t(-q, p) = -F_t(q, p) \quad \text{and} \quad F_{-t} = F_t^{-1}.$$

What is the generator of the induced unitary flow? Is it essentially self-adjoint on the C^∞ functions with compact support away from the line $q = 0$?

◇ **7.4-5.** Let M be an oriented Riemannian manifold and $L^2(\bigwedge^k(M))$ the space of L^2 k-forms with inner product $\langle \alpha, \beta \rangle = \int \alpha \wedge * \beta$. If X is a Killing field on M with a complete flow F_t, show that $i\pounds_X$ is a self-adjoint operator on $L^2(\bigwedge^k(M))$.

7.5 Introduction to Hodge–deRham Theory

Recall that a k-form α is called ***closed*** if $\mathbf{d}\alpha = 0$ and ***exact*** if $\alpha = \mathbf{d}\beta$ for some $k-1$ form β. Since $\mathbf{d}^2 = 0$, every exact form is closed, but the converse need not hold. Let

$$H^k(M) = \frac{\ker \mathbf{d}^k}{\operatorname{range} \mathbf{d}^{k-1}},$$

(where $\mathbf{d}^k$ denotes the exterior derivative on k-forms), and call it the k-th ***deRham cohomology group of*** M. (The group structure here is that if a real vector space.) The celebrated ***deRham theorem*** states that for a finite-dimensional compact manifold, these groups are isomorphic to the singular cohomology groups (with real coefficients) defined in algebraic topology; the isomorphism is given by integration. For proofs, see Singer and Thorpe [1976] or Warner [1983]. The original books of Hodge [1952] and deRham [1955] (translated as deRham [1984]) remain excellent sources of information as well. A special but important case of the deRham theorem is proved in Supplement 7.5B.

The scope of this section is to informally discuss the Hodge decomposition theory based on differential operators and to explain how it is related to the deRham cohomology groups. In addition, some topological applications of the theory are given, such as the Brouwer fixed-point theorem, and the degree of a map is defined. In the sequel, M will denote a compact oriented Riemannian manifold, and δ the codifferential operator. At first we assume M has no boundary. Later we will discuss the case in which M has a boundary.

7.5.1 Definition. *The* ***Laplace–deRham operator***

$$\Delta : \Omega^k(M) \to \Omega^k(M)$$

is defined by

$$\Delta = \mathbf{d}\delta + \delta\mathbf{d}.$$

A form for which $\Delta\alpha = 0$ is called ***harmonic***. *Let*

$$\mathcal{H}^k = \left\{ \alpha \in \Omega^k(M) \mid \Delta\alpha = 0 \right\}$$

denote the vector space of harmonic k-forms.

If $f \in \mathcal{H}^0(M)$, then

$$\Delta f = \mathbf{d}\delta f + \delta\mathbf{d} f = \delta\mathbf{d} f = -\operatorname{div}\operatorname{grad} f,$$

so $\Delta f = -\nabla^2 f$, where ∇^2 is the Laplace–Beltrami operator. This minus sign can be a source of confusion and one has to be careful.

Recall that the L^2-inner product in $\Omega^k(M)$ is defined by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$$

and that $\mathbf{d}$ and δ are adjoints with respect to this inner product. That is, $\langle \mathbf{d}\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$ for all $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$. Thus it follows that for $\alpha, \beta \in \Omega^k(M)$, we have

$$\begin{aligned}\langle \Delta\alpha, \beta \rangle &= \langle \mathbf{d}\delta\alpha, \beta \rangle + \langle \delta\mathbf{d}\alpha, \beta \rangle = \langle \delta\alpha, \delta\beta \rangle + \langle \mathbf{d}\alpha, \mathbf{d}\beta \rangle \\ &= \langle \alpha, \mathbf{d}\delta\beta \rangle + \langle \alpha, \delta\mathbf{d}\beta \rangle = \langle \alpha, \Delta\beta \rangle,\end{aligned}$$

and thus Δ is symmetric. This computation also shows that $\langle \Delta\alpha, \alpha \rangle \geq 0$ for all $\alpha \in \Omega^k(M)$.

7.5.2 Proposition. *Let M be a compact boundaryless oriented Riemannian manifold and $\alpha \in \Omega^k(M)$. Then $\Delta\alpha = 0$ iff $\delta\alpha = 0$ and $\mathbf{d}\alpha = 0$.*

Proof. It is obvious from the expression

$$\Delta\alpha = \mathbf{d}\delta\alpha + \delta\mathbf{d}\alpha$$

that if $\mathbf{d}\alpha = 0$ and $\delta\alpha = 0$, then $\Delta\alpha = 0$. Conversely, the previous computation shows that

$$0 = \langle \Delta\alpha, \alpha \rangle = \langle \mathbf{d}\alpha, \mathbf{d}\alpha \rangle + \langle \delta\alpha, \delta\alpha \rangle,$$

so the result follows. ■

7.5.3 Theorem (The Hodge Decomposition Theorem)**.** *Let M be a compact, boundaryless, oriented, Riemannian manifold and let $\omega \in \Omega^k(M)$. Then there is an $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^{k+1}(M)$, and $\gamma \in \Omega^k(M)$ such that*

$$\omega = \mathbf{d}\alpha + \delta\beta + \gamma,$$

where $\Delta(\gamma) = 0$. Furthermore, $\mathbf{d}\alpha$, $\delta\beta$, and γ are mutually L^2 orthogonal and so are uniquely determined. That is,

$$\Omega^k(M) = \mathbf{d}\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M) \oplus \mathcal{H}^k. \tag{7.5.1}$$

We can easily check that the spaces in the Hodge decomposition are orthogonal. For example, $\mathbf{d}\Omega^{k-1}(M)$ and $\delta\Omega^{k+1}(M)$ are orthogonal since

$$\langle \mathbf{d}\alpha, \delta\beta \rangle = \langle \mathbf{dd}\alpha, \beta \rangle = 0,$$

δ being the adjoint of $\mathbf{d}$ and $\mathbf{d}^2 = 0$.

The basic idea behind the proof of the Hodge theorem can be abstracted as follows. We consider a linear operator T on a Hilbert space E and assume that $T^2 = 0$. In our case $T = \mathbf{d}$ and E is the space of L^2 forms. (We ignore the fact that T is only densely defined.) Let T^* be the adjoint of T. Let

$$\mathcal{H} = \{\, x \in E \mid Tx = 0 \text{ and } T^*x = 0 \,\}.$$

We assert that

$$E = \operatorname{cl}(\operatorname{range} T) \oplus \operatorname{cl}(\operatorname{range} T^*) \oplus \mathcal{H} \tag{7.5.2}$$

which, apart from technical points on understanding the closures, is the essential content of the Hodge decomposition. To prove equation (7.5.2), note that the ranges of T and T^* are orthogonal because

$$\langle Tx, T^*y \rangle = \langle Tx^2, y \rangle = 0.$$

If $\mathcal{C}$ denotes the orthogonal complement of $\operatorname{cl}(\operatorname{range} T) \oplus \operatorname{cl}(\operatorname{range} T^*)$, then $\mathcal{H} \subset \mathcal{C}$. If $x \in \mathcal{C}$ then $\langle Ty, x \rangle = 0$ for all y implies $T^*x = 0$. Similarly, $Tx = 0$, so $\mathcal{C} \subset \mathcal{H}$ and hence $C = \mathcal{H}$.

The complete proof of the Hodge theorem requires elliptic estimates and may be found in Morrey [1966]. For more elementary expositions, consult Flanders [1963] and Warner [1983].

7.5.4 Corollary. *Let $\mathcal{H}^k$ denote the space of harmonic k-forms. Then the vector spaces $\mathcal{H}^k$ and H^k $(= \ker \mathbf{d}^k / \operatorname{range} \mathbf{d}^{k-1})$ are isomorphic.*

Proof. Map $\mathcal{H}^k \to \ker \mathbf{d}^k$ by inclusion and then to H^k by projection. We need to show that this map is an isomorphism. Suppose $\gamma \in \mathcal{H}^k$ and $[\gamma] = 0$ where $[\gamma] \in H^k$ is the class of γ. But $[\gamma] = 0$ means that γ is exact; $\gamma = \mathbf{d}\beta$. But since $\delta\gamma = 0$, γ is orthogonal to $\mathbf{d}\beta$; that is, γ is orthogonal to itself, so $\gamma = 0$. Thus the map $\gamma \mapsto [\gamma]$ is one-to-one. Next let $[\omega] \in \mathcal{H}^k$. We can, by the Hodge theorem, decompose $\omega = \mathbf{d}\alpha + \delta\beta + \gamma$, where $\gamma \in \mathcal{H}^k$. Since $\mathbf{d}\omega = 0$, $\mathbf{d}\delta\beta = 0$, so $0 = \langle \beta, \mathbf{d}\delta\beta \rangle = \langle \delta\beta, \delta\beta \rangle$, so $\delta\beta = 0$. Thus, $\omega = \mathbf{d}\alpha + \gamma$ and hence $[\omega] = [\gamma]$, so the map $\gamma \mapsto [\gamma]$ is onto. ■

The space $\mathcal{H}^k \cong H^k$ is finite dimensional. Again the proof relies on elliptic theory (the kernel of an elliptic operator on a compact manifold is finite dimensional).

The Hodge theorem plays a fundamental role in incompressible hydrodynamics, as we shall see in §8.2. It allows the introduction of the pressure for a given fluid state. It has applications to many other areas of mathematical physics and engineering as well; see for example, Fischer and Marsden [1979] and Wyatt, Chua, and Oster [1978].

Below we shall state a generalization of the Hodge theorem for some decomposition theorems for general elliptic operators (rather than the special case of the Laplacian). However, we first pause to discuss what happens if a boundary is present. This theory was worked out by Kodaira [1949], Duff and Spencer [1952], and Morrey [1966, Chapter 7]. Differentiability across the boundary is very delicate, but important. Some of the best results in this regard are due to Morrey.

Note that $\mathbf{d}$ and δ may not be adjoints in this case, because boundary terms arise when we integrate by parts (see Exercise 7.5-5). Hence we must impose certain boundary conditions. Let $\alpha \in \Omega^k(M)$. Then α is called ***parallel*** or ***tangent*** to ∂M if ***the normal part***, defined by

$$n\alpha = i^*(*\alpha)$$

is zero where $i : \partial M \to M$ is the inclusion map. Analogously, α is ***perpendicular*** or ***normal*** to ∂M if its ***tangent part***, defined by

$$t\alpha = i^*(\alpha)$$

is zero.

Let X be a vector field on M. Using the metric, we know when X is tangent or perpendicular to ∂M. Now X corresponds to the one–form $X^\flat$ and also to the $(n-1)$-form $\mathbf{i}_X\mu = *X^\flat$ (μ denotes the Riemannian volume form). One checks that *X is tangent to ∂M if and only if $X^\flat$ is tangent to ∂M iff $\mathbf{i}_X\mu$ is normal to ∂M. Similarly X is normal to ∂M iff $\mathbf{i}_X\mu$ is tangent to ∂M.* Set

$$\begin{aligned}
\Omega^k_t(M) &= \{\, \alpha \in \Omega^k(M) \mid \alpha \text{ is tangent to } \partial M \,\},\\
\Omega^k_n(M) &= \{\, \alpha \in \Omega^k(M) \mid \alpha \text{ is perpendicular to } \partial M \,\}, \text{ and}\\
\mathcal{H}^k(M) &= \{\, \alpha \in \Omega^k(M) \mid \mathbf{d}\alpha = 0,\ \delta\alpha = 0 \,\}.
\end{aligned}$$

The condition that $\mathbf{d}\alpha = 0$ and $\delta\alpha = 0$ is, in general, **stronger** than $\Delta\alpha = 0$ when M has a boundary. One calls elements of $\mathcal{H}^k$ ***harmonic fields***, after Kodaira [1949].

7.5.5 Theorem (Hodge Theorem for Manifolds with Boundary)**.** *Let M be a compact oriented Riemannian manifold with boundary. The following decomposition holds:*

$$\Omega^k(M) = \mathbf{d}\Omega^{k-1}_t(M) \oplus \delta\Omega^{k+1}_n(M) \oplus \mathcal{H}^k.$$

One can easily check from the following formula (obtained from Stokes' theorem):

$$\langle \mathbf{d}\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle + \int_{\partial M} \alpha \wedge *\beta$$

(see Exercise 7.5-5), that the summands in this decomposition are orthogonal.

Two other closely related decompositions of interest are

(i) $\Omega^k(M) = \mathbf{d}\Omega^{k-1}(M) \oplus D^k_t$ where $D^k_t = \{\, \alpha \in \Omega^k_t(M) \mid \delta\alpha = 0 \,\}$ are the ***co-closed k-forms tangent*** to ∂M and, dually

(ii) $\Omega^k(M) = \delta(\Omega^{k+1}(M)) \oplus C^k_n$ where C^k_n are the ***closed k-forms normal*** to ∂M.

To put the Hodge theorem in a general context, we give a brief discussion of differential operators and their symbols. (See Palais [1965a], Wells [1980], and Marsden and Hughes [1983] for more information and additional details on proofs.) Let E and F be vector bundles of M and let $C^\infty(E)$ denote the C^∞ sections of E. Assume M is Riemannian and that the fibers of E and F have inner products. A ***kth order differential operator*** is a linear map $\mathrm{D} : C^\infty(E) \to C^\infty(F)$ such that, if $f \in C^\infty(E)$ and f vanishes to kth order at $x \in M$, then $\mathrm{D}(f)(x) = 0$. It is not difficult to see that vanishing to kth order makes intrinsic sense independent of charts and that D is a kth order differential operator iff in local charts D has the form

$$\mathrm{D}(f) = \sum_{0 \le |j| \le k} \alpha_j \frac{\partial^{|j|} f}{\partial x^{j_1} \dots \partial x^{j_s}},$$

where $j = (j_1, \dots, j_s)$ is a multi-index and α_j is a C^∞ matrix-valued function of x (the matrix corresponding to linear maps of E to F).

The operator D has an ***adjoint operator*** D^* given in charts (with the standard Euclidean inner product on fibers) by

$$\mathrm{D}^*(h) = \sum_{0 \le |j| \le k} (-1)^{|j|} \frac{1}{\rho} \frac{\partial^{|j|}}{\partial x^{j_1} \dots \partial x^{j_s}} (\rho \alpha_j^t h),$$

where $\rho\, dx^1 \wedge \cdots \wedge dx^n$ is the volume element on M and α_j^t is the transpose of a_j. The crucial property of D^* is

$$\langle g, \mathrm{D}^* h \rangle = \langle \mathrm{D}\, g, h \rangle,$$

where $\langle\, , \rangle$ denotes the L^2 inner product, $g \in C_c^\infty(E)$, and $h \in C_c^\infty(F)$. That is, g and h are C^∞ sections with compact support. For example, we have the differential operators

$$\begin{aligned} \mathbf{d} : C^\infty(\Lambda^k) &\to C^\infty(\Lambda^{k+1}) \qquad &&\text{(first order)} \\ \delta : C^\infty(\Lambda^k) &\to C^\infty(\Lambda^{k-1}) \qquad &&\text{(first order)} \\ \Delta : C^\infty(\Lambda^k) &\to C^\infty(\Lambda^k) \qquad &&\text{(second order)} \end{aligned}$$

where $\mathbf{d}^* = \delta$, $\delta^* = \mathbf{d}$, and $\Delta^* = \Delta$. The ***symbol*** of D assigns to each $\xi \in T_x^* M$, a linear map $\sigma(\xi) : E_x \to F_x$ defined by

$$\sigma(\xi)(e) = \mathbf{D}\left(\frac{1}{k!} (g - g(x))^k f \right)(x),$$

where $g \in C^\infty(M, \mathbb{R})$, $\mathbf{d}g(x) = \xi$, and $f \in C^\infty(E)$, $f(x) = e$. By writing this out in coordinates one sees that $\sigma(\xi)$ so defined is independent of g and f and is a homogeneous polynomial expression in ξ of degree k obtained by substituting each ξ_j in place of $\partial/\partial x^j$ in the highest order terms. For example, if

$$\mathrm{D}(f) = \sum \alpha^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \text{(lower order terms)}, \quad \text{then} \quad \sigma(\xi) = \sum \alpha^{ij} \xi_i \xi_j$$

(α^{ij} is for each i, j a map of E_x to F_x). For real-valued functions, the classical definition of an elliptic operator is that the foregoing quadratic form be *definite*. This can be generalized as follows: D is called ***elliptic*** if $\sigma(\xi)$ is an isomorphism for each $\xi = 0$. To see that $\Delta : C^\infty(\bigwedge^k) \to C^\infty(\bigwedge^k)$ is elliptic one uses the following facts:

1. The symbol of $\mathbf{d}$ is $\sigma(\xi) = \xi \wedge$.

2. The symbol of δ is $\sigma(\xi) = \mathbf{i}_{\xi^\#}$.

3. The symbol is multiplicative: $\sigma(\xi)(\mathrm{D}_1 \circ \mathrm{D}_2) = \sigma(\xi)(\mathrm{D}_1) \circ \sigma(\xi)(\mathrm{D}_2)$.

From these, it follows by a straightforward calculation that the symbol of Δ is given by $\sigma(\xi)\alpha = \|\xi\|^2\alpha$, so Δ is elliptic. (Compute

$$\xi \wedge (\mathbf{i}_{\xi^\#}\alpha) + \mathbf{i}_{\xi^\#}(\xi \wedge \alpha)$$

applied to $(v_1, \ldots, v_k)$, noting that all but one term cancel.)

7.5.6 Theorem (Elliptic Splitting Theorem—Fredholm Alternative). *Let* D *be an elliptic operator as above. Then*

$$C^\infty(F) = \mathrm{D}(C^\infty(E)) \oplus \ker(\mathrm{D}^*)$$

Indeed this holds if it is merely assumed that either D *or* D^* *has injective symbol.*

The proof of this leans on elliptic estimates that are not discussed here. As in the Hodge theorem, the idea is that the L^2 orthogonal complement of range D is $\ker(\mathrm{D})^*$. This yields an L^2 splitting and we get a C^∞ splitting via elliptic estimates. The splitting in case D (resp., D^*) has injective symbol relies on the fact that then $\mathrm{D}^*\,\mathrm{D}$ (resp., $\mathrm{D}\,\mathrm{D}^*$) is elliptic.

For example, the equation $\mathrm{D}\,u = f$ is soluble iff f is orthogonal to $\ker(D^*)$. More specifically, $\Delta u = f$ is soluble if f is orthogonal to the constants; that is, $\int f d\mu = 0$.

The Hodge theorem is derived from the elliptic splitting theorem as follows. Since Δ is elliptic and symmetric

$$C^\infty(\textstyle\bigwedge^k(M)) = \mathrm{range}(\Delta) \oplus \ker(\Delta) = \mathrm{range}(\Delta) \oplus \mathcal{H}$$

Now write a k-form ω as

$$\omega = \Delta\rho + \gamma = \mathbf{d}\delta\rho + \delta\mathbf{d}\rho + \gamma,$$

so to get Theorem 7.5.3, we can choose $\alpha = \delta\rho$ and $\beta = \mathbf{d}\rho$.

Supplement 7.5A
Introduction to Degree Theory

One of the purposes of degree theory is to provide algebraic measures of the number of solutions of nonlinear equations. Its development rests on Stokes' theorem. It beautifully links calculus on manifolds with ideas on differential and algebraic topology.

All manifolds in this section are assumed to be finite dimensional, paracompact and Lindelöf. We begin with an extendability result.

7.5.7 Proposition. *Let V and N be orientable manifolds,* $\dim(V) = n+1$ *and* $\dim(N) = n$. *If $f : \partial V \to N$ is a smooth proper map that extends to a smooth map of V to N, then for every $\omega \in \Omega^n(N)$ with compact support,*

$$\int_{\partial V} f^*\omega = 0.$$

Proof. Let $F : V \to N$ be a smooth extension of f. Then by Stokes' theorem

$$\int_{\partial V} f^*\omega = \int_{\partial V} F^*\omega = \int_V \mathbf{d}F^*\omega = \int_V F^*\mathbf{d}\omega = 0.$$

since $\mathbf{d}\omega = 0$. ■

This proposition will be applied to the case $V = [0,1] \times M$. For this purpose let us recall the product orientation (see Exercise 6.5-14). If N and M are orientable manifolds (at most one of which has a boundary), then $N \times M$ is a manifold (with boundary), which is orientable in the following way. Let $\pi_1 : N \times M \to N$ and $\pi_2 : N \times M \to M$ be the canonical projections and $[\omega], [\eta]$ orientations on N and M respectively. Then the orientation of $N \times M$ is defined to be $[\pi_1^*\omega \wedge \pi_1^*\eta]$. Alternatively, if $v_1, \ldots, v_n \in T_xN$ and $w_1, \ldots, w_m \in T_yM$ are positively oriented bases in the respective tangent spaces, then

$$(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m) \in T_{(x,y)}(N \times M)$$

is defined to be a positively oriented basis in their product. Thus, for $[0,1] \times M$, a natural orientation will be given at every point $(t, x) \in [0,1] \times M$ by $(1,0), (0, v_1), \ldots, (0, v_m)$, where $v_1, \ldots, v_m \in T_xM$ is a positively oriented basis.

The boundary orientation of $[0,1] \times M$ is determined according to Definition 7.2.7 Since

$$\partial([0,1] \times M) = (\{0\} \times M) \cup (\{1\} \times M),$$

every element of this union is oriented by the orientation of M. On the other hand, this union is oriented by the boundary orientation of $[0,1] \times M$. Since the outward normal at $(1,x)$ is $(1,0)$, we see that a positively oriented basis of $T_{(1,x)}(\{1\} \times M)$ is given by

$$(0, v_1), \ldots, (0, v_m) \quad \text{for } v_1, \ldots, v_m \in T_xM,$$

a positively oriented basis. However, since the outward normal at $(0,x)$ is $(-1,0)$, a positively oriented basis of $T_{(0,x)}(\{0\} \times M)$ must consist of elements $(0, w_1), \ldots, (0, w_m)$ such that $(-1,0), (0, w_1), \ldots, (0, w_m)$ is positively oriented in $[0,1] \times M$, that is, defines the same orientation as

$$(1,0), (0, v_1), \ldots, (0, v_m), \quad \text{for } v_1, \ldots, v_m \in T_xM$$

a positively oriented basis. This means that $w_1, \ldots, w_m \in T_xM$ is negatively oriented (see Figure 7.5.1). Thus *the oriented manifold $\partial([0,1] \times M)$ is the disjoint union of $\{0\} \times M$, where M is negatively oriented, with $f\{1\} \times M$, where M is positively oriented.*

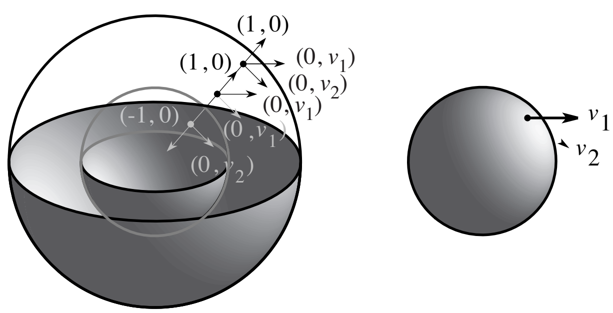

FIGURE 7.5.1. Orientation on spheres

7.5.8 Definition. *Two smooth mappings $f, g : M \to N$, are called C^r-**homotopic** if there is a C^r-map $F : [0,1] \times M \to N$ such that*

$$F(0, m) = f(m) \quad \text{and} \quad F(1, m) = g(m),$$

*for all $m \in M$. The homotopy F is called **proper** if F is a proper map; in this case f and g are said to be **properly C^r-homotopic maps**.*

Note that if f and g are properly homotopic then necessarily f and g are proper as restrictions of a proper map to the closed sets $\{0\} \times M$ and $\{1\} \times M$, respectively.

7.5.9 Proposition. *Let M and N be orientable n-manifolds, with M boundaryless, let $\omega \in \Omega^n(N)$ have compact support, and suppose $f, g : M \to N$ are properly smooth homotopic maps. Then*

$$\int_M f^*\omega = \int_M g^*\omega.$$

Proof. There are two ways to do this.

Method 1. Let $F : [0,1] \times M \to N$ be the proper homotopy between f and g. By the remarks preceding Definition 7.5.8, we have

$$\int_{\{0\}\times M} f^*\omega = -\int_M f^*\omega \quad \text{and} \quad \int_{\{1\}\times M} g^*\omega = \int_M g^*\omega,$$

so that

$$\int_M g^*\omega - \int_M f^*\omega = \int_{\partial([0,1]\times M)} (F|\partial([0,1] \times M))^*\omega = 0$$

by Proposition 7.5.7.

Method 2. By Theorem 6.4.16, $f^*\omega - g^*\omega = \mathbf{d}\eta$ for some $\eta \in \Omega^{n-1}(M)$ which has compact support since the homotopy between f and g is a proper map. Then by Stokes' theorem

$$\int_M f^*\omega - \int_M g^*\omega = \int_M \mathbf{d}\eta = 0.$$

■

Remark. Properness of f and g does not suffice in the hypothesis of Proposition 7.5.9. For example, if $M = N = \mathbb{R}$, $\omega = adx$ with $\alpha \geq 0$ a C^∞ function satisfying $\operatorname{supp}(a) \subset \,]-\infty, 0[$, then $f(x) = x$ and $g(x) = x^2$ are smoothly but not properly homotopic via $F(t,x) = (1-t)x + tx^2$ and $\int_{-\infty}^{+\infty} f^*\omega > 0$, while $\int_{-\infty}^{+\infty} g^*\omega = 0$ since $g^*\omega = 0$. ♦

7.5.10 Theorem (Degree Theorem)**.** *Let M and N be oriented n-manifolds, N connected, M boundaryless, and $f : M \to N$ a smooth proper map. Then there is an integer* $\deg(f)$ *constant on the proper homotopy class of f, called the **degree of** f such that for any $\eta \in \Omega^n(N)$ with compact support,*

$$\int_M f^*\eta = \deg(f) \int_N \eta. \tag{7.5.3}$$

If $x \in M$ is a regular point of f, let $\operatorname{sign}(T_x f)$ *be 1 or -1 depending on whether the isomorphism $T_x f : T_x M \to T_{f(x)} N$ preserves or reverses orientation. The integer* $\deg(f)$ *is given by*

$$\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(T_x f). \tag{7.5.4}$$

where y is an arbitrary regular value of f; if $y \notin f(M)$ the right hand side is by convention equal to zero.

Proof. By Proposition 7.5.9, $\int_M f^*\eta$ depends only on the proper homotropy class of f (and on η). By Sard's theorem, there is a regular value y of f. There are two possibilities: either $\mathcal{R}_f = N\backslash f(M)$ or not. If $\mathcal{R}_f = N\backslash f(M)$, then $T_x f$ is never onto for all $x \in M$. For any $v_1, \ldots, v_n \in T_x M$,

$$(f^*\eta)(x)(v_1, \ldots, v_n) = \eta(f(x))(T_x f(v_1), \ldots, T_x f(v_n)) = 0,$$

since $T_x f(v_1), \ldots, T_x f(v_n)$ are linearly dependent. Thus $\deg(f)$ exists and equals zero.

Assume $\mathcal{R}_f \cap f(M) \neq \varnothing$ and let $y \in \mathcal{R}_f \cap f(M)$. Since M and N have the same dimension, $f^{-1}(y)$ is a zero-dimensional submanifold of M, hence discrete. Properness of f implies that $f^{-1}(y)$ is also compact, that is, $f^{-1}(y) = \{x_1, \ldots, x_{k+l}\}$, where $T_{x_i} f$ is orientation preserving for $i = 1, \ldots, k$ and orientation reversing for $i = k+1, \ldots, k+l$. The inverse function theorem implies that there are open neighborhoods U_i of x_i and V of y such that

$$f^{-1}(V) = U_1 \cup \cdots \cup U_{k+l}, \quad U_i \cap U_j = \varnothing$$

and if $f|U_i : U_i \to V$ is a diffeomorphism. If $\operatorname{supp}(\eta) \subset V$, then by the change of variables formula

$$\int_M f^*\eta = \sum_{i=1}^{k+l} \int_{U_i} f^*\eta = (k-l)\int_V \eta = (k-l)\int_N \eta \tag{7.5.5}$$

and so the theorem is proved for $\operatorname{supp}(\eta) \subset V$.

To deal with a general η proceed in the following way. For the open neighborhood V of η, consider the collection of open subsets of N,

$$\mathcal{S} = \{\, \varphi(V) \mid \varphi \text{ is a diffeomorphism properly homotopic to the identity} \,\}.$$

We shall prove that $\mathcal{S}$ covers N. Let $n \in N$; we will show that there is a diffeomorphism φ properly homotopic to the identity such that $\varphi(n) = y$. Let $c : [0,1] \to N$ be a smooth curve with $c(0) = n$ and $c(1) = y$. As in Theorem 5.5.9, use a partition of unity to extend $c'(t)$ to a smooth vector field $X \in \mathfrak{X}(N)$ such that X vanishes outside a compact neighborhood of $c([0,1])$. The flow F_t of X is complete by Corollary 4.1.20 and is the identity outside the above compact neighborhood of $c([0,1])$. Thus the restriction $F : [0,1] \times N \to N$ is proper. Then $\varphi = F_1$ is a proper diffeomorphism properly homotopic to the identity on N and $\varphi(n) = F_1(n) = c(1) = y$.

Since $\mathcal{S}$ covers N, choose a partition of unity $\{(V_\alpha, h_\alpha)\}$ subordinate to $\mathcal{S}$ and let $\eta_\alpha = h_\alpha \eta$; thus, $\operatorname{supp}(\eta) \subset V_\alpha \subset \varphi_\alpha(V)$ for some φ_α. Since all φ_α are orientation preserving, the change of variables formula and equation (7.5.5) give

$$(k-l)\int_N \eta = (k-l)\sum_\alpha \int_{V_\alpha} \eta_\alpha = (k-l)\sum_\alpha \int_V \varphi_\alpha^* \eta_\alpha = \sum_\alpha \int_M f^*\varphi_\alpha^*\eta_\alpha.$$

Since φ_α is properly homotopic to the identity and f is proper, it follows that $\varphi_\alpha \circ f$ is properly homotopic to f. Thus by Proposition 7.5.9,

$$\int_M (\varphi_\alpha \circ f)^*\eta_\alpha = \int_M f^*\eta_\alpha$$

and therefore,

$$(k-l)\int_N \eta = \sum_\alpha \int_M f^*\eta_\alpha = \int_M f^*\eta. \qquad \blacksquare$$

Notice that by construction, if $\deg(f) \neq 0$, then f is onto, so $f(x) = y$ is solvable for x given y.

7.5.11 Corollary. *Let V and N be orientable manifolds with* $\dim(V) = n + 1$, *and* $\dim(N) = n$. *If* $f : \partial V \to N$ *extends to* V, *then* $\deg(f) = 0$.

This is a reformulation of Proposition 7.5.7. Similarly, Proposition 7.5.9 is equivalent to the following.

7.5.12 Corollary. *Let M, N be orientable n-manifolds, N connected, M boundaryless, and let $f, g : M \to N$ be smooth properly homotopic maps. Then* $\deg\,(f) = \deg(g)$.

This corollary is useful in three important applications. The first concerns vector fields on spheres.

7.5.13 Theorem (Hairy Ball Theorem). *Every vector field on an even dimensional sphere has a critical point.*

Proof. Let S^{2n} be embedded as the unit sphere in $\mathbb{R}^{2n+1}$ and $X \in \mathfrak{X}(S^{2n})$. Then X defines a map $f : S^{2n} \to \mathbb{R}^{2n+1}$ with components $f(x) = (f^1(x), \ldots, f^{2n+1}(x))$ satisfying

$$f^1(x)x^1 + \cdots + f^{2n+1}(x)x^{2n+1} = 0.$$

Here $f^i(x)$ are the components of X in $\mathbb{R}^{2n+1}$.

Assume that X has no critical point. Replacing f by $f/\|f\|$, we can assume that $f : S^{2n} \to S^{2n}$. The map

$$F : [0,1] \times S^{2n} \to S^{2n}, \quad F(t,x) = (\cos \pi t)x + (\sin \pi t)f(x)$$

is a smooth proper homotopy between $F(0,x) = x$ and $F(1,x) = -x$. That is, the identity Id is homotopic to the antipodal map $A : S^{2n} \to S^{2n}, A(x) = -x$. Thus by Corollary 7.5.12, $\deg A = 1$. However, since the Jacobian of A is -1 (this is the place where we use evenness of the dimension of the sphere), A is orientation reversing and thus by the Degree theorem 7.5.10, $\deg(A) = -1$, which is a contradiction. ■

The second application is to prove the existence of fixed points for maps of the unit ball to itself.

7.5.14 Theorem (Brouwer's Fixed-Point Theorem). *A smooth mapping of the closed unit ball of $\mathbb{R}^n$ into itself has a fixed point.*

Proof. Let B denote the closed unit ball in $\mathbb{R}^n$ and let $S^{n-1} = \partial B$ be its boundary, the unit sphere. If $f : B \to B$ has no fixed point, define $g(x) \in S^{n-1}$ to be the intersection of the line starting at $f(x)$ and going through x with S^{n-1}. The map $g : B \to S^{n-1}$ so defined is smooth and for $x \in S^{n-1}$, $g(x) = x$. If $n = 1$ this already gives a contradiction, since g must map $B = [-1,1]$ onto $\{-1,1\} = S^0$, which is disconnected. For $n \geq 2$, define a smooth proper homotopy $F : [0,1] \times S^{n-1} \to S^{n-1}$ by $F(t,x) = g(tx)$. Thus F is a homotopy between the constant map $c : S^{n-1} \to S^{n-1}$, $c(x) = g(0)$ and the identity of S^{n-1}. But $c^*\omega = 0$ for any $\omega \in \Omega^{n-1}(S^{n-1})$, so that by Theorem 7.5.10, $\deg c = 0$. On the other hand, by Corollary 7.5.12, $\deg(c) = 1$, which is false. ■

The Brouwer fixed point theorem is valid for continuous mappings and is proved in the following way. If f has no fixed points, then by compactness there exists a positive constant $K > 0$ such that $\|f(x) - x\| > K$ for all $x \in B$. Let $\epsilon < \min(K, 2)$ and choose $\delta > 0$ such that $2\delta/(1+\delta) < \epsilon$; that is, $\delta < \epsilon/(2-\epsilon)$. By the Weierstrass approximation theorem (see, for example, Marsden and Hoffman [1993]) there exists a polynomial mapping $q : \mathbb{R}^n \to \mathbb{R}^n$ such that $\|f(x) - q(x)\| < \delta$ for all $x \in B$. The image $q(B)$ lies inside the closed ball centered at 0 of radius $1 + \delta$, so that $p \equiv q/(1+\delta) : B \to B$ and

$$\|f(x) - p(x)\| \leq \left\| f(x) - \frac{f(x)}{1+\delta} \right\| + \left\| \frac{f(x)}{1+\delta} - \frac{q(x)}{1+\delta} \right\| \leq \frac{2\delta}{1+\delta} < \epsilon$$

for all $x \in B$. Since p is smooth by Theorem 7.5.14, it has a fixed point, say $x_0 \in B$. Then

$$0 < K < \|f(x_0) - x_0\| \leq \|f(x_0) - p(x_0)\| + \|p(x_0) - x_0\| \leq \epsilon,$$

which contradicts the choice $\epsilon < K$.

Brouwer's fixed point theorem is *false* in an open ball, for the open ball is diffeomorphic to $\mathbb{R}^n$ and translation provides a counterexample.

The proof we have given is not "constructive." For example, it is not clear how to base a numerical search on this proof, nor is it obvious that the fixed point we have found varies continuously with f. For these aspects, see Chow, Mallet-Paret, and Yorke [1978].

A third application of Corollary 7.5.12 is a topological proof of the fundamental theorem of algebra.

7.5.15 Theorem (The Fundamental Theorem of Algebra)**.** *Any polynomial $p : C \to \mathbb{C}$ of degree $n > 0$ has a root.*

Proof. Assume without loss of generality that $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, where $a_i \in \mathbb{C}$, and regard p as a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$. If p has no root, then we can define the smooth map $f(z) = p(z)/|p(z)|$ whose restriction to S^1 we denote by $g : S^1 \to S^1$.

Let $R > 0$ and define for $t \in [0, 1]$ and $z \in S^1$,

$$p_t(z) = (Rz)^n + t\left[a_{n-1}(Rz)^{n-1} + \cdots + a_0\right].$$

Since

$$p_t\frac{(z)}{(Rz)^n} = 1 + t\left[a_{n-1}(Rz)^{-1} + \cdots + \frac{a_0}{(Rz)^{-n}}\right]$$

and the coefficient of t converges to zero as $R \to \infty$, we conclude that for sufficiently large R, none of the p_t has zeros on S^1. Thus,

$$F : [0, 1] \times S^1 \to S^1 \quad \text{defined by} \quad F(t, z) = \frac{p_t(z)}{|p_t(z)|}$$

is a smooth proper homotopy of $d_n(z) = z^n$ with $g(Rz)$, which in turn is properly homotopic to $g(z)$.

On the other hand, $G : [0, 1] \times S^1 \to S^1$ defined by $G(t, z) = f(tz)$ is a proper homotopy of the constant mapping $c : S^1 \to S^1, c(z) = f(0)$ with g. Thus d_n is properly homotopic to a constant map and hence $\deg d_n = 0$ by Corollary 7.5.12. However, if S^1 is parameterized by arc length θ, $0 \leq \theta \leq 2\pi$, then d_n maps the segment $0 \leq \theta \leq 2\pi/n$ onto the segment $0 \leq \theta \leq 2\pi$ since d_n has the effect $e^{i\theta} \mapsto e^{in\theta}$. If ω denotes the corresponding volume form on S^1, the change of variables formula thus gives

$$\int_{S^1} d_n^*\omega = n\int_{S^1} \omega = 2\pi n, \quad \text{that is,} \quad \deg d_n = n,$$

which for $n \neq 0$ is a contradiction. ■

The fundamental theorem of algebra shows that any polynomial $p : \mathbb{C} \to \mathbb{C}$ of degree n can be written as

$$p(z) = c(z - z_1)^{k_1} \cdots (z - z_m)^{k_m},$$

where $z_1, \ldots, z_m$ are the distinct roots of $p, k_1, \ldots, k_m$ are their multiplicities, $k_1 + \cdots + k_m = n$, and $c \in \mathbb{C}$ is the coefficient of z^n in $p(z)$. The fundamental theorem of algebra can be refined to take into account multiplicities of roots in the following way.

7.5.16 Proposition. *Let D be a compact subset of $\mathbb{C}$ with open interior and smooth boundary ∂D. Assume that the polynomial $p : \mathbb{C} \to \mathbb{C}$ has no zeros on ∂D. Then the total number of zeros of p which lie in the interior of D, counting multiplicities, equals the degree of the map $p/|p| : \partial D \to S^1$.*

Proof. Let $z_1, \ldots, z_m$ be the roots of p in the interior of D with multiplicities $k(1), \ldots, k(m)$. Around each z_i construct an open disk D_i centered at z_i, $D_i \subset D$, such that

$$\partial D \cap \partial D_i = \varnothing \quad \text{and} \quad \partial D_i \cap \partial D_j = \varnothing,$$

for all $i \neq j$. Then

$$V = D \backslash (D_1 \cup \cdots \cup D_m)$$

is a smooth compact two-dimensional manifold whose boundary is $\partial D \cup \partial D_1 \cup \ldots \partial D_m$. The boundary orientation of ∂D_i induced by V is *opposite* to the usual boundary orientation of ∂D_i as the boundary of the disk D_i; see Figure 7.5.2. Since $p/|p|$ is defined on all of V, Corollary 7.5.11 implies that the degree of $p/|p| : \partial V \to S^1$ is zero. But the degree of a map defined on a disjoint union of manifolds is the sum of the individual degrees and thus the degree of $p/|p|$ on ∂D equals the sum of the degrees of $p/|p|$ on all ∂D_j. The proposition is therefore proved if we show that the degree of $p/|p|$ on ∂D_i is the multiplicity $k(i)$ of the root z_i.

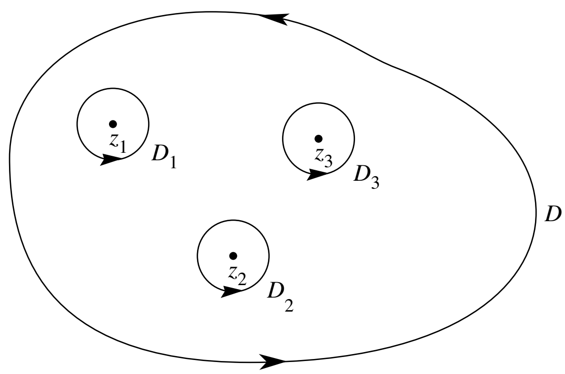

FIGURE 7.5.2. Relating degree with numbers of zeros

Let

$$r(z) = c \prod_{j=1, j\neq i}^{m} (z - z_j)^{k(j)}, \quad \text{so} \quad p(z) = (z - z_i)^{k(i)} r(z)$$

and the only zero of $p(z)$ in the disk D_i is z_i. Then $\varphi : z \in S^1 \to z_i + R_i z \in \partial D_i$, where R_i is the radius of D_i, is a diffeomorphism and therefore the degree of $p/|p| : \partial D_i \to S^1$ equals the degree of $(p \circ \varphi)/|p \circ \varphi| : S^1 \to S^1$. The homotopy $H : [0,1] \times S^1 \to S^1$ of $z^{k(i)} \arg(r(z_i))$ with $(p \circ \varphi)/|p \circ \varphi|$ given by

$$H(t, z) = \frac{z^{k(i)} r(z_i + tR_i z)}{|r(z_i + tR_i z)|}$$

is proper and smooth, since $z_i + tR_i z \in D_i$ for all $z \in S^1$, $t \in [0,1]$. Thus in ∂D_i we have

$$\deg \frac{p}{|p|} = \deg \frac{p \circ \varphi}{|p \circ \varphi|} = \deg z^{k(i)} = k(i). \qquad \blacksquare$$

A variant of the fundamental theorem of algebra is the following.

7.5.17 Proposition. *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be a C^1 proper map. Assume there is a closed subset $K \subset U$ such that for all $x \in U \backslash K$, the Jacobian $J(f)(x)$ does not change sign and is not identically equal to zero. Then f is surjective.*

Proof. The map f cannot be constant since the Jacobian $J(f)(x)$ is not identically zero for all $x \in U \backslash K$. For the same reason, f has a regular value $y \in f(U \backslash K)$, for if all values in $f(U \backslash K)$ are singular, $J(f)$ will vanish on $U \backslash K$. If $y \in f(U \backslash K)$ is a regular value of f then $\operatorname{sign}(T_x f)$ does not change for all $x \in f^{-1}(y)$ so by the degree theorem 7.5.10, $\deg(f) \neq 0$, which then implies that f is onto. $\blacksquare$

The orientation preserving character of proper diffeomorphisms is characterized in terms of the degree as follows.

7.5.18 Proposition. *Let M and N be oriented boundaryless connected manifolds and $f : M \to N$ a proper local diffeomorphism. Then* $\deg f = 1$*, if and only if f is an orientation preserving diffeomorphism.*

Proof. If f is an orientation preserving diffeomorphism, then $\deg(f) = 1$ by Theorem 7.5.10. Conversely, let f be a proper local diffeomorphism with $\deg f = 1$. Define

$$U_{\pm} = \{ m \in M \mid \operatorname{sign} T_m f = \pm 1 \} .$$

Since f is a local diffeomorphism, $U_{\pm}$ are open in M. Connectedness of M and

$$M = U_+ \cup U_-, \quad U_+ \cap U_- = \varnothing$$

imply that $M = U_+$ or $M = U_-$. Let us show that $U_- = \varnothing$. Since $\deg(f) = 1$, f is onto and hence if $n \in N$, $f^{-1}(n) \neq \varnothing$ is a discrete submanifold of M. Properness of f implies that

$$f^{-1}(n) = \{m(1), \ldots, m(k)\}.$$

Since f is a local diffeomorphism of a neighborhood U_i of $m(i)$ onto a neighborhood V of n, $\operatorname{sign} T_{m(i)} f$ is the same for all $i = 1, \ldots, k$ (for otherwise $J(f)$ must vanish somewhere). Thus $\deg(f) = \pm k$ according to whether $T_{m(i)} f$ preserves or reverses orientation. Since $\deg(f) = 1$, this implies $U_- = \varnothing$ and $k = 1$, that is, f is injective. Thus f is a bijective local diffeomorphism, that is, a diffeomorphism. ■

Supplement 7.5B
Zero and n-Dimensional Cohomology

Here we compute $H^0(M)$ and $H^n(M)$ for a connected n-manifold M. Recall that the ***cohomology groups*** are defined by

$$H^k(M) = \ker(\mathbf{d})^k / \operatorname{range}(\mathbf{d}^{k-1}),$$

where $\mathbf{d}^k : \Omega^k(M) \to \Omega^{k+1}(M)$ is the exterior differential. If $\Omega^k_c(M)$ denotes the k-forms with compact support, then $\mathbf{d}^k : \Omega^k_c(M) \to \Omega^{k+1}_c(M)$ and one forms in the same manner $H^k_c(M)$, the ***compactly supported cohomology groups*** of M.

Thus,

$$H^0(M) = \{ f \in \mathcal{F}(M) \mid \mathbf{d}f = 0 \} \cong \mathbb{R}$$

since any locally constant function on a connected space is constant. If M were not connected, then $H^0(M) = \mathbb{R}^c$, where c is the number of connected components of M. By the Poincaré lemma, if M is contractible, then $H^q(M) = 0$ for $q \neq 0$.

The rest of this supplement is devoted to the proof and applications of the following *special case of deRham's theorem.*

7.5.19 Theorem. *Let M be a boundaryless connected n-manifold.*

(i) *If M is orientable, then $H^n_c(M) \cong \mathbb{R}$, the isomorphism being given by integration: $[\omega] \mapsto \int_M \omega$. In particular $\omega \in \Omega^n_c(M)$ is exact iff $\int_M \omega = 0$.*

(ii) *If M is nonorientable, then $H_c^n(M) = 0$.*

(iii) *If M is non-compact, orientable or not, then $H^n(M) = 0$.*

Before starting the actual proof, let us discuss (i). The integration mapping $\int_M : \Omega^n(M) \to \mathbb{R}$ is linear and onto. To see that it is onto, let ω be an n-form with support in a chart in which the local expression is $\omega = f dx^1 \wedge \cdots \wedge dx^n$ with f a bump function. Then $\int_M \omega = \int_{\mathbb{R}^n} f(x)dx > 0$. Since we can multiply ω by any scalar, the integration map is onto. Any ω with nonzero integral cannot be exact by Stokes' theorem. This last remark also shows that integration induces a mapping, which we shall still call integration, $\int_M : H_c^n(M) \to \mathbb{R}$, which is linear and onto. Thus, in order to show that it is an isomorphism as (i) states, it is necessary and sufficient to prove it is injective, that is, to show that if $\int_M \omega = 0$ for $\omega \in \Omega^n(M)$, then ω is exact. The proof of this will be done in the following lemmas.

7.5.20 Lemma. *Theorem* 7.5.19 *holds for $M = S^1$.*

Proof. Let $p : \mathbb{R} \to S^1$ be given by $p(t) = e^{it}$ and $\omega \in \Omega^1(S^1)$. Then $p^*\omega = f\, dt$ for $f \in \mathcal{F}(\mathbb{R})$ a 2π-periodic function. Let F be an antiderivative of f. Since

$$0 = \int_{S_1} \omega = \int_t^{t+2\pi} f(s)ds = F(t + 2\pi) - F(t)$$

for all $t \in \mathbb{R}$, we conclude that F is also 2π-periodic, so it induces a unique map $G \in \mathcal{F}(S^1)$, determined by $p^*G = F$. Hence $p^*\omega = \mathbf{d}F = p^*\mathbf{d}G$ implies $\omega = \mathbf{d}G$ since p is a surjective submersion. ■

7.5.21 Lemma. *Theorem* 7.5.19 *holds for $M = S^n$, $n > 1$.*

Proof. This will be done by induction on n, the case $n = 1$ being the previous lemma. Write $S^n = N \cup S$, where $N = \{ x \in S^n \mid x^{n+1} \geq 0 \}$ is the closed northern hemisphere and $S = \{ x \in S^n \mid x^{n+1} \leq 0 \}$ the closed southern hemisphere. Then $N \cap S = S^{n-1}$ is oriented in two *different* ways as the boundary of N and S, respectively. Let

$$O_N = \{ x \in S^n \mid x^{n+1} > -\epsilon \}, \quad O_S = \{ x \in S^n \mid x^{n+1} < \epsilon \}$$

be open contractible neighborhoods of N and S, respectively. Thus by the Poincaré lemma, there exist $\alpha_N \in \Omega^{n-1}(O_N)$, $\alpha_S \in \Omega^{n-1}(O_S)$ such that $\mathbf{d}\alpha_N = \omega$ on O_N, $\mathbf{d}\alpha_S = \omega$ on O_S. Hence by hypothesis and Stokes' theorem,

$$\begin{aligned} 0 &= \int_{S^n} \omega = \int_N \omega + \int_S \omega = \int_N \mathbf{d}\alpha_N + \int_S \mathbf{d}\alpha_S = \int_{\partial N} i^*\alpha_N + \int_{\partial S} i^*\alpha_S \\ &= \int_{S^{n-1}} i^*\alpha_N - \int_{S^{n-1}} i^*\alpha_S \\ &= \int_{S^{n-1}} i^*(\alpha_N - \alpha_S) \end{aligned}$$

where $i : S^{n-1} \to S^n$ is the inclusion of S^{n-1} as the equator of S^n; the minus sign appears on the second integral because the orientations of S^{n-1} and ∂S are opposite. By induction, $i^*(\alpha_N - \alpha_S) \in \Omega^{n-1}(S^{n-1})$ is exact.

Let $O = O_N \cap O_S$ and note that the map $r : O \to S^{n-1}$, sending each $x \in S$ to $r(x) \in S^{n-1}$, the intersection of the meridian through x with the equator S^{n-1}, is smooth. Then $r \circ i$ is the identity on S^{n-1}. Also, $i \circ r$ is homotopic to the identity of O, the homotopy being given by sliding $x \in O$ along the meridian to $r(x)$. Since $\mathbf{d}(\alpha_N - \alpha_S) = \omega - \omega = 0$ on O, by Theorem 6.4.16 we conclude that $(\alpha_N - \alpha_S) - r^*i^*(\alpha_N - \alpha_S)$ is exact on O. But we just showed that $i^*(\alpha_N - \alpha_S) \in \Omega^{n-1}(S^{n-1})$ is exact, and hence $r^*i^*(\alpha_N - \alpha_S) \in \Omega^{n-1}(O)$ is also exact. Hence $\alpha_N - \alpha_S \in \Omega^{n-1}(O)$ is exact. Thus, there exists $\beta \in \Omega^{n-2}(O)$ such that $\alpha_N - \alpha_S = \mathbf{d}\beta$

on O. Now use a bump function to extend β to a form $\gamma \in \Omega^{n-2}(S^n)$ so that on O, $\beta = \gamma$, and $\gamma = 0$ on $S^n \backslash V$, where V is an open set such that $\mathrm{cl}(U) \subset V$. Then

$$\lambda(x) = \begin{cases} \alpha_N(x), & \text{if } x \in N, \\ \alpha_S(x) + \mathbf{d}\gamma, & \text{if } x \in S \end{cases}$$

is by construction C^∞ and $\mathbf{d}\lambda = \omega$. ■

7.5.22 Lemma. *A compactly supported n-form $\omega \in \Omega^n(\mathbb{R}^n)$ is the exterior derivative of a compactly supported $(n-1)$-form on $\mathbb{R}^n$ iff $\int_{\mathbb{R}^n} \omega = 0$.*

Proof. Let $\sigma : S^n \to \mathbb{R}^n$ be the stereographic projection from the north pole $(0, \ldots, 1) \in S^n$ onto $\mathbb{R}^n$ and assume without loss of generality that $(0, \ldots, 1) \notin \sigma^{-1}$ $(\mathrm{supp}\,\omega)$. By the previous lemma, $\sigma^*\omega = \mathbf{d}\alpha$, for some $\alpha \in \Omega^{n-1}(S^n)$ since

$$0 = \int_{\mathbb{R}^n} \omega = \int_S \sigma^*\omega$$

by the change of variables formula. But $\sigma^*\omega = \mathbf{d}\alpha$ is zero in a contractible neighborhood U of the north pole, so that by the Poincaré lemma, $\alpha = \mathbf{d}\beta$ on U, where $\beta \in \Omega^{n-2}(U)$. Now extend β to an $(n-2)$-form $\gamma \in \Omega^{n-2}(S^n)$ such that $\beta = \gamma$ on U and $\gamma = 0$ outside a neighborhood of $\mathrm{cl}(U)$. But then $\sigma_*(\alpha - \mathbf{d}\gamma)$ is compactly supported in $\mathbb{R}^n$ and $\mathbf{d}\sigma_*(\alpha - \mathbf{d}\gamma) = \sigma_*\mathbf{d}\alpha = \omega$. ■

7.5.23 Lemma. *Let M be a boundaryless connected n-manifold. Then $H^n_c(M)$ is at most one-dimensional.*

Proof. Let (U_0, φ_0) be a chart on M such that $\varphi_0(U_0)$ is the open unit ball B in $\mathbb{R}^n$. Let $\omega \in \Omega^n_c(M)$, satisfying $\mathrm{supp}(\omega) \subset U_1$, be the pull–back of a form $f dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(B)$ where $f \geq 0$ and $\int_{\mathbb{R}^n} f(x)dx = 1$. To prove the lemma, it is sufficient to show that for every $\eta \in \Omega^n_c(M)$ there exists a number $c \in \mathbb{R}$ such that $\eta - c\omega = \mathbf{d}\zeta$ for some $\zeta \in \Omega^{n-1}_c(M)$.

First assume $\eta \in \Omega^{n-1}_c(M)$ has $\mathrm{supp}(\eta)$ entirely contained in a chart (U, φ) and let $U_0, U_1, \ldots, U_k$ be a finite covering of a curve starting in U_0 and ending on $U_k = U$ such that $U_i \cap U_{i+1} = \varnothing$. Let $\alpha_i \in \Omega^n_c(U_i)$, $i = 1, \ldots, k-1$ be non-negative n-forms such that

$$\mathrm{supp}(\alpha_i) \subset U_i, \quad \mathrm{supp}(\alpha_i) \cap U_{i+1} \neq \varnothing, \quad \text{and} \quad \int_{\mathbb{R}^n} \varphi_{i^*}(\alpha_i) = 1.$$

Let $\alpha_0 = \omega$ and $\alpha_k = \eta$. But then

$$\int_{\mathbb{R}^n} \varphi_{i^*}(\alpha_{i-1}) \neq 0, \quad i = 1, \ldots, k$$

by the change-of-variables formula, so that with $c_i = -1/\int_{\mathbb{R}^n} \varphi_{i^*}(\alpha_{i-1})$ we have $\int_{\mathbb{R}^n} \varphi_{i^*}(\alpha_i c_i \alpha_{i-1}) = 0$. Thus by the previous lemma $\varphi_{i^*}(\alpha_i - c_i\alpha_{i-1})$ is the differential of an $(n-1)$-form supported in B. That is, there exists $\beta_i \in \Omega^{n-1}_c(M)$, β_i vanishing outside U_i such that

$$\alpha_i - c_i\alpha_{i-1} = \mathbf{d}\beta_i, \quad i = 1, \ldots, k.$$

Put $c = c_k \cdots c_1$ and

$$\beta = \beta_k + (c_k\beta_{k-1}) + (c_k c_{k-1}\beta_{k-2}) + \cdots + (c_{k-1} \cdots c_2\beta_1) \in \Omega^{n-1}(M).$$

Then

$$\begin{aligned} \eta - c\omega = \alpha_k - c\alpha_0 &= \alpha_k - c_k\alpha_{k-1} + c_k(\alpha_{k-1} - c_{k-1}\alpha_{k-2}) + \ldots \\ &\quad + (c_k \cdots c_2)(\alpha_1 - c_1\alpha_0) \\ &= \mathbf{d}\beta_k + c_k\mathbf{d}\beta_{k-1} + \cdots + (c_k \cdots c_2)\mathbf{d}\beta_1 = \mathbf{d}\beta. \end{aligned}$$

Let $\eta \in \Omega^n_c(M)$ be arbitrary and $\{\, \chi_i \mid i = 1, \ldots, k \,\}$ a partition of unity subordinate to the given atlas $\{\, (U_i, \varphi_i) \mid i = 1, \ldots, k \,\}$. Then $\chi_i \eta$ is compactly supported in U_i and hence there exist constants c_i and forms $\alpha_i \in \Omega^{n-1}_c(M)$ such that $\chi_i \eta - c_i \omega = \mathbf{d}\alpha_i$. If

$$c = \sum_{i=1}^{k} c_i \quad \text{and} \quad \alpha = \sum_{i=1}^{k} \alpha_i \in \Omega^{n-1}_c(M),$$

then

$$\eta - c\omega = \sum_{i=1}^{k} (\chi_i \eta - c_i \omega) = \sum_{i=1}^{k} \mathbf{d}\alpha_i = \mathbf{d}\alpha. \qquad \blacksquare$$

Proof of Theorem 7.5.19. (i) By the preceding lemma, $H^n_c(M)$ is zero- or one-dimensional. We have seen that $\int_M : H^n_c(M) \to \mathbb{R}$ is linear and onto so that necessarily $H^n_c(M)$ is one-dimensional; that is, $\int_M \omega = 0$ iff ω is exact.

(ii) Let $\widetilde{M}$ be the oriented double covering of M and $\pi : \widetilde{M} \to M$ the canonical projection. Define $\pi^\# : H^n(M) \to H^n(\widetilde{M})$ by $\pi^\#[\alpha] = [\pi^*\alpha]$. We shall first prove that $\pi^\#$ is the zero map. Let $\{U_i\}$ be an open covering of M by chart domains and $\{\chi_i\}$ a subordinate partition of unity. Let $\pi^{-1}(Ui) = U^1_i \cup U^2_i$. Then $\{\, U^j_i \mid j = 1, 2 \,\}$ is an open covering of $\widetilde{M}$ by chart domains and the maps $\psi^j_i = \chi_i \circ \pi/2 : U^j_i \to \mathbb{R}$, $j = 1, 2$, form a subordinate partition of unity on M. Let $\alpha \in \Omega^n_c(M)$. Then

$$\int_{\widetilde{M}} \pi^*\alpha = \sum_{i,j} \int_{U^j_i} \psi^1_i \pi^*\alpha = \sum_{i=1}^{k} \left(\int_{U^1_i} \psi^1_i \pi^*\alpha + \int_{U^2_i} \psi^2_i \pi^*\alpha \right) = 0,$$

each term vanishing since their push–forwards by the coordinate maps coincide on $\mathbb{R}^n$ and U^1_i and U^2_i have opposite orientations. By (i), we conclude that $\pi^*\alpha = \mathbf{d}\beta$ for some $\beta \in \Omega^{n-1}(\widetilde{M})$; that is, $\pi^\#[\alpha] = [\pi^*\alpha] = [0]$ for all $[\alpha] \in H^n_c(M)$.

We shall now prove that $\pi^\#$ is injective, which will show that $H^n_c(M) = 0$. Let $\alpha \in \Omega^n_c(M)$ be such that $\pi^*\alpha = \mathbf{d}\beta$ for some $\beta \in \Omega^{n-1}c(\widetilde{M})$ and let $r : \widetilde{M} \to \widetilde{M}$ be the diffeomorphism associating to $(m, [\omega]) \in \widetilde{M}$ the point $(m, [-\omega]) \in \widetilde{M}$. Then clearly $\pi \circ r = \pi$ so that

$$\mathbf{d}(r^*\beta) = r^*(\mathbf{d}\beta) = r^*\pi^*\alpha = (\pi \circ r)^*\alpha = \pi^*\alpha = \mathbf{d}\beta.$$

Define $\tilde{\gamma} \in \Omega^{n-1}_c(M)$ by setting $\tilde{\gamma} = (1/2)(\beta + r^*\beta)$ and note that $r^*\tilde{\gamma} = \tilde{\gamma}$ and

$$\mathbf{d}\tilde{\gamma} = \frac{\mathbf{d}\beta + \mathbf{d}r^*\beta}{2} = \mathbf{d}\beta = \pi^*\alpha.$$

But $\tilde{\gamma}$ projects to a well-defined form $\gamma \in \Omega^{n-1}_c(M)$ such that $\pi^*\gamma = \tilde{\gamma}$, since $r^*\tilde{\gamma} = \tilde{\gamma}$. Thus $\pi^*\alpha = \mathbf{d}\tilde{\gamma} = \mathbf{d}\pi^*\gamma = \pi^*\mathbf{d}\gamma$, which implies that $\alpha = \mathbf{d}\gamma$, since π is a surjective submersion.

(iii) Assume first that $\omega \in \Omega^n_c(M)$ has its support contained in a relatively compact chart domain U_1 of M. Then out of a finite open relatively compact covering of $\mathrm{cl}(U_1)$ by chart domains, pick a relatively compact chart domain U_2 which does not intersect $\mathrm{supp}(\omega)$. Working with $\mathrm{cl}(U_2)\backslash U_1$, find a relatively compact chart domain U_3 such that

$$U_1 \cap U_3 = \varnothing, \quad U_2 \cap U_3 \neq \varnothing, \quad U_3 \cap (M \backslash (U_1 \cap U_2)) \neq \varnothing.$$

Proceed inductively to find a sequence $\{U_n\}$ of relatively compact chart domains such that

$$U_n \cap U_{n+1} \neq \varnothing, \quad U_n \cap U_{n-1} \neq \varnothing, \quad U_n \cap U_m = \varnothing$$

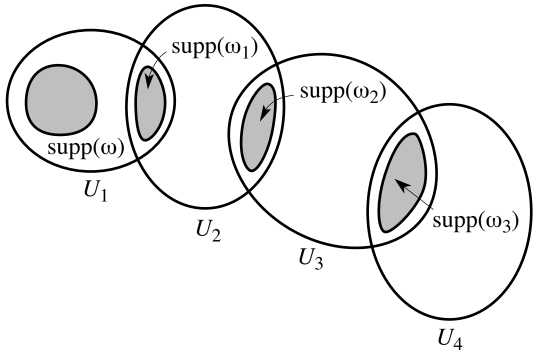

FIGURE 7.5.3. A chain of open sets

for all $m \neq n-1, n, n+1$, and such that $\operatorname{supp}(\omega) \subset U_1$, $U_2 \cap \operatorname{supp}(\omega) = \varnothing$. Since M is not compact, this sequence can be chosen to be infinite; see Figure 7.5.3.

Now choose in each $U_n \cap U_{n+1}$ an n-form ω_n with compact support such that

$$\int_{U_1} \omega = \int_{U_1} \omega_1 = \int_{U_2} \omega_2 = \cdots = \int_{U_n} \omega_n = \cdots .$$

Since $H_c^n(U_n) = \mathbb{R}$ by (i), U_n being orientable, ω_{n-1} and ω_n define the same cohomology class, that is, there is $\eta_n \in \Omega^{n-1}c(U_n)$ such that $\omega_{n-1} = \omega_n + \mathbf{d}\eta_n$. If we let $\omega_0 = \omega$, we get recursively

$$\omega = \mathbf{d}\eta_1 + \omega_1 = \mathbf{d}(\eta_1 + \eta_2) + \omega_2 = \cdots = \mathbf{d}\left(\sum_{i=1}^{n} \eta_i\right) + \omega_n = \cdots .$$

We claim that $\omega = \mathbf{d}(\sum_{n \geq 1} \eta_n)$, where the sum is finite since any point of the manifold belongs to at most two U_n's. Thus, if $p \in \bigcup_{n \geq 1} U_n$, let $p \in U_n$ so that

$$\begin{aligned} \mathbf{d}(\Sigma_{n \geq 1} \eta_n)(p) &= \mathbf{d}\eta_{n-1}(p) + \mathbf{d}\eta_n(p) + \mathbf{d}\eta_{n+1}(p) \\ &= \mathbf{d}\eta_{n-1}(p) + \omega_{n-1}(p) - \omega_{n+1}(p) \\ &= \mathbf{d}\eta_{n-1}(p) + \omega_{n-1}(p) - \omega_{n+2}(p) - \mathbf{d}\eta_{n+2}(p) \end{aligned}$$

with the convention $\eta_0 = 0$. Since $U_n \cap U_{n+2} = \varnothing$ and $\operatorname{supp} \omega_{n+2}$, $\operatorname{supp} \eta_{n+2} \subset U_{n+2}$, it follows that the last two terms vanish. Thus,

$$\mathbf{d}\left(\sum_{n \geq 1} \eta_n\right)(p) = \mathbf{d}\eta_{n-1}(p) + \omega_{n-1}(p).$$

If $n = 1$, this proves the desired equality. If $n \geq 2$, then

$$\mathbf{d}\left(\sum_{n \geq 1} \eta_n\right)(p) = \mathbf{d}\eta_{n-1}(p) + \omega_{n-1}(p) = \omega_{n-2}(p)$$

and $U_n \cap U_{n-2} = \varnothing$ implies that $\mathbf{d}(\sum_{n \geq 1} \eta_n) = 0$. Since also $\omega(p) = 0$ in this case, the desired equality holds again. Finally, if $p \notin \bigcup_{n \geq 1} U_n$, then both sides of the equality are zero and we showed that ω is exact, $\omega = \mathbf{d}(\sum_{n \geq 1} \eta_n)$, with $\operatorname{supp}(\sum_{n \geq 1} \eta_n) \subset \bigcup_{n \geq 1} U_n$.

Now if $\omega \in \Omega^n(M)$, let $\{(U_i, g_i)\}$ be a partition of unity subordinate to a locally finite atlas of M whose chart domains are relatively compact. Thus $\operatorname{supp}(g_i\omega) \subset U_i$ and by what we just proved, $g_i\omega = \mathbf{d}\eta_i$, with supp$(\eta_i)$ contained in the union of the chain of open sets $\{U_n^i\}$, $U_1^i = U_i$, as described above. Refine each such chain, such that all its elements are one of the U_j's. Since at most two of the U_n^i intersect for each fixed i, it follows that the sum $\Sigma_i \eta_i$ is locally finite and therefore $h = \sum_i \eta_i \in \Omega^{n-1}(M)$. Finally, $\omega = \sum_i g_i \omega = \sum_i \mathbf{d}\eta_i = \mathbf{d}\eta$, thus showing that ω is exact and hence $H^n(M) = 0$. ■

One can use this result as an alternative method to introduce the ***degree*** of a proper map $f : M \to N$ between oriented n-manifolds; that is, that integer deg(f) such that

$$\int_M f^*\eta = \deg(f) \int_N \eta$$

for any $\eta \in \Omega_c^n(N)$. Indeed, since the isomorphism $H_c^n(N) \cong \mathbb{R}$ is given by $[\eta] \mapsto \int_N \eta$, the linear map $[\eta] \mapsto \int_M f^*\eta$ of $H_c^n(N)$ to $\mathbb{R}$ must be some *real* multiple of this isomorphism:

$$\int_M f^*\eta = \deg(f) \int_N \eta$$

for all $\eta \in \Omega_c^n(N)$ and some real deg(f).

To prove that deg(f) is an integer in this context and that the formula (7.5.3) for deg(f) is independent of the regular value y, note that if y is any regular value of f and $x \in f^{-1}(y)$, then there exist compact neighborhoods V of y and U of x such that $f|U : U \to V$ is a diffeomorphism. Since $f^{-1}(y)$ is compact and discrete, it must be finite, say $f^{-1}(y) = \{x_1, \ldots, x_k\}$. This shows that $f^{-1}(V) = U_1 \cup \cdots \cup U_k$ with all U_i disjoint and the sum in the degree formula is finite. Shrink V if necessary to lie in a chart domain. Now choose $\eta \in \Omega_c^n(N)$ satisfying supp$(\eta) \subset V$. Then

$$\int_M f^*\eta = \sum_{x_i \in f^{-1}(y)} \int_{U_i} f^*\eta = \left\{ \sum_{x_i \in f^{-1}(y)} \operatorname{sign}(T_{x_i} f) \right\} \int_N \eta$$

by the change of variables formula in $\mathbb{R}^n$, so the claim follows.

Degree theory can be extended to infinite dimensions as well and has important applications to partial differential equations and bifurcations. This theory is similar in spirit to the above and was developed by Leray and Schauder in the 1930s. See Chow and Hale [1982], Nirenberg [1974], and Elworthy and Tromba [1970b] for modern accounts.

Exercises

◇ **7.5-1** (Poincaré duality). Show that $*$ induces an isomorphism $* : H^k \to H^{n-k}$ and $H_c^k \to H_c^{n-k}$.

◇ **7.5-2.** (For students knowing some algebraic topology.) Develop some basic properties of deRham cohomology groups such as homotopy invariance, exact sequences, Mayer–Vietoris sequences and excision. Use this to compute the cohomology of some standard simple spaces (tori, spheres, projective spaces).

◇ **7.5-3.** (i) Show that any smooth vector field X on a compact Riemannian manifold (M, g) can be written uniquely as

$$X = Y + \operatorname{grad} p$$

where Y has zero divergence (and is parallel to ∂M if M has boundary).

(ii) Show directly that the equation

$$\Delta p = -\operatorname{div} X, \quad (\operatorname{grad} p) \cdot n = X \cdot n$$

is formally soluble using the ideas of the Fredholm alternative.

◇ **7.5-4.** Show that any symmetric two-tensor h on a compact Riemannian manifold (M, g) can be uniquely decomposed in the form

$$h = \pounds_X g + k.$$

where $\delta k = 0$, δ being the divergence of g, defined by $\delta k = (\pounds_{(\cdot)} g)^* k$, where $(\pounds_{(\cdot)} g)^*$ is the adjoint of the operator $X \mapsto \pounds_X g$. (See Berger and Ebin [1969] and Cantor [1981] for more information.)

◇ **7.5-5.** Let $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$, where M is a compact oriented Riemannian manifold with boundary. Show that

(i)
$$\langle \mathbf{d}\alpha, \beta \rangle - \langle \alpha, \delta\beta \rangle = \int_{\partial M} \alpha \wedge *\beta.$$

HINT: Show that $*\delta\beta = (-1)^k \mathbf{d} * \beta$ and use Stokes theorem or Corollary 7.2.13.

(ii)
$$\langle \mathbf{d}\delta\alpha, \beta \rangle - \langle \delta\alpha, \delta\beta \rangle = \int_{\partial M} \delta\alpha \wedge *\beta$$
$$\langle \mathbf{d}\alpha, \mathbf{d}\beta \rangle - \langle \alpha, \delta\mathbf{d}\beta \rangle = \int_{\partial M} \alpha \wedge *\mathbf{d}\beta$$

(iii) (**Green's formula**)

$$\langle \Delta\alpha, \beta \rangle - \langle \alpha, \Delta\beta \rangle = \int_{\partial M} (\delta\alpha \wedge *\beta - \mathbf{d}\beta \wedge *\alpha + \alpha \wedge *\mathbf{d}\beta - \beta \wedge *\mathbf{d}\alpha).$$

HINT: Show first that

$$\langle \Delta\alpha, \beta \rangle - \langle \mathbf{d}\alpha, \mathbf{d}\beta \rangle - \langle \delta\alpha, \delta\beta \rangle = \int_{\partial M} (\delta\alpha \wedge *\beta - \beta \wedge *\mathbf{d}\alpha).$$

◇ **7.5-6.** (For students knowing algebraic topology.) Define relative cohomology groups and relate them to the Hodge decomposition for manifolds with boundary.

◇ **7.5-7.** Prove the local formulas

$$(\delta\alpha)_{i_1 \cdots i_k} = \frac{1}{k+1} |\det[g_{rs}]|^{-1/2} g_{i_1 r_1 \cdots i_k r_k} \frac{\partial}{\partial x^l}$$
$$\left(\sum_{p=1}^{k+1} (-1)^p g^{r_1 j_1} \cdots g^{r_{p-1} j_{p-1}} g^{l j_p} g^{r_p j_{p+1}} \cdots \right.$$
$$\left. g^{r_k j_{k+1}} \beta_{j_1 \cdots j_{k+1}} |\det[g_{rs}]|^{1/2} \right)$$
$$(\delta\alpha)^{r_1 \cdots r_k} = \frac{1}{k+1} |\det[g_{ij}]|^{-1/2} \frac{\partial}{\partial x^l}$$
$$\left(\sum_{p=1}^{k+1} (-1)^p \alpha^{r_1 \cdots r_{p-1} l r_p \cdots r_k} |\det[g_{ij}]|^{1/2} \right)$$

where $i_1 < \cdots < i_k$ and $\alpha \in \Omega^{k+1}(M)$ according to the following guidelines. First prove the second formula. Work in a chart (U, φ) with $\varphi(U) = B_3(0)$ = open ball of radius 3, and prove the formula on $\varphi^{-1}(B_1(0))$. For this, choose a function χ on $\mathbb{R}^n$ with $\operatorname{supp}(\chi) \subset B_3(0)$ and $\chi | B_1(0) \equiv 1$. Then extend $\chi \varphi_* \alpha$ to $\mathbb{R}^n$, denote it by α' and consider the set $B_4(0)$.

(i) Show from Exercise 7.5-5(i) that $\langle \mathbf{d}\beta, \alpha' \rangle = \langle \beta, \delta\alpha' \rangle$ for any $\beta \in \Omega^{k+1}(B_4(0))$.

(ii) In the explicit expression for $\langle \mathbf{d}\beta, \alpha' \rangle$, perform an integration by parts and justify it.

(iii) Find the expression for $\delta\alpha'$ by comparing $\langle \beta, \Delta\alpha' \rangle$ with the expression found in (ii) and argue that it must hold on $\varphi^{-1}(B_1(0))$.

◇ **7.5-8.** Let $\varphi : M \to M$ be a diffeomorphism of an oriented Riemannian manifold (M, g) and let δ_g denote the codifferential corresponding to the metric g and $\langle\, ,\rangle_g$ the inner product on $\Omega^k(M)$ corresponding to the metric g. Show that

(i) $\langle \alpha, \beta \rangle_g = \langle \varphi^*\alpha, \varphi^*\beta \rangle_{\varphi^* g}$ for $\alpha, \beta \in \Omega^k(M)$ and

(ii) $\delta_{\varphi^* g}(\varphi^*\alpha) = \varphi^*(\delta_g \alpha)$ for $\alpha \in \Omega^k(M)$.

Hint: Use the fact that $\mathbf{d}$ and δ are adjoints.

◇ **7.5-9.** (i) Let c_1 and c_2 be two differentiably homotopic curves and $\omega \in \Omega^1(M)$ a closed one-form. Show that

$$\int_{c_1} \omega = \int_{c_2} \omega.$$

(ii) Let M be simply connected. Show that $H^1(M) = 0$.

Hint: For $m_0 \in M$, let c be a curve from m_0 to $m \in M$. Then $f(m) = \int_c \omega$ is well defined by (i) and $\mathbf{d}f = \omega$.

(iii) Show that $H^1(S^1) \neq 0$ by exhibiting a closed one-form that is not exact.

◇ **7.5-10.** The ***Hopf degree theorem*** states that f and $g : M^n \to S^n$ are homotopic iff they have the same degree. By consulting references if necessary, prove this theorem in the context of Supplements 7.5A and B.
Hint: Consult Guillemin and Pollack [1974] and Hirsch [1976].

◇ **7.5-11.** What does the degree of a map have to do with Exercise 7.2-4 on integration over the fiber? Give some examples and a discussion.

◇ **7.5-12.** Show that the equations

$$z^{13} + \sin(|z|^2)z^7 + 3z^4 + 2 = 0$$
$$z^8 + \cos(|z|^2)z^5 + 5\log(|z|^2)z^4 + 53 = 0$$

have a root.

◇ **7.5-13.** Let $f : M \to N$ where M and N are compact orientable boundaryless manifolds and N is contractible. Show that $\deg(f) = 0$. Conclude that the only contractible compact manifold (orientable or not) is the one-point space.
Hint: Show that the oriented double covering of a contractible non-orientable manifold is contractible.

◇ **7.5-14.** Show that every smooth map $f : S^n \to \mathbb{T}^n$, $n > 1$ has degree zero. Conclude that S^n and $\mathbb{T}^n$ are not diffeomorphic if $n > 1$.
Hint: Show that f is homotopic to a constant map.