6 Differential Forms

Differential k-forms are tensor fields of type $(0, k)$ that are completely antisymmetric. Such tensor fields arise in many applications in physics, engineering, and mathematics. A hint at why this is so is the fact that the classical operations of grad, div, and curl and the theorems of Green, Gauss, and Stokes can all be expressed concisely in terms of differential forms and an operator on differential forms to be studied in this chapter, the exterior derivative $\mathbf{d}$. However, identities like $\nabla \times (\nabla f) = 0$ and $\nabla \cdot (\nabla \times X) = 0$ are elegantly phrased as the single identity $\mathbf{d}^2 = 0$. However, the examples of Hamiltonian mechanics and Maxwell's equations (see Chapter 8) show that their applicability goes well beyond this.

The goal of the chapter is to develop the calculus of differential forms, due largely to Cartan [1945]. The exterior derivative operator $\mathbf{d}$ plays a central role; its properties and the expression of the Lie derivative in terms of it will be developed.

6.1 Exterior Algebra

We begin with the exterior algebra of a vector space and extend this fiberwise to a vector bundle. As with tensor fields, the most important case is the tangent bundle of a manifold, which is considered in the next section.

We first recall a few facts about the permutation group on k elements; some of these facts have already been discussed in §2.2. Proofs of the results that we cite are obtainable from virtually any elementary algebra book. The ***permutation group*** on k elements, denoted S_k, consists of all bijections $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ usually given in the form of a table

$$\begin{pmatrix} 1 & \cdots & k \\ \sigma(1) & \cdots & \sigma(k) \end{pmatrix},$$

together with the structure of a group under composition of maps. Clearly, S_k has order $k!$. Letting $\{-1, 1\}$ have its natural multiplicative group structure, there is a homomorphism denoted $\operatorname{sign} : S_k \to \{-1, 1\}$; that is, for $\sigma, \tau \in S_k$, $\operatorname{sign}(\sigma \circ \tau) = (\operatorname{sign}\sigma)(\operatorname{sign}\tau)$. A permutation σ is called ***even*** when $\operatorname{sign}\sigma = +1$ and ***odd*** when $\operatorname{sign}\sigma = -1$. This homomorphism can be described as follows. A ***transposition*** is a permutation that swaps two elements of $\{1, \ldots, k\}$, leaving the remainder fixed. An even (odd) permutation can be written as the product of an even (odd) number of transpositions. The expression of σ as a product of transpositions is not unique, but the number of transpositions is always even or odd corresponding to σ being even or odd.

If $\mathbf{E}$ and $\mathbf{F}$ are Banach spaces, an element of $T_k^0(\mathbf{E},\mathbf{F}) = L^k(\mathbf{E};\mathbf{F})$; that is, a k-multilinear continuous mapping of $\mathbf{E}\times\cdots\times\mathbf{E}\to\mathbf{F}$ is called ***skew symmetric*** when

$$t(e_1,\ldots,e_k) = (\operatorname{sign}\sigma)\, t(e_{\sigma(1)},\ldots,e_{\sigma(k)})$$

for all $e_1,\ldots,e_k\in\mathbf{E}$ and $\sigma\in S_k$. This is equivalent to saying that $t(e_1,\ldots,e_k)$ changes sign when any two of $e_1,\ldots,e_k$ are swapped. The subspace of skew symmetric elements of $L^k(\mathbf{E};\mathbf{F})$ is denoted $L_a^k(\mathbf{E};\mathbf{F})$ (the subscript a stands for "alternating"). Some additional shorthand will be useful. Namely, let $\bigwedge^0(\mathbf{E},\mathbf{F}) = \mathbf{F}$, $\bigwedge^1(\mathbf{E},\mathbf{F}) = L(\mathbf{E},\mathbf{F})$ and in general, $\bigwedge^k(\mathbf{E},\mathbf{F}) = L_a^k(\mathbf{E};\mathbf{F})$, be the vector space of skew symmetric $\mathbf{F}$-valued multilinear maps or exterior $\mathbf{F}$-valued k-forms on $\mathbf{E}$. If $\mathbf{F}=\mathbb{R}$, we write $\bigwedge^0(\mathbf{E})=\mathbb{R}$, $\bigwedge^1(\mathbf{E})=\mathbf{E}^*$ and $\bigwedge^k(\mathbf{E}) = L_a^k(\mathbf{E};\mathbb{R})$; elements of $\bigwedge^k(\mathbf{E})$ are called ***exterior k-forms***. Some authors write $\bigwedge^k(\mathbf{E}^*)$ where we write $\bigwedge^k(\mathbf{E})$.

To form elements of $\bigwedge^k(\mathbf{E},\mathbf{F})$ from elements of $T_k^0(\mathbf{E};\mathbf{F})$, we can skew-symmetrize the latter. For example, if $t\in T_2^0(\mathbf{E})$, the two tensor $\mathbf{A}t$ defined by

$$(\mathbf{A}t)\,(e_1,e_2) = \frac{1}{2}\left[t\,(e_1,e_2) - t\,(e_2,e_1)\right]$$

is skew symmetric and if t is already skew, $\mathbf{A}t$ coincides with t. More generally, we make the following definition.

6.1.1 Definition. *The* ***alternation mapping*** $\mathbf{A} : T_k^0(\mathbf{E},\mathbf{F})\to T_k^0(\mathbf{E},\mathbf{F})$ *(for notational simplicity we do not index the* $\mathbf{A}$ *with* $\mathbf{E}$, $\mathbf{F}$ *or* k*) is defined by*

$$\mathbf{A}t(e_1,\ldots,e_k) = \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sign}\sigma)\, t(e_{\sigma(1)},\ldots,e_{\sigma(k)}),$$

where the sum is over all $k!$ *elements of* S_k*.*

6.1.2 Proposition. $\mathbf{A}$ *is a linear mapping onto* $\bigwedge^k(\mathbf{E},\mathbf{F})$, $\mathbf{A}|\bigwedge^k(\mathbf{E},\mathbf{F})$ *is the identity, and* $\mathbf{A}\circ\mathbf{A}=\mathbf{A}$.

Proof. Linearity of $\mathbf{A}$ is clear from the definition. If $t\in\bigwedge^k(\mathbf{E},\mathbf{F})$, then

$$\begin{aligned}\mathbf{A}t(e_1,\ldots,e_k) &= \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sign}\sigma)\, t(e_{\sigma(1)},\ldots,e_{\sigma(k)})\\ &= \frac{1}{k!}\sum_{\sigma\in S_k} t(e_1,\ldots,e_k) = t(e_1,\ldots,e_k)\end{aligned}$$

since S_k has order $k!$. This proves the first two assertions, and the last follows from them. ■

From $\mathbf{A}=\mathbf{A}\circ\mathbf{A}$, it follows that $\|\mathbf{A}\|\le\|\mathbf{A}\|^2$, and so, as $\mathbf{A}\neq 0$, $\|\mathbf{A}\|\ge 1$. From the definition of $\mathbf{A}$, we see $\|\mathbf{A}\|\le 1$; thus $\|\mathbf{A}\|=1$. In particular, $\mathbf{A}$ is continuous.

6.1.3 Definition. *If* $\alpha\in T_k^0(\mathbf{E})$ *and* $\beta\in T_l^0(\mathbf{E})$*, define their* ***wedge product*** $\alpha\wedge\beta\in\bigwedge^{k+l}(\mathbf{E})$ *by*

$$\alpha\wedge\beta = \frac{(k+l)!}{k!l!}\mathbf{A}(\alpha\otimes\beta).$$

For $\mathbf{F}$-valued forms, we can also define $\wedge$, where $\otimes$ is taken with respect to a given bilinear form $B\in L(\mathbf{F}_1,\mathbf{F}_2;\mathbf{F}_3)$. Since $\mathbf{A}$ and $\otimes$ are continuous, so is $\wedge$. There are several possible conventions for defining the wedge product $\wedge$. The one here conforms to Spivak [1979], and Bourbaki [1971] but not to Kobayashi and Nomizu [1963] or Guillemin and Pollack [1974]. See Exercise 6.1-7 for the possible conventions. Our definition of $\alpha\wedge\beta$ is the one that eliminates the largest number of constants encountered later.

A (k, l)-***shuffle*** is a permutation σ of $\{1, 2, \ldots, k + l\}$ such that

$$\sigma(1) < \cdots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \cdots < \sigma(k+l).$$

The reason for the name "shuffles" is that these are the kind of permutations made when a deck of $k + l$ cards is shuffled, with k cards held in one hand and l in the other.

The reader should prove that for α a k-form and β an l-form, we have

$$\begin{aligned}(\alpha \wedge \beta)&(e_1, \ldots, e_{k+l}) \\ &= \sum (\operatorname{sign} \sigma) \alpha(e_{\sigma(1)}, \ldots, e_{\sigma(k)}) \beta(e_{\sigma(k+1)}, \ldots, e_{\sigma(k+l)})\end{aligned} \tag{6.1.1}$$

where the sum is over all (k, l) shuffles σ. Formula (6.1.1) is a convenient way to compute wedge products, as we see in the following examples.

6.1.4 Examples.

A. If α is a two-form and β is a one-form, then

$$(\alpha \wedge \beta)(e_1, e_2, e_3) = \alpha(e_1, e_2)\beta(e_3) - \alpha(e_1, e_3)\beta(e_2) + \alpha(e_2, e_3)\beta(e_1)$$

Indeed the only $(2, 1)$ shuffles in S_3 are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

of which only the second one has sign -1.

B. If α and β are one-forms, then

$$(\alpha \wedge \beta)(e_1, e_2) = \alpha(e_1)\beta(e_2) - \alpha(e_2)\beta(e_1)$$

since S_2 consists of two $(1, 1)$ shuffles. ♦

6.1.5 Proposition. *For $\alpha \in T_k^0(\mathbf{E})$, $\beta \in T_l^0(\mathbf{E})$, and $\gamma \in T_m^0(\mathbf{E})$, we have*

(i) $\alpha \wedge \beta = \mathbf{A}\alpha \wedge \beta = \alpha \wedge \mathbf{A}\beta$;

(ii) $\wedge$ *is bilinear*;

(iii) $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$;

(iv) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma = \dfrac{(k+l+m)!}{k!l!m!} \mathbf{A}(\alpha \otimes \beta \otimes \gamma)$.

Proof. For (i), first note that if $\sigma \in S_k$ and we define

$$\sigma t\left(e_1, \ldots, e_k\right) = t\left(e_{\sigma(1)}, \ldots, e_{\sigma(k)}\right),$$

then $\mathbf{A}(\sigma t) = (\operatorname{sign} \sigma)\mathbf{A}t$, because

$$\begin{aligned}\mathbf{A}(\sigma t)(e_1, \ldots, e_k) &= \frac{1}{k!} \sum_{\rho \in S_k} (\operatorname{sign} \rho)\, t(e_{\rho\sigma(1)}, \ldots, e_{\rho\sigma(k)}) \\ &= \frac{1}{k!} \sum_{\rho \in S_k} (\operatorname{sign} \sigma)(\operatorname{sign} \rho\sigma)\, t(e_{\rho\sigma(1)}, \ldots, e_{\rho\sigma(k)}) \\ &= (\operatorname{sign} \sigma)\mathbf{A}t(e_1, \ldots, e_k)\end{aligned}$$

since $\rho \mapsto \rho\sigma$ is a bijection. Therefore, since

$$\mathbf{A}t = \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sign}\sigma)\sigma t,$$

we get

$$\begin{aligned}\mathbf{A}(\mathbf{A}\alpha\otimes\beta) &= \mathbf{A}\left(\frac{1}{k!}\sum_{\tau\in S_k}(\operatorname{sign}\tau)(\tau\alpha\otimes\beta)\right)\\ &= \frac{1}{k!}\sum_{\tau\in S_k}(\operatorname{sign}\tau)\mathbf{A}(\tau\alpha\otimes\beta) \qquad \text{(by linearity of } \mathbf{A})\\ &= \frac{1}{k!}\sum_{\tau\in S_k}(\operatorname{sign}\tau')\mathbf{A}\tau'(\alpha\otimes\beta),\end{aligned}$$

where $\tau' \in S_{k+l}$ is defined by

$$\tau'(1,\dots,k,\dots,k+l) = (\tau(1),\dots,\tau(k),k+1,\dots,k+l),$$

so sign $\tau =$ sign τ' and $\tau\alpha\otimes\beta = \tau'(\alpha\otimes\beta)$. Thus the preceding expression for $\mathbf{A}(\mathbf{A}\alpha\otimes\beta)$ becomes

$$\frac{1}{k!}\sum_{\tau\in S_k}(\operatorname{sign}\tau')(\operatorname{sign}\tau')\mathbf{A}(\alpha\otimes\beta) = \mathbf{A}(\alpha\otimes\beta)\frac{1}{k!}\sum_{\tau\in S_k}1 = \mathbf{A}(\alpha\otimes\beta).$$

Thus, $\mathbf{A}(\mathbf{A}\alpha\otimes\beta) = \mathbf{A}(\alpha\otimes\beta)$ which is equivalent to $(\mathbf{A}\alpha)\wedge\beta = \alpha\wedge\beta$. The other equality in (i) is similar. (ii) is clear since $\otimes$ is bilinear and $\mathbf{A}$ is linear.

For (iii), let $\sigma_0 \in S_{k+l}$ be given by

$$\sigma_0\,(1,\dots,k+l) = (k+1,\dots,k+l,1,\dots,k)\,.$$

Then

$$(\alpha\otimes\beta)\,(e_1,\dots,e_{k+l}) = (\beta\otimes\alpha)\left(e_{\sigma_0(1)},\dots,e_{\sigma_0(k+l)}\right).$$

Hence, by the proof of (i), $\mathbf{A}(\alpha\otimes\beta) = (\operatorname{sign}\sigma_0)\mathbf{A}(\beta\otimes\alpha)$. But sign $\sigma_0 = (-1)^{kl}$. Finally, for (iv),

$$\begin{aligned}\alpha\wedge(\beta\wedge\gamma) &= \frac{(k+l+m)!}{k!(l+m)!}\mathbf{A}(\alpha\otimes(\beta\wedge\gamma))\\ &= \frac{(k+l+m)!}{k!(l+m)!}\frac{(l+m)!}{l!m!}\mathbf{A}(\alpha\otimes\mathbf{A}(\beta\otimes\gamma))\\ &= \frac{(k+l+m)!}{k!l!m!}\mathbf{A}(\alpha\otimes\beta\otimes\gamma)\end{aligned}$$

since $\mathbf{A}(\alpha\otimes\mathbf{A}\beta) = \mathbf{A}(\alpha\otimes\beta)$, which was proved in (i), and by associativity of $\otimes$. We calculate $(\alpha\wedge\beta)\wedge\gamma$ in the same way. ■

Conclusions (i)–(iii) hold (with identical proofs) for $\mathbf{F}$-valued forms when the wedge product is taken with respect to a given bilinear mapping B. Associativity can also be generalized under suitable assumptions on the bilinear mappings, such as requiring $\mathbf{F}$ to be an associative algebra under B. Because of associativity, $\alpha\wedge\beta\wedge\gamma$ can be written with no ambiguity.

6.1.6 Examples.

A. If α^i, $i = 1, \ldots, k$ are one-forms, then

$$\begin{aligned}\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right)(e_1, \ldots, e_k) &= \sum_\sigma (\operatorname{sign}\sigma)\, \alpha^1\left(e_{\alpha(1)}\right) \cdots \alpha^k\left(e_{\alpha(k)}\right) \\ &= \det\left[\alpha^i\left(e_j\right)\right].\end{aligned}$$

Indeed, repeated application of Proposition 6.1.5(iv) gives

$$\gamma^1 \wedge \cdots \wedge \gamma^k = \frac{(d_1 + \cdots + d_k)!}{d_1! \cdots d_k!} \mathbf{A}(\gamma^1 \otimes \cdots \otimes \gamma^k), \tag{6.1.2}$$

where γ_i is a d_i-form on $\mathbf{E}$. In particular, if α_i is a one form, equation 6.1.2 gives

$$\alpha_1 \wedge \cdots \wedge \alpha_k = k! \mathbf{A}(\alpha_1 \otimes \cdots \otimes \alpha_k), \tag{6.1.2'}$$

which yields the stated formula after using the definition of $\mathbf{A}$. If $e_1, \ldots, e_n$ and $e^1, \ldots, e^n$ are dual bases, observe that as a special case,

$$\left(e^1 \wedge \cdots \wedge e^k\right)(e_1, \ldots, e_k) = 1.$$

B. If at least one or α or β is of even degree, then Proposition 6.1.5(iii) says that $\alpha \wedge \beta = \beta \wedge \alpha$. If both are of odd degree, then $\alpha \wedge \beta = -\beta \wedge \alpha$. Thus, if α is a one-form, then $\alpha \wedge \alpha = 0$. But if α is a two-form, then in general $\alpha \wedge \alpha \neq 0$. For example, if $\alpha = e^1 \wedge e^2 + e^3 \wedge e^4 \in \bigwedge^2\left(\mathbb{R}^4\right)$ where e^1, e^2, e^3, e^4 is the standard dual basis of $\mathbb{R}^4$, then $\alpha \wedge \alpha = 2e^1 \wedge e^2 \wedge e^3 \wedge e^4 \neq 0$.

C. The properties listed in Proposition 6.1.5 make the computations of wedge products similar to polynomial multiplication, care being taken with commutativity. For example, if $\alpha^1, \ldots, \alpha^5$ are one forms on $\mathbb{R}^5$,

$$\alpha = 2\alpha^1 \wedge \alpha^3 + \alpha^2 \wedge \alpha^3 - 3\alpha^3 \wedge \alpha^4 \in \bigwedge^2\left(\mathbb{R}^5\right)$$

and

$$\beta = -\alpha^1 \wedge \alpha^2 \wedge \alpha^5 + 2\alpha^1 \wedge \alpha^3 \wedge \alpha^4 \in \bigwedge^3\left(\mathbb{R}^5\right),$$

then the wedge product $\alpha \wedge \beta$ is computed using the bilinearity and commutation properties of $\wedge$:

$$\begin{aligned}\alpha \wedge \beta = &-2(\alpha^1 \wedge \alpha^3) \wedge (\alpha^1 \wedge \alpha^2 \wedge \alpha^5) - (\alpha^2 \wedge \alpha^3) \wedge (\alpha^1 \wedge \alpha^2 \wedge \alpha^5) \\ &+ 3(\alpha^3 \wedge \alpha^4) \wedge (\alpha^1 \wedge \alpha^2 \wedge \alpha^5) + 4(\alpha^1 \wedge \alpha^3) \wedge (\alpha^1 \wedge \alpha^3 \wedge \alpha^4) \\ &+ 2(\alpha^2 \wedge \alpha^3) \wedge (\alpha^1 \wedge \alpha^3 \wedge \alpha^4) - 6(\alpha^3 \wedge \alpha^4) \wedge (\alpha^1 \wedge \alpha^3 \wedge \alpha^4) \\ = &\ 3\alpha^3 \wedge \alpha^4 \wedge \alpha^1 \wedge \alpha^2 \wedge \alpha^5 = 3\alpha^1 \wedge \alpha^2 \wedge \alpha^3 \wedge \alpha^4 \wedge \alpha^5.\end{aligned}$$

♦

To express the wedge product in coordinate notation, suppose $\mathbf{E}$ is finite dimensional with basis $e_1, \ldots, e_n$. The components of $t \in T_k^0(\mathbf{E})$ are the real numbers

$$t_{i_1 \cdots i_k} = t(e_{i_1}, \ldots, e_{i_k}), \quad 1 \leq i_1, \ldots, i_k \leq n. \tag{6.1.3}$$

For $t \in \bigwedge^k(\mathbf{E})$, equation (6.1.3) is antisymmetric in its indices $i_1, \ldots, i_k$. For example, $t \in \bigwedge^2(\mathbf{E})$ yields t_{ij}, a skew symmetric $n \times n$ matrix. From Definition 6.1.1 of the alternation mapping and equation 6.1.3, we have

$$(\mathbf{A}t)_{i_1 \ldots i_k} = \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sign} \sigma) t_{\sigma(i_1) \cdots \sigma(i_k)};$$

that is, $\mathbf{A}t$ antisymmetrizes the components of t. For example, if $t \in T_2^0(\mathbf{E})$, then

$$(\mathbf{A}t)_{ij} = \frac{t_{ij} - t_{ji}}{2}.$$

If $\alpha \in \bigwedge^k(\mathbf{E})$ and $\beta \in \bigwedge^l(\mathbf{E})$, then equations (6.1.1) and (6.1.3) yield

$$(\alpha \wedge \beta)_{i_1 \cdots i_{k+l}} = \sum (\operatorname{sign} \sigma) \alpha_{\sigma(i_1) \ldots \sigma(i_k)} \beta_{\sigma(i_{k+1}) \ldots \sigma(i_{k+l})}$$

where the sum is over all the (k, l)-shuffles in S_{k+l}.

6.1.7 Definition. *The direct sum of the spaces $\bigwedge^k(\mathbf{E})$ $(i = 0, 1, 2, \ldots)$ together with its structure of real vector space and multiplication induced by $\wedge$, is called the **exterior algebra** of $\mathbf{E}$, or the **Grassmann algebra** of $\mathbf{E}$. It is denoted by $\bigwedge(\mathbf{E})$.*

Thus $\bigwedge(\mathbf{E})$ is a ***graded associative algebra***, that is, an algebra in which every element has a degree (a k-form has degree k), and the degree map is additive on products (by Proposition 6.1.2 and Definition 6.1.3). Elements of $\bigwedge(\mathbf{E})$ may be written as finite sums of increasing degree exactly as one writes a polynomial as a sum of monomials. Thus if $a, b, c \in \mathbb{R}$, $\alpha \in \bigwedge^1(\mathbf{E})$ and $\beta \in \bigwedge^2(\mathbf{E})$ then $a + b\alpha + c\beta$ makes sense in $\bigwedge(\mathbf{E})$. The one-form α can be understood as an element of $\bigwedge^1(\mathbf{E})$ and also of $\bigwedge(\mathbf{E})$, where α is identified with $0 + \alpha + 0 + 0 + \cdots$.

6.1.8 Proposition. *Suppose $\mathbf{E}$ is finite dimensional and $n = \dim \mathbf{E}$. Then for $k > n$, $\bigwedge^k(\mathbf{E}) = \{0\}$, while for $0 < k \leq n$, $\bigwedge^k(\mathbf{E})$ has dimension $n!/(n-k)!k!$. The exterior algebra over $\mathbf{E}$ has dimension 2^n. If $\{e_1, \ldots, e_n\}$ is an (ordered) basis of $\mathbf{E}$ and $\{e^1, \ldots, e^n\}$ its dual basis, a basis of $\bigwedge^k(\mathbf{E})$ is*

$$\left\{ e^{i_1} \wedge \cdots \wedge e^{i_k} \mid 1 \leq i_1 < i_k < \cdots < i_k \leq n \right\}. \tag{6.1.4}$$

Proof. First we show that the indicated wedge products span $\bigwedge^k(\mathbf{E})$. If $\alpha \in \bigwedge^k(\mathbf{E})$, then from Proposition 5.1.2,

$$\alpha = \alpha(e_{i_1}, \ldots, e_{i_k}) e^{i_1} \otimes \cdots \otimes e^{i_k},$$

where the summation convention indicates that this should be summed over all choices of $i_1, \ldots, i_k$ between 1 and n. If the linear operator $\mathbf{A}$ s applied to this sum and equation (6.1.2) is used, we get

$$\begin{aligned} \alpha = \mathbf{A}\alpha &= \alpha(e_{i_1}, \ldots, e_{i_k}) \mathbf{A}(e^{i_1} \otimes \cdots \otimes e^{i_k}) \\ &= \alpha(e_{i_1}, \ldots, e_{i_k}) \frac{1}{k!} e^{i_1} \wedge \cdots \wedge e^{i_k}. \end{aligned}$$

The sum still runs over all choices of the $i_1, \ldots, i_k$ and we want only distinct, ordered ones. However, since α is skew symmetric, the coefficient in α is 0 if $i_1, \ldots, i_k$ are not distinct. If they are distinct and $\sigma \in S_k$, then

$$\alpha(e_{i_1}, \ldots, e_{i_k}) e^{i_1} \wedge \cdots \wedge e^{i_k} = \alpha(e_{\sigma(i_1)}, \ldots, e_{\sigma(i_k)}) e^{\sigma(i_1)} \wedge \cdots \wedge e^{\sigma(i_k)},$$

since both α and the wedge product change by a factor of sign σ. Since there are $k!$ of these rearrangements, we are left with

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha(e_{i_1}, \ldots, e_{i_k}) e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

This shows that equation (6.1.4) spans $\bigwedge^k(\mathbf{E})$.

Secondly, we show that the elements in equation (6.1.4) are linearly independent. Suppose that

$$\sum_{i_1<\cdots<i_k} \alpha_{i_1,\ldots,i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} = 0.$$

For fixed $i'_1, \ldots, i'_k$, let $j'_{k+1}, \ldots, j'_n$ denote the complementary set of indices, $j'_{k+1} < \cdots < j'_n$. Then

$$\sum_{i_1<\cdots<i_k} \alpha_{i_1\cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{j'_{k+1}} \wedge \cdots \wedge e^{j'_n} = 0.$$

However, this reduces to

$$\alpha_{i'_1\cdots i'_k} e^1 \wedge \cdots \wedge e^n = 0.$$

But $e^1 \wedge \cdots \wedge e^n \neq 0$, as $(e^1 \wedge \cdots \wedge e^n)(e_1, \ldots, e_n) = 1$ by Example 6.1.6A. Hence the coefficients are zero. ■

6.1.9 Corollary. *If* $\dim \mathbf{E} = n$, *then* $\dim \bigwedge^n(\mathbf{E}) = 1$. *If* $\{\alpha^1, \ldots, \alpha^n\}$ *is a basis for* $\mathbf{E}^*$, *then* $\alpha^1 \wedge \cdots \wedge \alpha^n$ *spans* $\bigwedge^n(\mathbf{E})$.

Proof. This follows from Proposition 6.1.8. ■

6.1.10 Corollary. *Let* $\alpha^1, \ldots, \alpha^k \in \mathbf{E}^*$. *Then* $\alpha^1, \ldots, \alpha^k$ *are linearly dependent iff* $\alpha^1 \wedge \cdots \wedge \alpha^n$ *spans* $\bigwedge^n(\mathbf{E})$.

Proof. If $\alpha^1, \ldots, \alpha^k$ are linearly dependent, then

$$\alpha^i = \sum_{j\neq 1} c_j \alpha^j$$

for some i. Since $\alpha \wedge \alpha = 0$, for α a one-form, we see that $\alpha^1 \wedge \cdots \wedge \alpha^k = 0$. Conversely, if $\alpha^1 \wedge \cdots \wedge \alpha^k = 0$, then by Corollary 6.1.9, $\alpha^1, \ldots, \alpha^k$ is not a basis for span $\{\alpha^1, \ldots, \alpha^k\}$. Therefore $k > \dim(\operatorname{span}\{\alpha^1, \ldots, \alpha^k\})$ and so $\alpha^1, \ldots, \alpha^k$ are linearly dependent. ■

6.1.11 Corollary. *Let* $\theta \in \bigwedge^1(\mathbf{E})$ *and* $\alpha \in \bigwedge^k(\mathbf{E})$. *Then* $\theta \wedge \alpha = 0$ *iff there exists* $\beta \in \bigwedge^{k-1}(\mathbf{E})$ *such that* $\alpha = \theta \wedge \beta$.

Proof. Clearly, if $\alpha = \theta \wedge \beta$, then $\theta \wedge \alpha = 0$. Conversely, assume $\theta \wedge \alpha = 0$, $\theta \neq 0$ and choose a basis $\{e_i\}_{i\in I}$ of $\mathbf{E}$ such that for some $k \in I$, $e^k = \theta$. If

$$\sum_{i_1<\cdots<i_k} \alpha_{i_1\ldots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} = 0.$$

From $\theta \wedge \alpha = 0$ it follows that all summands not involving e^k are zero. Now factor e^k out of the remaining terms and call the resulting $(k-1)$-form β. ■

6.1.12 Examples.

A. Let $\mathbf{E} = \mathbb{R}^2$, $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$ and $\{e^1, e^2\}$ the dual basis. Any element ω of $\bigwedge^1(\mathbb{R}^2)$ can be written uniquely as $\omega = \omega_1 e^1 + \omega_2 e^2$, and any element ω of $\bigwedge(\mathbb{R}^2)$ can be written uniquely as $\omega = \omega_{12} e^1 \wedge e^2$.

B. Let $\mathbf{E} = \mathbb{R}^3$, $\{e_1, e_2, e_3\}$ be the standard basis, and $\{e^1, e^2, e^3\}$ the dual basis. Any element $\omega \in \bigwedge^1(\mathbb{R}^3)$ can be written uniquely as

$$\omega = \omega_1 e^1 + \omega_2 e^2 + \omega_3 e^3.$$

Similarly, any elements $\eta \in \bigwedge^2(\mathbb{R}^3)$ and $\xi \in \bigwedge^3(\mathbb{R}^3)$ can be uniquely written as

$$\eta = \eta_{12} e^1 \wedge e^2 + \eta_{13} e^1 \wedge e^3 + \eta_{23} e^2 \wedge e^3$$

and

$$\xi = \xi_{123} e^1 \wedge e^2 \wedge e^3.$$

C. Since $\mathbb{R}^3$, $\bigwedge^1(\mathbb{R}^3)$, and $\bigwedge^2(\mathbb{R}^3)$ all have the same dimension, they are isomorphic. An isomorphism $\mathbb{R}^3 \cong \bigwedge^1(\mathbb{R}^3) = (\mathbb{R}^3)^*$ is the standard one associated to a given basis: $e_i \mapsto e^i$, $i = 1, 2, 3$. An isomorphism of $\bigwedge^1(\mathbb{R}^3)$ with $\bigwedge^2(\mathbb{R}^3)$ is determined by

$$e^1 \mapsto e^2 \wedge e^3, \quad e^2 \mapsto e^3 \wedge e^1, \quad \text{and} \quad e^3 \mapsto e^1 \wedge e^2.$$

This isomorphism is usually denoted by $* : \bigwedge^1(\mathbb{R}^3) \mapsto \bigwedge^2(\mathbb{R}^3)$; we shall study this map in general in the next section under the name ***Hodge star operator***.

The standard isomorphism of $\mathbb{R}^3$ with $\bigwedge^1(\mathbb{R}^3) = (\mathbb{R}^3)^*$ is given by the index lowering action $^\flat$ of the standard metric on $\mathbb{R}^3$; that is, $^\flat(e_i) = e^i$. Then $* \circ {}^\flat : \mathbb{R}^3 \to \bigwedge^2(\mathbb{R}^3)$ has the following property:

$$\left(* \circ {}^\flat\right)(e \times f) = {}^\flat(e) \wedge {}^\flat(f) \tag{6.1.5}$$

for all $v, w \in \mathbb{R}^3$, where $\times$ denotes the usual cross-product of vectors; that is,

$$v \times w = \left(v^2 w^3 - v^3 w^2\right) e_1 + \left(v^3 w^1 - v^1 w^3\right) e_2 + \left(v^1 w^2 - v^2 w^1\right) e_3.$$

The relation (6.1.5) follows from the definitions and the fact that if $\alpha = \alpha_1 e^1 + \alpha_2 e^2 + \alpha_3 e^3$ and $\beta = \beta_1 e^1 + \beta_2 e^2 + \beta_3 e^3$, then

$$\begin{aligned}\alpha \wedge \beta = {} & (\alpha_2 \beta_3 - \alpha_3 \beta_2) e^2 \wedge e^3 \\ & + (\alpha_3 \beta_1 - \alpha_1 \beta_3) e^3 \wedge e^1 + (\alpha_1 \beta_2 - \alpha_2 \beta_1) e^1 \wedge e^2.\end{aligned}$$

♦

Exercises

◇ **6.1-1.** Compute $\alpha \wedge \alpha$, $\alpha \wedge \beta$, $\beta \wedge \beta$, and $\beta \wedge \alpha \wedge \beta$ for $\alpha = 2e^1 \wedge e^3 - e^2 \wedge e^3 \in \bigwedge^2(\mathbb{R}^3)$ and $\beta = -e^1 + e^2 - 2e^3$ where $\{e^1, e^2, e^3\}$ is a basis of $(\mathbb{R}^3)^*$.

◇ **6.1-2.** If $k!$ is omitted in the definition of $\mathbf{A}$ in Definition 6.1.1, show that $\wedge$ fails to be associative.

◇ **6.1-3.** Let $v_1, \ldots, v_k$ be linearly dependent vectors. Show that for each $\alpha \in \bigwedge^r(\mathbf{E})$, we have $\alpha(v_1, \ldots, v_k) = 0$.

◇ **6.1-4.** Let $\mathbf{E}$ be finite dimensional. Show that $\bigwedge^k(\mathbf{E}^*)$ is isomorphic to $\left(\bigwedge^k(\mathbf{E})\right)^*$.

Hint: Define $\varphi : \left(\bigwedge^k(\mathbf{E})\right)^* \to \bigwedge^k(\mathbf{E}^*)$ by

$$\varphi(\sigma)\left(\alpha^1, \ldots, \alpha^k\right) = \sigma\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right)$$

and construct its inverse using the basis in Proposition 6.1.8.

◇ **6.1-5.** Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbf{E}$ with dual basis $\{e^1, \ldots, e^n\}$ and let $\{f_1, \ldots, f_m\}$ be a basis of $\mathbf{F}$. Show the following:

(i) every $\beta \in \bigwedge^k(\mathbf{E}, \mathbf{F})$ can be uniquely written as $\beta = \Sigma_{1 \le i \le m} \beta_i f_i$ for $\beta_i \in \bigwedge^k(\mathbf{E})$, where

$$(\gamma f)(v_1, \ldots, v_k) = \gamma(v_1, \ldots, v_k) f \in \mathbf{F}$$

for $v_1, \ldots, v_k \in \mathbf{E}$, $f \in \mathbf{F}$, and $\gamma \in \bigwedge^k(\mathbf{E})$;

(ii) $\{ (e^{i_1} \wedge \cdots \wedge e^{i_k}) f_j \mid i_1 < \cdots < i_k \}$ is a basis of $\bigwedge^k(\mathbf{E}, \mathbf{F})$ and thus $\dim\left(\bigwedge^k(\mathbf{E}, \mathbf{F})\right) = \dfrac{mn!}{(n-k)!k!}$;

(iii) $\dim(\bigwedge(\mathbf{E}, \mathbf{F})) = m2^n$;

(iv) if $B \in L(\mathbb{R}, \mathbf{F}; \mathbf{F})$, where $B(t, f) = tf$ and $\wedge$ is the wedge product defined by B, regarded as a map $\wedge : \bigwedge^1(\mathbf{E}) \times \bigwedge^k(\mathbf{E}, \mathbf{F}) \to \bigwedge^{k+1}(\mathbf{E}, \mathbf{F})$ show that

$$\alpha \wedge \beta = \sum_{1 \le i \le m} (\alpha \wedge \beta_i) f_i.$$

If $\mathbf{E} = \mathbb{R}^3$, $\mathbf{F} = \mathbb{R}^2$,

$$\begin{aligned} &\alpha = e^1 \wedge e^2 - 2e^1 \wedge e^3, \text{ and} \\ &\beta = \left(e^1 \wedge e^3\right) f_1 + 2\left(e^2 \wedge e^3\right) f_2 - \left(e^1 \wedge e^2\right) f_3, \end{aligned}$$

compute $\alpha \wedge \beta$.

◇ **6.1-6.** Let $\{e_1, \ldots, e_k\}$ and $\{f_1, \ldots, f_k\}$ be linearly independent sets of vectors. Show that they span the same k-dimensional subspace iff

$$f_1 \wedge \cdots \wedge f_k = a e_1 \wedge \cdots \wedge e_k,$$

where $a \neq 0$. (Give a definition of $f_1 \wedge \cdots \wedge f_k$ as part of your answer.) Show that in fact

$$a = \det \varphi, \quad \text{where } \varphi : \operatorname{span}\{e_1, \ldots, e_k\} \to \operatorname{span}\{f_1, ..., f_k\}$$

is determined by $\varphi(e_i) = f_i, i = 1, \ldots, k$. Use this to relate $\bigwedge^k$ with G_k in Example 3.1.8G.

◇ **6.1-7** (P. Chernoff and J. Robbin)**.** Let $\wedge'$ be another wedge product on forms that is associative and satisfies $\alpha \wedge' \beta = c(k, l) \alpha \wedge \beta$, where α is a k-form and β is an one-form, $c(k, l)$ is a scalar, and forms of degree zero act as scalars.

(i) Prove the "cocycle identity" $c(k, l)c(k + l, m) = c(k, l + m)c(l, m)$.

(ii) Define $\psi(l)$ inductively by $\psi(0) = \psi(1) = 1$ and $\psi(l + 1) = c(1, l)\psi(l)$. Show that $c(k, l) = \psi(k + l)/\psi(k)\psi(l)$. Deduce that $c(k, l) = c(l, k)$; that is, $\wedge'$ satisfies $\alpha \wedge' \beta = (-1)^{kl} \beta \wedge' \alpha$ automatically.

(iii) Show that c given in (ii) yields an associative wedge product. ($\psi(k) = 1/k!$ converts our wedge product convention to that of Kobayashi and Nomizu [1963]).

6.2 Determinants, Volumes, and the Hodge Star Operator

According to linear algebra, the determinant of an $n \times n$ matrix is a skew-symmetric function of its rows or columns. Thus, if $x_1, \ldots, x_n \in \mathbb{R}^n$, and we define ω by

$$\omega(x_1, \ldots, x_n) = \det[x_1, \ldots, x_n]$$

where $[x_1, \ldots, x_n]$ denotes the $n \times n$ matrix whose columns are $x_1, \ldots, x_n$, then ω is an element of $\bigwedge^n(\mathbb{R}^n)$. We also recall from linear algebra that $\det[x_1, \ldots, x_n]$ is the oriented volume of the parallelepiped P spanned by $x_1, \ldots, x_n$ (Figure 6.2.1) and that if x_i has components x_i^j, the determinant is given by

$$\det[x_1, \ldots, x_n] = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) x^1_{\sigma(1)} \cdots x^n_{\sigma(s)}.$$

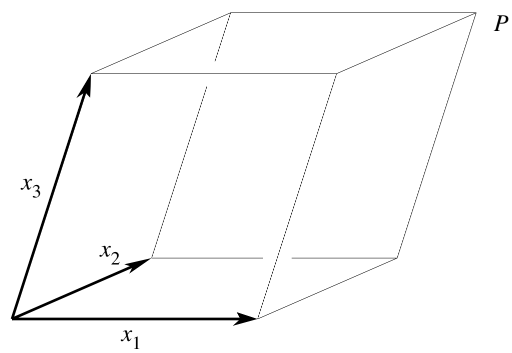

FIGURE 6.2.1. Volume$(P) = \det[x_1, x_2, x_3]$

In this section determinants and volumes are approached from the point of view of exterior algebra. Throughout this section $\mathbf{E}$ is assumed to be a finite-dimensional vector space and we denote its dimension by $\dim \mathbf{E} = n$.

If $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, it is shown in linear algebra that det φ is the oriented volume of the image of the unit cube under φ (see Figure 6.2.2). In fact det φ is a measure of how φ changes

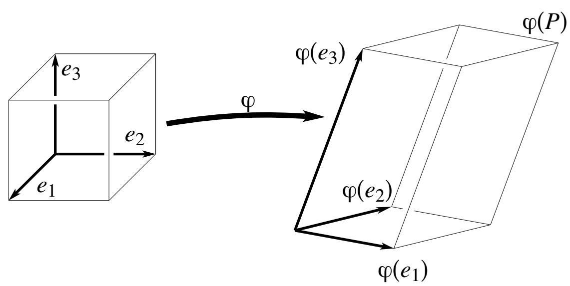

FIGURE 6.2.2. Image of a cube under a linear map

volumes. In advanced calculus, this fact is the basis for introducing the Jacobian determinant in the change of variables formula for multiple integrals. This background will lead the exposition to the development of the Jacobian determinant of a mapping of manifolds.

Definition of the Determinant. Recall that the pull-back $\varphi^*\alpha$ of $\alpha \in T^0_k(\mathbf{F})$ by $\varphi \in L(\mathbf{E}, \mathbf{F})$ is the element of $T^0_k(\mathbf{E})$ defined by

$$(\varphi^*\alpha)(e_1, \ldots, e_k) = \alpha(\varphi(e_1), \ldots, \varphi(e_k)).$$

If $\varphi \in \mathrm{GL}(\mathbf{E}, \mathbf{F})$, then $\varphi_* = (\varphi^{-1})^*$ denotes the push-forward. The following proposition is a consequence of the definitions and Proposition 5.1.9. (The same results hold for Banach space valued forms.)

6.2.1 Proposition. *Let $\varphi \in L(\mathbf{E}, \mathbf{F})$ and $\psi \in L(\mathbf{F}, \mathbf{G})$*

(i) $\varphi^* : T_k^0(\mathbf{F}) \to T_k^0(\mathbf{E})$ *is linear, and* $\varphi^*(\bigwedge(\mathbf{F})) \subset \bigwedge^k(\mathbf{E})$.

(ii) $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

(iii) *If φ is the identity, so is φ^*.*

(iv) *If* $\varphi \in \mathrm{GL}(\mathbf{E}, \mathbf{F})$, *then* $\varphi^* \in \mathrm{GL}\left(T_k^0(\mathbf{F}), T_k^0(\mathbf{F})\right)$, $\left(\varphi^{-1}\right)^* = \varphi_*$, *and* $(\varphi^*)^{-1} = \left(\varphi^{-1}\right)_*$; *if* $\psi \in \mathrm{GL}(\mathbf{F}, \mathbf{G})$, *then* $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

(v) *If* $\alpha \in \bigwedge^k(\mathbf{F})$ *and* $\beta \in \bigwedge^l(\mathbf{F})$, *then* $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$.

For example, if

$$\beta = \beta_{a_1 \ldots a_k} f^{a_1} \wedge \cdots \wedge f^{a_k} \in \bigwedge^k(\mathbf{F}) \quad (\text{sum over } a_1 < \cdots < a_k)$$

and $\varphi \in L(\mathbf{E}, \mathbf{F})$ is given by the matrix $[A_i^a]$, that is, relative to ordered bases $\{e_1, \ldots, e_n\}$ of $\mathbf{E}$ and $\{f_1, \ldots, f_m\}$ of $\mathbf{F}$, one has $\varphi(e_i) = A_i^a f_a$, then

$$\begin{aligned}(\varphi^*\beta) &= \beta_{a_1 \cdots a_k} \varphi^*(f^{a_1}) \wedge \cdots \wedge \varphi^*(f^{a_k}) \quad (\text{sum over } a_1 < \cdots < a_k) \\ &= \beta_{a_1 \cdots a_k} A_{j_1}^{a_1} e^{j_1} \wedge \cdots \wedge A_{j_k}^{a_k} e^{j_k} \\ &= \beta_{a_1 \cdots a_k} A_{j_1}^{a_1} \cdots A_{j_k}^{a_k} e^{j} \wedge \cdots \wedge e^{j_k} \\ &= k! \beta_{a_1 \cdots a_k} A_{j_1}^{a_1} \cdots A_{j_k}^{a_k} e^{j_1} \wedge \cdots \wedge e^{j_k}, j_1 < \cdots < j_k.\end{aligned}$$

Recall that $\varphi^* : \bigwedge^n(\mathbf{E}) \to \bigwedge^n(\mathbf{E})$ is a linear mapping and $\bigwedge^n(\mathbf{E})$ is one-dimensional. Thus, if ω_0 is a basis and $\omega = c\omega_0$, then $\varphi^*\omega = c\varphi^*\omega_0 = b\omega$ for some constant b, clearly unique.

6.2.2 Definition. *Let* $\dim(\mathbf{E}) = n$ *and* $\varphi \in L(\mathbf{E}, \mathbf{E})$. *The unique constant* $\det \varphi$, *such that* $\varphi^* : \bigwedge^n(\mathbf{E}) \to \bigwedge^n(\mathbf{E})$ *satisfies*

$$\varphi^*\omega = (\det \varphi)\omega$$

for all $\omega \in \bigwedge^n(\mathbf{E})$ *is called the **determinant** of* φ.

The definition shows that the determinant does not depend on the choice of basis of $\mathbf{E}$, nor does it depend on a norm on $\mathbf{E}$. To compute $\det \varphi$, choose a basis $\{e_1, \ldots, e_n\}$ of $\mathbf{E}$ with dual basis $\{e^1, \ldots, e^n\}$. Let $\varphi \in L(\mathbf{E}, \mathbf{E})$ have the matrix $[A_i^j]$; that is, $\varphi(e_i) = \Sigma_{1 \le j \le n} A_i^j e_j$. By Example 6.1.6A,

$$\begin{aligned}\varphi^*(e^1 \wedge \cdots \wedge e^n)(e_1, \ldots, e_n) &= (e^1 \wedge \cdots \wedge e^n)(\varphi(e_1), \ldots, \varphi(e_n)) \\ &= \det[e^j(\varphi(e_i))] = \det[A_i^j].\end{aligned}$$

Since $(e^1 \wedge \cdots \wedge e^n)(e_1, \ldots, e_n) = 1$ we get $\det \varphi = \det[A_i^j]$, the classical expression of the determinant of a matrix with $x_1, \ldots, x_n$ as columns, where x_i has components A_i^j. Thus the definition of the determinant in Definition 6.2.2 coincides with the classical one.

Properties of the Determinant. From properties of pull-back, we deduce corresponding properties of the determinant, all of which are well known from linear algebra.

6.2.3 Proposition. *Let* $\varphi, \psi \in L(\mathbf{E}, \mathbf{E})$. *Then*

(i) $\det(\varphi \circ \psi) = (\det \varphi)(\det \psi)$;

(ii) *if φ is the identity,* $\det \varphi = 1$;

(iii) φ *is an isomorphism iff* $\det \varphi \neq 0$, *and in this case* $\det(\varphi^{-1}) = (\det \varphi)^{-1}$.

Proof. To prove (i), note first that $(\varphi \circ \psi)^*\omega = \det(\varphi \circ \psi)\omega$; but $(\varphi \circ \psi)^*\omega = (\psi^* \circ \varphi^*)\omega$. Hence,

$$(\varphi \circ \psi)^* \omega = \psi^*(\det \varphi)\omega = (\det \psi)(\det \varphi)\omega$$

so (i) follows. Part (ii) follows at once from the definition. For (iii), suppose φ is an isomorphism with inverse φ^{-1}. Therefore, by (i) and (ii),

$$1 = \det(\varphi \circ \varphi^{-1}) = (\det \varphi)(\det \varphi^{-1}),$$

and, in particular, $\det \varphi \neq 0$. Conversely, if φ is not an isomorphism there is an $e_1 \neq 0$ satisfying $\varphi(e_1) = 0$. Extend to a basis $\{e_1, e_2, \ldots, e_n\}$. Then for all n-forms ω, we have

$$(\varphi^*\omega)(e_1, \ldots, e_n) = \omega(0, \varphi(e_2), \ldots, \varphi(e_n)) = 0.$$

Hence, $\det \varphi = 0$. ■

In Chapter 2 we saw that if $\mathbf{E}$ and $\mathbf{F}$ are finite dimensional, one convenient norm giving the topology of $L(\mathbf{E}, \mathbf{F})$ is the operator norm:

$$\|\varphi\| = \sup\{\|\varphi(e)\| \mid \|e\| = 1\} = \sup\left\{\frac{\|\varphi(e)\|}{\|e\|} | e \neq 0\right\}$$

where $\|e\|$ is a norm on $\mathbf{E}$. (See §2.2.) Hence, for any $e \in \mathbf{E}$,

$$\|\varphi(e)\| \leq \|\varphi\| \, \|e\|.$$

6.2.4 Proposition. *The map* $\det : L(\mathbf{E}, \mathbf{E}) \to \mathbb{R}$ *is continuous.*

Proof. This is clear from the component formula for the determinant, but let us also give a coordinate free proof. Note that

$$\begin{aligned} \|\omega\| &= \sup\{ |\omega(e_1, \ldots, e_n)| \mid \|e_1\| = \cdots = \|e_n\| = 1 \} \\ &= \sup\{ |\omega(e_1, \ldots, e_n)| / \|e_1\| \cdots \|e_n\| \mid e_1, \ldots, e_n \neq 0 \} \end{aligned}$$

is a norm on $\bigwedge^n (\mathbf{E})$ and $|\omega(e_1, \ldots, e_n)| \leq \|\omega\| \, \|e_1\| \cdots \|e_n\|$. Then, for $\varphi, \psi \in L(\mathbf{E}, \mathbf{E})$,

$$\begin{aligned} &| \det \varphi - \det \psi| \, \|\omega\| \\ &\quad = \|\varphi^*\omega - \psi^*\omega\| \\ &\quad = \sup\{ |\omega(\varphi(e_1), \ldots, \varphi(e_n)) - \omega(\psi(e_1), \ldots, \psi(e_n))| \mid \|e_1\| = \cdots \\ &\qquad\qquad = \|e_n\| = 1 \} \\ &\quad \leq \sup\{ |\omega(\varphi(e_1) - \psi(e_1), \varphi(e_2), \ldots, \varphi(e_n))| + \cdots \\ &\qquad + |\omega(\psi(e_1), \psi(e_2), \ldots, \varphi(e_n) - \psi(e_n))| \mid \|e_1\| = \cdots = \|e_n\| = 1 \} \\ &\quad \leq \|\omega\| \, \|\varphi - \psi\| \{ \|\varphi\|^{n-1} + \|\varphi\|^{n-2}\|\psi\| + \cdots + \|\psi\|^{n-1} \} \\ &\quad \leq \|\omega\| \, \|\varphi - \psi\|(\|\varphi\| + \|\psi\|)^{n-1}. \end{aligned}$$

Consequently,

$$| \det \varphi - \det \psi| \leq \|\varphi - \psi\|(\|\varphi\| + \|\psi\|)^{n-1}$$

from which the result follows. ■

In Chapter 2 we saw that the set of isomorphisms of $\mathbf{E}$ to $\mathbf{F}$ form an open subset of $L(\mathbf{E}, \mathbf{F})$. Using the determinant, we can give an alternate proof in the finite-dimensional case.

6.2.5 Proposition. *Suppose that* $\mathbf{E}$ *and* $\mathbf{F}$ *are finite-dimensional and let* $\mathrm{GL}(\mathbf{E},\mathbf{F})$ *denote those* $\varphi \in L(\mathbf{E},\mathbf{F})$ *that are isomorphisms. Then* $\mathrm{GL}(\mathbf{E},\mathbf{F})$ *is an open subset of* $L(\mathbf{E},\mathbf{F})$*.*

Proof. If $\mathrm{GL}(\mathbf{E},\mathbf{F}) = \varnothing$, the conclusion is true. Otherwise, there is an isomorphism $\psi \in \mathrm{GL}(\mathbf{E},\mathbf{F})$. A map φ in $L(\mathbf{E},\mathbf{F})$ is an isomorphism if and only if $\psi^{-1}\circ\varphi$ is also. This happens precisely when $\det(\psi^{-1}\circ\varphi) \neq 0$. Therefore, $\mathrm{GL}(\mathbf{E},\mathbf{F})$ is the inverse image of $\mathbb{R}\backslash\{0\}$ under the map taking φ to $\det(\psi^{-1}\circ\varphi)$. Since this is continuous and $\mathbb{R}\backslash\{0\}$ is open, $\mathrm{GL}(\mathbf{E},\mathbf{F})$ is also open. ■

Orientation. The basis elements of $\bigwedge^n(\mathbf{E})$ enable us to define orientation or "handedness" of a vector space.

6.2.6 Definition. *The nonzero elements of the one-dimensional space* $\bigwedge^n(\mathbf{E})$ *are called* ***volume elements****. If* ω_1 *and* ω_2 *are volume elements, we say* ω_1 *and* ω_2 *are* ***equivalent*** *iff there is a* $c > 0$ *such that* $\omega_1 = c\omega_2$*. An equivalence class* $[\omega]$ *of volume elements on* $\mathbf{E}$ *is called an* ***orientation*** *on* $\mathbf{E}$*. An* ***oriented vector space*** $(\mathbf{E},[\omega])$ *is a vector space* $\mathbf{E}$ *together with an orientation* $[\omega]$ *on* $\mathbf{E}$*;* $[-\omega]$ *is called the* ***reverse orientation****. A basis* $\{e_1,\ldots,e_n\}$ *of the oriented vector space* $(\mathbf{E},[\omega])$ *is called* ***positively*** *(resp.,* ***negatively****) oriented, if* $\omega(e_1,\ldots,e_n) > 0$ *(resp.,* < 0*).*

The last statement is independent of the representative of the orientation $[\omega]$, for if $\omega' \in [\omega]$, then $\omega' = c\omega$ for some $c > 0$, and thus $\omega'(e_1,\ldots,e_n)$ and $\omega(e_1,\ldots,e_n)$ have the same sign. Also note that a vector space $\mathbf{E}$ has exactly two orientations: one given by selecting an arbitrary dual basis $\{e^1,\ldots,e^n\}$ and taking $[e^1\wedge\cdots\wedge e^n]$; the other is its reverse orientation.

This definition of orientation is related to the concept of orientation from calculus as follows. In $\mathbb{R}^3$, a right-handed coordinate system like the one in Figure 6.2.1 is by convention positively oriented, as are all other right-handed systems. On the other hand, any left-handed coordinate system, obtained for example from the one in Figure 6.2.1 by interchanging x_1 and x_2, is by convention negatively oriented. Thus one would call a positive orientation in $\mathbb{R}^3$ the set of all right-handed coordinate systems. The key to the abstraction of this construction for any vector space lies in the observation that the determinant of the change of ordered basis of two right-handed systems in $\mathbb{R}^3$ is always strictly positive. Thus, if $\mathbf{E}$ is an n-dimensional vector space, define an equivalence relation on the set of ordered bases in the following way: two bases $\{e_1,\ldots,e_n\}$ and $\{e'_1,\ldots,e'_n\}$ are equivalent iff $\det\varphi > 0$, where $\varphi \in \mathrm{GL}(\mathbf{E})$ is given by $\varphi(e_i) = e'_i$, $i = 1,\ldots,n$. We can relate n-forms to the bases by associating to a basis $\{e_1,\ldots,e_n\}$ and its dual basis $\{e^1,\ldots,e^n\}$ the n-form $\omega = e^1\wedge\cdots\wedge e^n$. The following proposition shows that this association gives an identification of the corresponding equivalence classes.

6.2.7 Proposition. *An orientation in a vector space is uniquely determined by an equivalence class of ordered bases.*

Proof. If $[\omega]$ is an orientation of $\mathbf{E}$ there exists a basis $\{e_1,\ldots,e_n\}$ such that $\omega(e_1,\ldots,e_n) \neq 0$ since $\omega \neq 0$ in $\bigwedge^n(\mathbf{E})$. Changing the sign of e_1 if necessary, we can find a basis that is positively oriented. Let $\{e'_1,\ldots,e'_n\}$ be an equivalent basis and $\varphi \in \mathrm{GL}(\mathbf{E})$, defined by $\varphi(e_i) = e'_i$, $i = 1,\ldots,n$ be the change of basis isomorphism. Then if $\omega' \in [\omega]$, there exists $c > 0$ such that $\omega' = c\omega$, so we get

$$\begin{aligned}\omega'(e'_1,\ldots,e'_n) &= c\omega(\varphi(e_1),\ldots,\varphi(e_n)) = c(\varphi^*\omega)(e_1,\ldots,e_n)\\ &= c(\det\varphi)\omega(e_1,\ldots,e_n) > 0.\end{aligned}$$

That is, $[\omega]$ uniquely determines the equivalence class of $\{e_1,\ldots,e_n\}$.

Conversely, let $\{e_1,\ldots,e_n\}$ be a basis of $\mathbf{E}$ and let $\omega = e^1\wedge\cdots\wedge e^n$, where $\{e^1,\ldots,e^n\}$ is the dual basis. As before, $\omega'(e'_1,\ldots,e'_n) > 0$ for any $\omega' \in [\omega]$ and $\{e'_1,\ldots,e'_n\}$ equivalent to $\{e_1,\ldots,e_n\}$; thus, the equivalence class of the ordered basis $\{e_1,\ldots,e_n\}$ uniquely determines the orientation $[\omega]$. ■

Volume Elements in Inner Product Spaces. An important point is that to get a particular volume element on $\mathbf{E}$ requires additional structure, although the determinant does not. The idea is based on the fact that in $\mathbb{R}^3$ the volume of the parallelepiped $P = P(x_1, x_2, x_3)$ spanned by three positively oriented vectors x_1, x_2, and x_3 can be expressed independent of any basis as

$$\operatorname{Vol}(P) = (\det[\langle x_i, x_j \rangle])^{1/2},$$

where $[\langle x_i, x_j \rangle]$ denotes the symmetric 3×3 matrix whose entries are $\langle x_i, x_j \rangle$. If x_1, x_2, and x_3 are negatively oriented, $\det[\langle x_i, x_j \rangle] < 0$ and so the formula has to be modified to

$$\operatorname{Vol}(P) = (|\det[\langle x_i, x_j \rangle]|)^{1/2}. \tag{6.2.1}$$

Densities. The above argument suggests that besides the volumes, there are quantities involving absolute values of volume elements that are also important. This leads to the notion of densities.

6.2.8 Definition. *Let α be a real number. A continuous mapping $\rho : \mathbf{E} \times \cdots \times \mathbf{E} \to \mathbb{R}$ (n factors of $\mathbf{E}$ for $\mathbf{E}$ an n-dimensional vector space) is called an α-**density** if*

$$\rho(\varphi(v_1), \ldots, \varphi(v_n)) = |\det \varphi|^\alpha \rho(v_1, \ldots, v_n),$$

*for all $v_1, \ldots, v_n \in \mathbf{E}$ and all $\varphi \in L(\mathbf{E}, \mathbf{E})$. Let $|\bigwedge|^\alpha(\mathbf{E})$ denote the α-densities on $\mathbf{E}$. With $\alpha = 1$, 1-densities on $\mathbf{E}$ are simply called **densities** and $|\bigwedge|^1(\mathbf{E})$ is denoted by $|\bigwedge|(\mathbf{E})$.*

The determinant of φ in this definition is taken with respect to any volume element of $\mathbf{E}$. As we saw in Definition 6.2.2, this is independent of the choice of the volume element. Note that $|\bigwedge|^\alpha(\mathbf{E})$ is one-dimensional. Indeed, if ρ_1 and $\rho_2 \in |\bigwedge|^\alpha(\mathbf{E})$, $\rho_1 \neq 0$, and $\{e_1, \ldots, e_n\}$ is a basis of $\mathbf{E}$, then $\rho_2(e_1, \ldots, e_n) = a\rho_1(e_1, \ldots, e_n)$, for some constant $a \in \mathbb{R}$. If $v_1, \ldots, v_n \in \mathbf{E}$, let $v_i = \varphi(e_i)$, defining $\varphi \in L(\mathbf{E}, \mathbf{E})$. Then

$$\begin{aligned}\rho_2(v_1, \ldots, v_n) &= |\det \varphi|^\alpha \rho_2(e_1, \ldots, e_n) \\ &= a|\det \varphi|^\alpha \rho_1(e_1, \ldots, e_n) = a\rho_1(v_1, \ldots, v_n);\end{aligned}$$

that is, $\rho_2 = a\rho_1$.

Alpha-densities can be constructed from volume elements as follows. If $\omega \in \bigwedge^n(\mathbf{E})$, define $|\omega|^\alpha \in |\bigwedge|^\alpha(\mathbf{E})$ by

$$|\omega|^\alpha (e_1, \ldots, e_n) = |\omega(e_1, \ldots, e_n)|^\alpha$$

where $e_1, \ldots, e_n \in \mathbf{E}$. This association defines an isomorphism of $\bigwedge^n(\mathbf{E})$ with $|\bigwedge|^\alpha(\mathbf{E})$. Thus one often uses the notation $|\omega|^\alpha$ for α-densities.

Volume Elements in Inner Product Spaces. We shall construct canonical volume elements (and hence α-densities) for vector spaces carrying a bilinear symmetric nondegenerate covariant two-tensor, and in particular for inner product spaces. First we recall a fact from linear algebra.

6.2.9 Proposition. *Let $\mathbf{E}$ be an n-dimensional vector space and $g = \langle\,,\rangle \in T^0_2(\mathbf{E})$ be symmetric and of rank r; that is, the map $e \in \mathbf{E} \mapsto g(e, \cdot) \in \mathbf{E}^*$ has r-dimensional range. Then there is an ordered basis $\{e_1, \ldots, e_n\}$ of $\mathbf{E}$ with dual basis $\{e^1, \ldots, e^n\}$ such that*

$$g = \sum_{i=1}^{r} c_i e^i \otimes e^i,$$

where $c_i = \pm 1$ and $r \le n$, or equivalently, the matrix of g is

$$\begin{bmatrix} c_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & c_r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

*This basis $\{e_1, \ldots, e_n\}$ is called a g-**orthonormal basis**. Moreover, the number of basis vectors for which $g(e_i, e_i) = 1$ (resp., $g(e_i, e_i) = -1$) is unique and equals the maximal dimension of any subspace on which g is positive (resp., negative) definite. The number $s =$ the number of $+1$'s minus the number of -1's is called the **signature** of g. The number of -1's is called the **index** of g and is denoted $\operatorname{Ind}(g)$.*

Proof (Gram–Schmidt argument). Since g is symmetric, the following polarization identity holds:

$$g(e, f) = \frac{1}{4} g(e + f, e + f) - g(e - f, e - f).$$

Thus if $g \neq 0$, there is an $e_1 \in \mathbf{E}$ such that $g(e_1, e_1) \neq 0$. Rescaling, we can assume $c_1 = g(e_1, e_1) = \pm 1$. Let $\mathbf{E}_1$ be the span of e_1 and $\mathbf{E}_2 = \{ e \in \mathbf{E} \mid g(e_1, e) = 0 \}$. Clearly $\mathbf{E}_1 \cap \mathbf{E}_2 = \{0\}$. Also, if $z \in \mathbf{E}$, then $z - c_1 g(z, e_1) e_1 \in \mathbf{E}_2$ so that $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ and thus $\mathbf{E} = \mathbf{E}_1 \oplus \mathbf{E}_2$. Now if $g \neq 0$ on $\mathbf{E}_2$, there is an $e_2 \in \mathbf{E}_2$ such that $g(e_2, e_2) = c_2 = \pm 1$. Continue inductively to complete the proof.

For the second part, in the basis $\{e_1, \ldots, e_n\}$ just found, let

$$\mathbf{E}_1 = \operatorname{span} \{ e_i \mid g(e_i, e_i) = 1 \}, \quad \mathbf{E}_2 = \operatorname{span} \{ e_i \mid g(e_i, e_i) = -1 \}$$

and

$$\ker g = \{ e \mid g(e, e') = 0 \text{ for all } e' \in \mathbf{E} \}.$$

Note that $\ker g = \operatorname{span}\{ e_i \mid g(e_i, e_i) = 0 \}$ and thus $\mathbf{E} = \mathbf{E}_1 \oplus \mathbf{E}_2 \oplus \ker g$. Let $\mathbf{F}$ be any subspace of $\mathbf{E}$ on which g is positive definite. Then clearly $\mathbf{F} \cap \ker g = \{0\}$. We also have $\mathbf{E}_2 \cap \mathbf{F} = \{0\}$ since any $v \in \mathbf{E}_2 \cap \mathbf{F}$, $v \neq 0$, must simultaneously satisfy $g(v, v) > 0$ and $g(v, v) < 0$. Thus $\mathbf{F} \cap (\mathbf{E}_2 \oplus \ker g) = \{0\}$ and consequently $\dim \mathbf{F} \le \dim \mathbf{E}_1$. A similar argument shows that $\dim \mathbf{E}_2$ is the maximal dimension of any subspace of $\mathbf{E}$ on which g is negative definite. ■

Note that the number of ones in the diagonal representation of g is $(r+s)/2$ and the number of minus-ones is $\operatorname{Ind}(g) = (r - s)/2$. Nondegeneracy of g means that $r = n$. In this case $e \in \mathbf{E}$ may be written

$$e = \sum_{i=1,\ldots,n} \frac{g(e, e_i)}{c_i} e_i,$$

where $c_i = g(e_i, e_i) = \pm 1$ and $\{e_i\}$ is a g-orthonormal basis. For g a positive definite inner product, $r = n$ and $\operatorname{Ind}(g) = 0$; for g a Lorentz inner product $r = n$ and $\operatorname{Ind}(g) = 1$.

6.2.10 Proposition. *Let $\mathbf{E}$ be an n-dimensional vector space and $g \in T^0_2(\mathbf{E})$ be nondegenerate and symmetric.*

(i) *If $[\omega]$ is an orientation of $\mathbf{E}$ there exists a unique volume element $\mu = \mu(g) \in [\omega]$, called the g-**volume**, such that $\mu(e_1, \ldots, e_n) = 1$ for all positively oriented g-orthonormal bases $\{e_1, \ldots, e_n\}$ of $\mathbf{E}$. In fact, if $\{e^1, \ldots, e^n\}$ is the dual basis, then $\mu = e^1 \wedge \cdots \wedge e^n$. More generally, if $\{f_1, \ldots, f_n\}$ is a positively oriented basis with dual basis $\{f^1, \ldots, f^n\}$, then*

$$\mu = |\det [g(f_i, f_j)]|^{1/2} f^1 \wedge \cdots \wedge f^n.$$

(ii) *There is a unique α-density $|\mu|^\alpha$, called the g-α-**density**, with the property that*

$$|\mu|^\alpha(e_1, \ldots, e_n) = 1$$

for all g-orthonormal bases $\{e_1, \ldots, e_n\}$ of $\mathbf{E}$. If $\{e^1, \ldots, e^n\}$ is the dual basis, then $|\mu|^\alpha = |e^1 \wedge \cdots \wedge e^n|^\alpha$. More generally, if $v_1, \ldots, v_n \in \mathbf{E}$ are positively oriented, then

$$|\mu|^\alpha (v_1, \ldots, v_n) = |\det [g(v_i, v_j)]|^{\alpha/2}.$$

Proof. First we establish a relation between the determinants of the following three matrices: $[g(e_i, e_j)] = \operatorname{diag}(c_1, \ldots, c_n)$ (see Proposition 6.2.9), $[g(f_i, f_j)]$ for an arbitrary basis $\{f_1, \ldots, f_n\}$, and the matrix representation of $\varphi \in \mathrm{GL}(\mathbf{E})$ where $\varphi(e_i) = f_i = A_i^j e_j$. By Proposition 6.2.9, we have

$$\begin{aligned} g(f_i, f_j) &= \left(\sum_{p=1}^n c_p e^p \otimes e^p\right)\left(A_i^k e_k, A_j^l e_l\right) \\ &= c_p \delta_k^p \delta_l^p A_i^k A_j^l = c_p A_i^p A_j^p \qquad \text{(sum on } p). \end{aligned}$$

Thus,

$$\det[g(f_i, f_j)] = (\det\varphi)^2 \det[g(e_i, e_j)]. \tag{6.2.2}$$

By Proposition 6.2.9, $|\det[g(e_i, e_j)]| = 1$.

(i) Clearly if $\{e_1, \ldots, e_n\}$ is positively oriented and g-orthonormal, then $\mu(e_1, \ldots, e_n) = 1$ uniquely determines $\mu \in [\omega]$ by multilinearity. Suppose that $\{f_1, \ldots, f_n\}$ is another positively oriented g-orthonormal basis. If $\varphi \in \mathrm{GL}(\mathbf{E})$ where $\varphi(e_i) = f_i$, $i = 1, \ldots, n$, then by equation (6.2.2) and Proposition 6.2.9, it follows that $|\det\varphi| = 1$. But

$$0 < \mu(f_1, \ldots, f_n) = (\varphi^*\mu)(e_1, \ldots, e_n) = \det\varphi,$$

so that $\det\varphi = 1$. The second statement in (i) follows from the third.

For the third statement of (i), note that by equation (6.2.2)

$$\mu(f_1, \ldots, f_n) = \det\varphi = |\det[g(f_i, f_j)]|^{1/2}.$$

(ii) follows from (i) and the remarks following Definition 6.2.8. ■

A covariant symmetric nondegenerate two-tensor g on $\mathbf{E}$ induces one on $\bigwedge^k(\mathbf{E})$ for every $k = 1, \ldots, n$ in the following way. Let

$$\alpha = \alpha_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \quad \text{and} \quad \beta = \beta_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \in \textstyle\bigwedge^k(\mathbf{E}),$$

(sum over $i_1 < \cdots < i_k$) and let

$$\beta^{i_1 \cdots i_k} = g^{i_1 j_1} \cdots g^{i_k j_k} \beta_{i_1 \cdots j_k}$$

(sum over *all* $j_i, \ldots, j_k$) be the components of the associated contravariant k-tensor, where $[g^{kj}]$ denotes the inverse of the matrix with entries $g_{ij} = g(e_i, e_j)$. Then put

$$g^{(k)}(\alpha, \beta) = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \cdots i_k} \beta^{i_1 \cdots i_k}. \tag{6.2.3}$$

If there is no danger of confusion, we will write $\langle \alpha, \beta \rangle = g^{(k)}(\alpha, \beta)$. We now show that this definition does not depend on the basis. If $\{f_1, \ldots, f_n\}$ is another ordered basis of $\mathbf{E}$, let

$$\alpha = \alpha'_{a_i \cdots a_k} f^{a_1} \wedge \cdots \wedge f^{a_k} \quad \text{and} \quad \beta = \beta'_{a_1 \cdots a_k} f^{a_1} \wedge \cdots \wedge f^{a_k}.$$

The the identity map on $\mathbf{E}$ has matrix representation relative to the bases $\{e_1,\ldots,e_n\}$ and $\{f_1,\ldots,f_n\}$ given by $e_i = A^a_i f_a$. If $B = A^{-1}$ we have by Proposition 5.1.7,

$$\begin{aligned}\alpha'_{a_1\cdots a_k}\beta'^{a_1\cdots a_k} &= \alpha_{i_1\cdots i_k}B^{i_1}_{a_1}\cdots B^{i_k}_{a_k}A^{a_1}_{j_1}\cdots A^{a_k}_{j_k}\beta^{j_1\cdots j_k}\\ &= \alpha_{a_1\cdots a_k}\delta^{i_1}_{j_1}\cdots\delta^{i_k}_{j_k}\beta^{i_1\cdots j_k} = \alpha_{i_1\cdots i_k}\beta^{j_1\cdots j_k}.\end{aligned}$$

So defined, $g^{(k)}$ is clearly bilinear. It is also symmetric since

$$\begin{aligned}\beta_{i_1\cdots i_k}\alpha^{i_1\cdots i_k} &= g_{i_1j_1}\cdots g_{i_kj_k}\beta^{j_1\cdots j_k}g^{i_1l_1}\ldots g^{i_kl_k}\alpha_{l_1\cdots l_k}\\ &= \delta^{l_1}_{j_1}\cdots\delta^{l_k}_{j_k}\alpha_{l_1\cdots l_k}\beta^{j_1\cdots j_k} = \alpha_{j_1\cdots j_k}\beta^{j_1\cdots j_k},\end{aligned}$$

where $[g^{ij}] = [g_{ij}]^{-1}$, and $g_{ij} = g(e_i, e_j)$. Notice that $g^{(k)}$ is also nondegenerate since if $g^{(k)}(\alpha,\beta) = 0$ for all $\beta \in \bigwedge^k(\mathbf{E})$, choosing for β all elements of a basis, show that $\alpha_{i_1\cdots i_k} = 0$, that is, that $\alpha = 0$. The following has thus been proved.

6.2.11 Proposition. *A nondegenerate symmetric covariant two-tensor $g = \langle\,,\rangle$ on the finite-dimensional vector space $\mathbf{E}$ induces a similar tensor on $\bigwedge^k(\mathbf{E})$ for all $k = 1,\ldots,n$. Moreover, if $\{e_1,\ldots,e_n\}$ is a g-orthonormal basis of $\mathbf{E}$ in which*

$$g = \sum_{i=1}^n c_i e^i \otimes e^i, \quad c_i = \pm 1,$$

then the basis

$$\{\, e^{i_1}\wedge\cdots\wedge e^{i_k} \mid i_1 < \cdots < i_k \,\}$$

is orthonormal with respect to $g^{(k)} = \langle\,,\rangle$, and

$$\langle e^{i_1}\wedge\cdots\wedge e^{i_k}, e^{i_1}\wedge\cdots\wedge e^{i_k}\rangle = c_{i_1}\cdots c_{i_k}\ (=\pm 1). \tag{6.2.4}$$

Hodge Star Operator. This operator will be introduced with the aid of the g-volume μ on $\mathbf{E}$.

6.2.12 Proposition. *Let $\mathbf{E}$ be an oriented n-dimensional vector space and $g = \langle\,,\rangle \in T^0_2(\mathbf{E})$ a given symmetric and nondegenerate tensor. Let μ be the corresponding volume element of $\mathbf{E}$. Then there is a unique isomorphism $* : \bigwedge^k(\mathbf{E}) \to \bigwedge^{n-k}(\mathbf{E})$ satisfying*

$$\alpha \wedge *\beta = \langle\alpha,\beta\rangle\mu \quad \textit{for } \alpha,\beta \in \textstyle\bigwedge^k(\mathbf{E}). \tag{6.2.5}$$

If $\{e_1,\ldots,e_n\}$ is a positively oriented g-orthonormal basis of $\mathbf{E}$ and $\{e^1,\ldots,e^n\}$ is its dual basis, then

$$*(e^{\sigma(1)}\wedge\cdots\wedge e^{\sigma(k)}) = c_{\sigma(1)}\ldots c_{\sigma(k)}\operatorname{sign}(\sigma)(e^{\sigma(k+1)}\wedge\cdots\wedge e^{\sigma(n)}) \tag{6.2.6}$$

where $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$.

Proof. First uniqueness is proved. Let $*$ satisfy equation (6.2.5) and let $\beta = e^{\sigma(1)}\wedge\cdots\wedge e^{\sigma(k)}$ and α be one of the g-orthonormal basis vectors

$$e^{i_1}\wedge\cdots\wedge e^{i_k} \text{ of } \textstyle\bigwedge^k(\mathbf{E}), \quad i_1 < \cdots < i_k.$$

By equation (6.2.5), $\alpha\wedge *\beta = 0$ unless $(i_1,\ldots,i_k) = (\sigma(1),\ldots,\sigma(k))$. Thus,

$$*\beta = a e^{\sigma(k+1)}\wedge\cdots\wedge e^{\sigma(n)}$$

for a constant a. But then $\beta\wedge *\beta = a\operatorname{sign}(\sigma)\mu$ and by equation (6.2.4), $\langle\beta,\beta\rangle = c_{\sigma(1)}\ldots c_{\sigma(k)}$. Hence $a = c_{\sigma(1)}\ldots c_{\sigma(k)}\operatorname{sign}(\sigma)$ and so $*$ must satisfy equation (6.2.6). Thus $*$ is unique.

Define $*$ by equation (6.2.6), recalling that $e^{\sigma(1)}\wedge\cdots\wedge e^{\sigma(k)}$ for $\sigma(1) < \cdots < \sigma(k)$ forms a $g^{(k)}$-orthonormal basis of $\bigwedge^k(\mathbf{E})$. As before, equation (6.2.5) is then verified using this basis. Clearly $*$ defined by equation (6.2.6) is an isomorphism, as it maps the g-orthonormal basis of $\bigwedge^k(\mathbf{E})$ to that of $\bigwedge^{n-k}(\mathbf{E})$. ■

6.2.13 Proposition. *Let* $\mathbf{E}$ *be an oriented* n*-dimensional vector space,* $g = \langle\, , \rangle \in T_2^0(\mathbf{E})$ *symmetric and nondegenerate of signature* s*, and* μ *the associated* g*-volume of* $\mathbf{E}$*. The Hodge star operator satisfies the following properties for* $\alpha, \beta \in \bigwedge^k(\mathbf{E})$*:*

$$\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle \mu, \tag{6.2.7}$$

$$*1 = \mu, \; *\mu = (-1)^{\mathrm{Ind}(g)}, \tag{6.2.8}$$

$$** \alpha = (-1)^{\mathrm{Ind}(g)}(-1)^{k(n-k)}\alpha, \tag{6.2.9}$$

$$\langle \alpha, \beta \rangle = (-1)^{\mathrm{Ind}(g)} \langle *\alpha, *\beta \rangle. \tag{6.2.10}$$

Proof. Equation (6.2.7) follows from equation (6.2.5) by symmetry of $\langle \alpha, \beta \rangle$. Equations (6.2.8) follow directly from equation (6.2.6), with $k = 0, n$, respectively, and $\sigma =$ identity (note that $c_1 \ldots c_n = (-1)^{\mathrm{Ind}(g)}$). To verify equation (6.2.9), it suffices to take $\alpha = e^{\sigma(1)} \wedge \cdots \wedge e^{\sigma(k)}$. By equation (6.2.6),

$$*(e^{\sigma(k+1)} \wedge \cdots \wedge e^{\sigma(n)}) = b e^{\sigma(1)} \wedge \cdots \wedge e^{\sigma(k)}$$

for a constant b. To find b use equation (6.2.5) with $\alpha = \beta = e^{\sigma(k+1)} \wedge \cdots \wedge e^{\sigma(n)}$ to give (see equation (6.2.4))

$$b e^{\sigma(k+1)} \wedge \cdots \wedge e^{\sigma(n)} \wedge e^{\sigma(1)} \wedge \cdots \wedge e^{\sigma(k)} = c_{\sigma(k+1)} \ldots c_{\sigma(n)} \mu.$$

Hence $b = c_{\sigma(k+1)} \ldots c_{\sigma(n)} (-1)^{k(n-k)} \operatorname{sign}(\sigma)$. Thus, equation (6.2.6) implies

$$\begin{aligned} ** (e^{\sigma(1)} \wedge \cdots \wedge e^{\sigma(n)}) &= c_{\sigma(1)} \ldots c_{\sigma(k)} (\operatorname{sign}(\sigma)) * (e^{\sigma(k+1)} \wedge \cdots \wedge e^{\sigma(n)}) \\ &= c_{\sigma(1)} \ldots c_{\sigma(k)} c_{\sigma(k+1)} \ldots c_{\sigma(n)} (\operatorname{sign}(\sigma))^2 \\ &\quad (-1)^{k(n-k)} e^{\sigma(1)} \wedge \cdots \wedge e^{\sigma(k)} \\ &= (-1)^{\mathrm{Ind}(g)} (-1)^{k(n-k)} e^{\sigma(1)} \wedge \cdots \wedge e^{\sigma(k)}. \end{aligned}$$

Finally for equation (6.2.10), we use equations (6.2.7) and (6.2.9) to give

$$\begin{aligned} \langle *\alpha, *\beta \rangle \mu &= *\alpha \wedge ** \beta = (-1)^{\mathrm{Ind}(g)} (-1)^{k(n-k)} * \alpha \wedge \beta \\ &= (-1)^{\mathrm{Ind}(g)} \beta \wedge *\alpha = (-1)^{\mathrm{Ind}(g)} \langle \alpha, \beta \rangle \mu. \end{aligned}$$ ■

6.2.14 Examples.

A. The Hodge operator on $\bigwedge^1(\mathbb{R}^3)$ where $\mathbb{R}^3$ has the standard metric and dual basis is given from equation (6.2.6) by $*e^1 = e^2 \wedge e^3, *e^2 = -e^1 \wedge e^3$, and $*e^3 = e^1 \wedge e^2$. (This is the isomorphism considered in Example 6.1.12B.)

B. Using equation (6.2.5), we compute $*$ in an arbitrary oriented basis. Write

$$*(e^{i_1} \wedge \cdots \wedge e^{i_k}) = c^{i_1 \cdots j_k}_{j_{k+1} \cdots j_n} e^{j_{k+1}} \wedge \cdots \wedge e^{j_n}$$

(sum over $j_{k+1} < \cdots < j_n$) and apply equation (6.2.5) with

$$\beta = e^{i_1} \wedge \cdots \wedge e^{i_k} \quad \text{and} \quad \alpha = e^{j_1} \wedge \cdots \wedge e^{j_k}$$

where $\{j_1, \ldots, j_k\}$ is a complementary set of indices to $\{j_{k+1}, \ldots, j_n\}$. One gets

$$c^{i_1 \cdots i_k}_{j_{k+1} \cdots j_n} = g^{i_1 j_1} \cdots g^{i_k j_k} |\det[g_{ij}]|^{1/2} \operatorname{sign} \begin{pmatrix} 1 \cdots n \\ j_1 \cdots j_n \end{pmatrix}.$$

Hence

$$\begin{aligned} * \left(e^{i_1} \wedge \cdots \wedge e^{i_k}\right) & \\ = |\det[g_{ij}]|^{1/2} \sum \operatorname{sign} & \begin{pmatrix} 1 \cdots n \\ j_1 \cdots j_n \end{pmatrix} g^{i_1 j_1} \cdots g^{i_k j_k} e^{j_{k+1}} \wedge \cdots \wedge e^{j_n}, \end{aligned} \tag{6.2.11}$$

where the sum is over all $(k, n-k)$ shuffles

$$\begin{pmatrix} 1 \cdots n \\ j_1 \cdots j_n \end{pmatrix}.$$

C. In particular, if $k = 1$, equation (6.2.11) yields

$$*e^i = |\det([g_{ij}])|^{1/2} \sum_{j=1}^{n} (-1)^{j-1} g^{ij} e^1 \wedge \cdots \wedge \hat{e}^j \wedge \cdots \wedge e^n \tag{6.2.12}$$

since $\operatorname{sign}(j_1, j_2, \ldots, j_n) = (-1)^{j-1}$, for $j_2 < \cdots < j_n$, $j_1 = j$, and where $\hat{e}^j$ means that e^j is deleted.

D. From B we can compute the components of $*\alpha$, where $\alpha \in \bigwedge^k(\mathbf{E})$, relative to any oriented basis: write $\alpha = \alpha_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}$ (sum over $i_1 < \cdots < i_k$) and apply equation (6.2.11) to give

$$(*\alpha) = |\det[g_{ij}]|^{1/2} \sum \operatorname{sign} \begin{pmatrix} 1 \cdots n \\ j_1 \cdots j_n \end{pmatrix} \alpha_{i_1 \cdots i_k} g^{i_1 j_1} \cdots g^{i_k j_k} e^{j_{k+1}} \wedge \cdots \wedge e^{j_n}.$$

Hence

$$(*\alpha)_{j_{k+1} \cdots j_n} = |\det[g_{ij}]|^{1/2} \sum \alpha_{i_1 \cdots i_k} g^{i_1 j_1} \cdots g^{i_k j_k} \operatorname{sign} \begin{pmatrix} 1 \cdots n \\ j_1 \cdots j_n \end{pmatrix} \tag{6.2.13}$$

for $j_{k+1} < \cdots < j_n$ and where the sum is over all complementary indices $j_1 < \cdots < j_k$.

E. Consider $\mathbb{R}^4$ with the Lorentz inner product, which in the standard basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$ has the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Let $\{e^1, e^2, e^3, e^4\}$ be the dual basis. The Hodge operator on $\bigwedge^1(\mathbb{R}^4)$ is given by

$$\begin{aligned} *e^1 &= e^2 \wedge e^3 \wedge e^4, & *e^2 &= -e^1 \wedge e^3 \wedge e^4, \\ *e^3 &= e^1 \wedge e^2 \wedge e^4, & *e^4 &= e^1 \wedge e^2 \wedge e^3, \end{aligned}$$

and on $\bigwedge^2(\mathbb{R}^4)$ by

$$\begin{aligned} *(e^1 \wedge e^2) &= e^3 \wedge e^4, & *(e^1 \wedge e^3) &= -e^2 \wedge e^4, & *(e^2 \wedge e^3) &= e^1 \wedge e^4 \\ *(e^1 \wedge e^4) &= -e^2 \wedge e^3, & *(e^2 \wedge e^4) &= e^1 \wedge e^3, & *(e^3 \wedge e^4) &= -e^1 \wedge e^2. \end{aligned}$$

If $\mathbb{R}^4$ had been endowed with the usual Euclidean inner product, the formulas for $*e^4$, $*(e^1 \wedge e^4)$, $*(e^2 \wedge e^4)$, and $*(e^3 \wedge e^4)$ would have opposite signs. The Hodge $*$ operator on $\bigwedge^3(\mathbb{R}^4)$ follows from the formulas on $\bigwedge^1(\mathbb{R}^4)$ and the fact that for $\beta \in \bigwedge^1(\mathbb{R}^4)$, $**\beta = \beta$ (from formula (6.2.9)). Thus we obtain

$$\begin{aligned} *(e^2 \wedge e^3 \wedge e^4) &= e^1, & *(e^1 \wedge e^3 \wedge e^4) &= -e^2, \\ *(e^1 \wedge e^2 \wedge e^4) &= e^3, & *(e^1 \wedge e^2 \wedge e^3) &= e^4. \end{aligned}$$

F. If β is a one form and $v_1, v_2, \ldots, v_n$ is a positively oriented orthonormal basis, then

$$(*\beta)\,(v_2, \ldots, v_n) = \beta\,(v_1)\,.$$

This follows from equation (6.2.5) taking $\alpha = v^1$, the first element in the dual basis and using the orthonormality of $v_1, \ldots, v_n$. ♦

Exercises

◇ **6.2-1.** Let $\{e^1, e^2, e^3\}$ be the standard dual basis of $\mathbb{R}^3$ and

$$\alpha = e^1 \wedge e^2 - 2e^2 \wedge e^3 \in \bigwedge^2(\mathbb{R}^3), \quad \beta = 3e^1 - e^2 + 2e^3 \in \bigwedge^1(\mathbb{R}^3),$$

and $\varphi \in L\left(\mathbb{R}^2, \mathbb{R}^3\right)$ have the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}$$

Compute $\varphi^*\alpha$. With the aid of the standard metrics in $\mathbb{R}^2$ and $\mathbb{R}^3$, compute $*\alpha$, $*\beta$, $*(\varphi^*\alpha)$, and $*(\varphi^*\beta)$. do you get any equalities? Explain.

◇ **6.2-2.** A map $\varphi \in L(\mathbf{E}, \mathbf{F})$, where $(\mathbf{E}, \omega), (\mathbf{F}, \mu)$ are oriented vector spaces with chosen volume elements, is called ***volume preserving*** if $\varphi^*\mu = \omega$. Show that if $\mathbf{E}$ and $\mathbf{F}$ have the same (finite) dimension, then φ is an isomorphism.

◇ **6.2-3.** A map $\varphi \in L(\mathbf{E}, \mathbf{F})$, where $(\mathbf{E}, [\mu])$ and $(\mathbf{F}, [\omega])$ are oriented vector spaces, is called ***orientation preserving*** if $\varphi^*\mu \in [\omega]$. If $\dim \mathbf{E} = \dim \mathbf{F}$, and φ is orientation preserving, show that φ is an isomorphism. Given an example for $\mathbf{F} = \mathbf{E} = \mathbb{R}^3$ of an orientation-preserving but not volume-preserving map.

◇ **6.2-4.** Let $\mathbf{E}$ and $\mathbf{F}$ be n-dimensional real vector spaces with nondegenerate symmetric two-tensors, $g \in T^0_2(\mathbf{E})$ and $h \in T^0_2(\mathbf{F})$. Then $\varphi \in L(\mathbf{E}, \mathbf{F})$ is called an ***isometry*** if $h(\varphi(e), \varphi(e')) = g(e, e')$ for all $e, e' \in \mathbf{E}$.

(i) Show that an isometry is an isomorphism.

(ii) Consider on $\mathbf{E}$ and $\mathbf{F}$ the g- and h-volumes $\mu(g)$ and $\mu(h)$. Show that if φ is an orientation-preserving isometry, then φ^* commutes with the Hodge star operator, that is, the following diagram commutes:

$$\begin{array}{ccc} \bigwedge^k(\mathbf{F}) & \xrightarrow{\;\;*\;\;} & \bigwedge^{n-k}(\mathbf{F}) \\ \varphi^* \downarrow & & \downarrow \varphi^* \\ \bigwedge^k(\mathbf{E}) & \xrightarrow[\;\;*\;\;]{} & \bigwedge^{n-k}(\mathbf{E}) \end{array}$$

If φ is orientation reversing, show that $*(\varphi^*\alpha) = -\varphi^*(*\alpha)$ for $\alpha \in \bigwedge^k(\mathbf{F})$.

◇ **6.2-5.** Let g be an inner product and $\{f_1, f_2, f_3\}$ be a positively oriented basis of $\mathbb{R}^3$. Denote by $^\flat$ and $^\#$ the index lowering and raising actions defined by g.

(i) Show that for any vectors $u, v \in \mathbb{R}^3$

$$[*(u^\flat \wedge v^\flat)]^\# = \operatorname{sign}\begin{pmatrix} 1\,2\,3 \\ i\,j\,k \end{pmatrix} |\det[g(f_a, f_b)]|^{1/2} u^i v^j g^{kl} f_l.$$

(ii) Show that if g is the standard dot-product in $\mathbb{R}^3$ the formula in (i) reduces to the cross-product of u and v.

(iii) Generalize (i) to define the cross-product of $n-1$ vectors $u_1, \ldots, u_{n-1}$ in an oriented n-dimensional inner product space $(\mathbf{E}, g)$, and find its coordinate expression.

◇ **6.2-6.** Let $\mathbf{E}$ be an n-dimensional oriented vector space and let $g \in T_2^0(\mathbf{E})$ be symmetric and non-degenerate of signature s. Using the g-volume, define the Hodge star operator $* : \bigwedge^k(\mathbf{E}; \mathbf{F}) \to \bigwedge^{n-k}(\mathbf{E}; \mathbf{F})$, where $\mathbf{F}$ is another finite-dimension vector space by

$$*\alpha = (*\alpha^i) f_i,$$

where $\alpha^i \in \bigwedge^k(\mathbf{E})$, $\{f_1, \ldots, f_m\}$ is a basis of $\mathbf{F}$ and $\alpha = \alpha^i f_i$. Show the following.

(i) The definition is independent of the basis of $\mathbf{F}$.

(ii) $** = (-1)^{(n-s)/2+k(n-k)}$ on $\bigwedge^k(\mathbf{E}; \mathbf{F})$.

(iii) If $h \in T_2^0(\mathbf{F})$ and if we let $h'(f, \alpha) = (*a^i) h(f, f_i)$, then $*h'(f, \alpha) = h'(f, *\alpha)$.

(iv) If $\wedge$ is the wedge product in $\bigwedge(\mathbf{E}; \mathbf{F})$ with respect to a given bilinear form on $\mathbf{F}$, then for $\alpha, \beta \in \bigwedge^k(\mathbf{E}; \mathbf{F})$,

$$(*\alpha) \wedge \beta = (*\beta) \wedge \alpha \quad \text{and} \quad \alpha \wedge (*\beta) = \beta \wedge (*\alpha).$$

(v) Show how g and h induce a symmetric nondegenerate covariant two-tensor on $\bigwedge^k(\mathbf{E}; \mathbf{F})$ and find formulas analogous to equations (6.2.7)–(6.2.10).

◇ **6.2-7.** Prove the following identities in $\mathbb{R}^3$ using the Hodge star operator:

$$\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2 \quad \text{and} \quad u \times (v \times w) = (u \cdot w) v - (u \cdot v) w.$$

◇ **6.2-8.** (i) Prove the following identity for the Hodge star operator:

$$\langle *\alpha, \beta \rangle = \langle \alpha \wedge \beta, \mu \rangle,$$

where $\alpha \in \bigwedge^k(\mathbf{E})$ and $\beta \in \bigwedge^{n-k}(\mathbf{E})$.

(ii) Prove the basic properties of $*$ using (i) as the definition.

◇ **6.2-9.** Let $\mathbf{E}$ be an oriented vector space and $S \subset T_2^0(\mathbf{E})$ be the set of nondegenerate symmetric two–tensors of a fixed signature s.

(i) Show that S is open.

(ii) Show that the map vol : $g \mapsto \mu(g)$ assigning to each $g \in S$ its g-volume element is differentiable and has derivative at g given by $h \mapsto (\operatorname{trace} h)\mu(g)/2$.

6.3 Differential Forms

The exterior algebra will now be extended from vector spaces to vector bundles and in particular to the tangent bundle.

Exterior Forms on Local Vector Bundles. First of all, we need to consider the action of local bundle maps. As in Chapter 3, $U \times \mathbf{F}$ denotes a local vector bundle, where U is open in a Banach space $\mathbf{E}$ and $\mathbf{F}$ is a Banach space. From $U \times \mathbf{F}$, we construct the local vector bundle $U \times \bigwedge^k(\mathbf{F})$. Now we want to piece these local objects together into a global one.

6.3.1 Definition. *Let $\varphi : U \times \mathbf{F} \to U' \times \mathbf{F}'$ be a local vector bundle map that is an isomorphism on each fiber. Then define $\varphi_* : U \times \bigwedge^k(\mathbf{F}) \to U' \times \bigwedge^k(\mathbf{F}')$ by $(u, \omega) \mapsto (\varphi_0(u), \varphi_{u^*}\omega)$, where φ_u is the second factor of φ (an isomorphism for each u).*

6.3.2 Definition. *If $\varphi : U \times \mathbf{F} \to U' \times \mathbf{F}'$ is a local vector bundle map that is an isomorphism on each fiber, then so is φ_*. Moreover, if φ is a local vector bundle isomorphism, so is φ_*.*

Proof. This is a special case of Proposition 5.2.4. ■

The Exterior Algebra of a Vector Bundle. Given a vector bundle, we can form the exterior algebra fiberwise.

6.3.3 Definition. *Suppose $\pi : E \to B$ is a vector bundle. Define*

$$\bigwedge^k(E)|A = \bigcup_{b \in A} \bigwedge^k(E_b)$$

where A is a subset of B and $E_b = \pi^{-1}(b)$ is the fiber over $b \in B$. Let $\bigwedge^k(E)|B = \bigwedge^k(E)$ and define $\bigwedge^k(\pi) : \bigwedge^k(E) \to B$ by $\bigwedge^k(\pi)(t) = b$ if $t \in \bigwedge^k(E_b)$.

6.3.4 Theorem. *Assume $\{E|U_i, \varphi_i\}$ is a vector bundle atlas for the vector bundle π, where $\varphi_i : E|U_i \to U_i' \times \mathbf{F}_i'$. Then $\{\bigwedge^k(E)|U_i, \varphi_{i^*}\}$ is a vector bundle atlas of $\bigwedge^k(\pi) : \bigwedge^k(E) \to B$, where $\varphi_{i^*} : \bigwedge^k(E)|U_i \to U_i' \times \bigwedge^k(\mathbf{F}')$ is defined by $\varphi_{i^*}|E_b = (\varphi_i|E_b)_*$.*

Proof. We must verify **VB1** and **VB2** in Definition 3.4.4 of a vector bundle. Condition **VB1** is clear; for **VB2** let φ_i, φ_j be two charts for π, so that $\varphi_i \circ \varphi_j^{-1}$ is a local vector bundle isomorphism on its domain. But then, $\varphi_{i^*} \circ \varphi_{j^*}^{-1} = (\varphi_i \circ \varphi_j^{-1})_*$, which is a local vector bundle isomorphism by Definition 6.3.2. ■

Because of this theorem, the vector bundle structure of $\pi : E \to B$ induces naturally a vector bundle structure on $\bigwedge^k(E) \to B$.

Differential Forms on Manifolds. We now specialize to the important case when $\pi : E \to B$ is the tangent bundle. If $\tau_M : TM \to M$ is the tangent bundle of a manifold M, let

$$\bigwedge^k(M) = \bigwedge^k(TM) \quad \textit{and} \quad \bigwedge^k_M = \bigwedge^k(\tau_M),$$

so $\bigwedge^k_M : \bigwedge^k(M) \to M$ is the vector bundle of exterior k forms on the tangent spaces of M. Also, let $\Omega^0(M) = \mathcal{F}(M)$, $\Omega^1(M) = \mathcal{T}^0_1(M)$, and $\Omega^k(M) = \Gamma^\infty(\bigwedge^k_M)$, $k = 2, 3, \ldots$.

6.3.5 Proposition. *Regarding $\mathcal{T}^0_k(M)$ as an $\mathcal{F}(M)$ module, $\Omega^k(M)$ is an $\mathcal{F}(M)$ submodule; that is, $\Omega^k(M)$ is a subspace of $\mathcal{T}^0_k(M)$ and if $f \in \mathcal{F}(M)$ and $\alpha \in \Omega^k(M)$, then $f\alpha \in \Omega^k(M)$.*

Proof. If $\alpha_1, \alpha_2 \in \Omega^k(M)$ and $f \in \mathcal{F}(M)$, then we must show $f\alpha_1 + \alpha_2 \in \Omega^k(M)$. This follows from the fact that for each $m \in M$, the exterior algebra on T_mM is a vector space. ■

6.3.6 Proposition. *If $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, $k, l = 0, 1, \ldots$, define $\alpha \wedge \beta : M \to \bigwedge^{k+l}(M)$ by*

$$(\alpha \wedge \beta)(m) = \alpha(m) \wedge \beta(m).$$

Then $\alpha \wedge \beta \in \Omega^{k+l}(M)$, and $\wedge$ is bilinear and associative.

Proof. First, $\wedge$ is bilinear and associative since it is true pointwise. To show $\alpha \wedge \beta$ is of class C^∞, consider the local representative of $\alpha \wedge \beta$ in natural charts. This is a map of the form $(\alpha \wedge \beta)_\varphi = B \circ (\alpha_\varphi \times \beta_\varphi)$, with $\alpha_\varphi, \beta_\varphi \in C^\infty$ and $B = \wedge$, which is bilinear and continuous. Thus $(\alpha \wedge \beta)_\varphi$ is C^∞ by the Leibniz rule. ∎

6.3.7 Definition. *Let $\Omega(M)$ denote the direct sum of the spaces $\Omega^k(M)$, $k = 0, 1, \ldots$, together with its structure as an (infinite-dimensional) real vector space and with the multiplication $\wedge$ extended componentwise to $\Omega(M)$. (If $\dim M = n < \infty$, the direct sum need only be taken for $k = 0, 1, \ldots, n$.) We call $\Omega(M)$ the algebra of **exterior differential forms** on M. Elements of $\Omega^k(M)$ are called k-**forms**. In particular, elements of $\mathfrak{X}^*(M)$ are called **one-forms**.*

Note that we generally regard $\Omega(M)$ as a real vector space rather than an $\mathcal{F}(M)$ module (as with $\mathcal{T}(M)$). The reason is that $\mathcal{F}(M) = \Omega^0(M)$ is included in the direct sum, and $f \wedge \alpha = f \otimes \alpha = f\alpha$.

6.3.8 Examples.

A. A one-form θ on a manifold M assigns to each $m \in M$ a linear functional on T_mM.

B. A two-form ω on a manifold assigns to each $m \in M$ a skew symmetric bilinear map

$$\omega_m : T_mM \times T_mM \to \mathbb{R}.$$

C. For an n-manifold M, a tensor field $t \in \mathcal{T}^r_s(M)$ has the local expression

$$t(u) = t^{i_1 \cdots i_r}_{j_1 \cdots j_s}(u) \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s},$$

where $u \in U$, (U, φ) is a local chart on M, and

$$t^{i_1 \cdots i_r}_{j_1 \cdots j_s}(u) = t\left(dx^{i_1}, \cdots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_s}}\right)(u).$$

The proof of Proposition 6.1.8 gives the local expression for $\omega \in \bigwedge^k(M)$, namely

$$\omega(u) = \omega_{i_1 \cdots i_k}(u) dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad i_1 < \cdots < i_k,$$

where

$$\omega_{i_1 \cdots i_k}(u) = \omega\left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_k}}\right)(u).$$

D. In $\Omega(M)$, the addition of forms of different degree is "purely formal" as in the case $M = E$. Thus, for example, if M is a two-manifold (a surface) and (x, y) are local coordinates on $U \subset M$, a typical element of $\Omega(M)$ has the local expression $f + g\, dx + h\, dy + k\, dx \wedge dy$, for $f, g, h, k \in \mathcal{F}(U)$.

E. As in §6.1, we have an isomorphism of vector bundles $* : \bigwedge^1(\mathbb{R}^3) \to \bigwedge^2(\mathbb{R}^3)$ given by

$$dx^1 \mapsto dx^2 \wedge dx^3, \quad dx^2 \mapsto dx^3 \wedge dx^1, \quad dx^3 \mapsto dx^1 \wedge dx^2.$$

On the other hand, the index lowering action given by the standard Riemannian metric on $\mathbb{R}^3$ defines a vector bundle isomorphism $^\flat : T(\mathbb{R}^3) \to T^*(\mathbb{R}^3) = \bigwedge^1(\mathbb{R}^3)$. These two isomorphisms applied pointwise define maps

$$* : \mathfrak{X}^*(\mathbb{R}^3) \to \Omega^2, \quad \alpha \mapsto *\,\alpha$$

and

$$^\flat : \mathfrak{X}(\mathbb{R}^3) \to \mathfrak{X}^*(\mathbb{R}^3), \quad X \mapsto X^\flat.$$

Then Example 6.1.12C implies

$$*[(X \times Y)^\flat] = X^\flat \wedge Y^\flat$$

for any vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^3)$ where $X \times Y$ denotes the usual cross–product of vector fields on $\mathbb{R}^3$ from calculus. That is,

$$\begin{aligned} X \times Y = &\left(X^2Y^3 - X^3Y^2\right)\frac{\partial}{\partial x^1} + \left(X^3Y^1 - X^1Y^3\right)\frac{\partial}{\partial x^2} \\ &+ \left(X^1Y^2 - X^2Y^1\right)\frac{\partial}{\partial x^3} \end{aligned}$$

where $X = X^i \dfrac{\partial}{\partial x^i}$ and $Y = Y^i \dfrac{\partial}{\partial x^i}$, $i = 1, 2, 3$.

F. The wedge product is taken in $\Omega(M)$ in the same way as in the algebraic case. For example, if $M = \mathbb{R}^3$, $\alpha = dx^1 - x^1 dx^2 \in \Omega^1(M)$ and $\beta = x^2 dx^1 \wedge dx^3 - dx^2 \wedge dx^3$, then

$$\begin{aligned} \alpha \wedge \beta &= (dx^1 - x^1 dx^2) \wedge (x^2 dx^1 \wedge dx^3 - dx^2 \wedge dx^3) \\ &= 0 - x^1x^2 dx^2 \wedge dx^1 \wedge dx^3 - dx^1 \wedge dx^2 \wedge dx^3 + 0 \\ &= (x^1x^2 - 1) dx^1 \wedge dx^2 \wedge dx^3. \end{aligned}$$

♦

Pull-back and Push-forward of Forms. We can now extend the pull-back and push-forward operations from the context of vector spaces and linear maps to that of manifolds and nonlinear maps.

6.3.9 Definition. *Suppose $F : M \to N$ is a C^∞ mapping of manifolds. For $\omega \in \Omega^k(N)$, define $F^*\omega : M \to \bigwedge^k(M)$ by $F^*\omega(m) = (T_mF)^* \circ \omega \circ F(m)$; that is,*

$$(F^*\omega)_m(v_1, \ldots, v_k) = \omega_{F(m)}(T_mF \cdot v_1, \ldots, T_mF \cdot v_k),$$

*where $v_1, \ldots, v_k \in T_mM$; for $g \in \Omega^0(N)$, $F^*g = g \circ F$. We say $F^*\omega$ is the **pull-back** of ω by F.* (See Figure 6.3.1.)

6.3.10 Proposition. *Let $F : M \to N$ and $G : N \to W$ be C^∞ mappings of manifolds. Then*

(i) $F^* : \Omega^k(N) \to \Omega^k(M)$;

(ii) $(G \circ F)^* = F^* \circ G^*$;

(iii) *if $H : M \to M$ is the identity, then $H^* : \Omega^k(M) \to \Omega^k(M)$ is the identity;*

(iv) *if F is a diffeomorphism, then F^* is an isomorphism and*

$$(F^*)^{-1} = \left(F^{-1}\right)^*;$$

(v) $F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$ *for $\alpha \in \Omega^k(N)$ and $\beta \in \Omega^l(N)$.*

Proof. Choose charts $(U, \varphi), (V, \psi)$ of M and N so that $F(U) \subset V$. Then the local representative $F_{\varphi\psi} = \psi \circ F \circ \varphi^{-1}$ is of class C^∞, as is $\omega_\psi = (T\psi)^* \circ \omega \circ \psi^{-1}$. The local representative of $F^*\omega$ is

$$(F^*\omega)_\varphi(u) = (T\varphi)^* \circ F^*\omega \circ \varphi^{-1}(u) = (T_uF_{\varphi\psi})^* \circ \omega_\psi \circ F_{\varphi\psi}(u)$$

which is of class C^∞ by the composite mapping theorem.

For (ii), note that it holds for the local representatives; (iii) follows from the definition; (iv) follows in the usual way from (ii) and (iii); and (v) follows from the corresponding pointwise result. ■

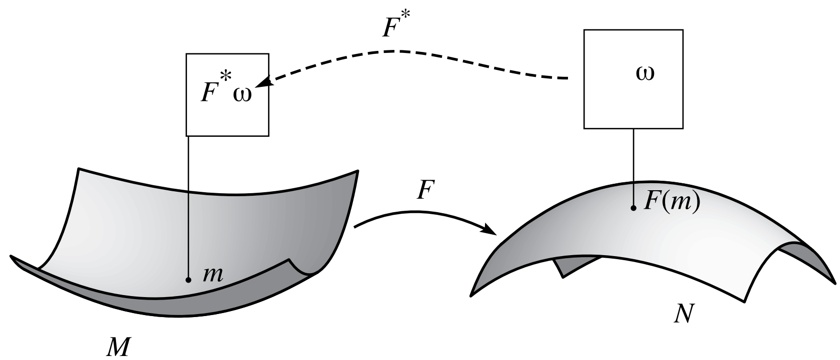

FIGURE 6.3.1. Pulling back forms

Vector Bundle Valued Forms (Optional). We close this section with a few optional remarks about vector-bundle-valued forms. As before, the idea is to globalize vector-valued exterior forms.

6.3.11 Definition. *Let $\pi : \hat{E} \to B$, $\rho : \hat{F} \to B$ be vector bundles over the same base. Define*

$$\bigwedge^k(\hat{E};\hat{F}) = L\left(\bigwedge^k(\hat{E}),\hat{F}\right),$$

*the vector bundle with base B of vector bundle homomorphisms over the identity from $\bigwedge^k(\hat{E})$ to $\hat{F}$. If $\hat{E} = TB$, $\bigwedge^k(TB;\hat{F})$ is denoted by $\bigwedge^k(B;\hat{F})$ and is called the vector bundle of F-**valued** k-**forms on** M. If $\hat{F} = B \times F$, we denote it by $\bigwedge^k(B,F)$ and call its elements vector-valued k-**forms on** M. The spaces of sections of these bundles are denoted respectively by $\Omega^k(\hat{E};\hat{F}), \Omega^k(B;\hat{F})$ and $\Omega^k(B;F)$. Finally, $\Omega(\hat{E};\hat{F})$ (resp., $\Omega(B;\hat{F}), \Omega(B,F)$) denotes the direct sum of $\Omega^k(\hat{E};\hat{F})$, $k = 1,2,\ldots,n$, together with its structure of an infinite-dimensional real vector space and $\mathcal{F}(B)$-module.*

Thus, $\alpha \in \Omega^k(\hat{E};\hat{F})$ is a smooth assignment to the points b of B of skew symmetric k-linear maps $\alpha_b : \hat{E}_b \times \cdots \times \hat{E}_b \to \hat{F}_b$. In particular, if all manifolds and bundles are finite dimensional, then $\alpha \in \Omega^k(M,\mathbb{R}^p)$ may be uniquely written in the form $\alpha = \Sigma_{i=1,\ldots,p}\alpha^i e_j$, where $\alpha^1,\ldots,\alpha^p \in \Omega^k(M)$, and $\{e_1,\ldots,e_p\}$ is the standard basis of $\mathbb{R}^p$. Thus $\alpha \in \Omega^k(E,\mathbb{R}^p)$ is written in local coordinates as

$$(\alpha^1_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \ldots, \alpha^p_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k})$$

for $i_1 < \cdots < i_k$. Proposition 6.3.10(i)–(iv) and its proof have straightforward generalizations to vector-bundle-valued forms on M. The wedge product requires additional structure to be defined, namely a smooth assignment $b \mapsto g_b$ of a symmetric bilinear map $g_b : \hat{F}_b \times \hat{F}_b \to \hat{F}_b$ for each $b \in B$. With this structure, Proposition 6.3.10(v) also carries over.

Exercises

◇ **6.3-1.** Show that for a vector bundle $\pi : E \to B$, $\bigwedge^k(E)$ is a (smooth) subbundle of $T^0_k(E)$. Generalize to vector-bundle-valued tensors and forms.

◇ **6.3-2.** Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $\varphi(x,y,z) = (x^2, yz)$. For

$$\alpha = v^2du + dv \in \Omega^1\left(\mathbb{R}^2\right) \quad \text{and} \quad \beta = uvdu \wedge dv \in \Omega^2\left(\mathbb{R}^2\right),$$

compute $\alpha \wedge \beta$, $\varphi^*\alpha$, $\varphi^*\beta$, and $\varphi^*(\alpha \wedge \beta)$.

◇ **6.3-3** (E. Cartan's lemma). Let M be an n-manifold and suppose that $\alpha^1, \ldots, \alpha^k \in \Omega^1(M), k \leq n$, are pointwise linearly independent. Show that $\beta^1, \ldots, \beta^k \in \Omega^1(M)$ satisfy $\Sigma_{1\leq i\leq k}\alpha^i \wedge \beta^i = 0$ iff there exist C^∞ functions $a_i^j \in \mathcal{F}(M)$ satisfying $a_i^j = a_j^i$ such that $\beta^j = a_i^j \beta^i$.
HINT: Work in a local chart and show first that α^i can be chosen to be dx^i; the symmetry of the matrix $[a_i^j]$ follows from antisymmetry of $\wedge$ and the given condition.

◇ **6.3-4.** A (strong) ***bundle metric*** g on a vector bundle $\pi : E \to B$ is a smooth section of $L_s^2(E; \mathbb{R})$ such that $g(b)$ is an inner product on E_b for every $b \in B$ which is (strongly) nondegenerate, that is, $e_b \in E_b \mapsto g(b)(e_b, \cdot) \in E_b^*$ is an isomorphism of Banach spaces.

(i) Show that the model of the fiber of E is a Hilbertizable space.

(ii) If $F \subset E$ is a subbundle of E, show that $F^\perp = \bigcup_{b\in B} F_b^\perp$ is a subbundle of E, where we define

$$F_b^\perp = \{ e_b \in E_b \mid g(b)\,(e_b, f_b) = 0 \text{ for all } f_b \in F_b \}.$$

(iii) Show that $E = F \oplus F^\perp$.

◇ **6.3-5.** Assume the vector bundle $\pi : E \to B$ has a strong bundle metric.

(i) If $\sigma : B \to E$ is a smooth nowhere vanishing section of E, let $F_b = \text{span}\{\sigma(b)\}$, $F = \bigcup_{b\in B} F_b$. Show that F is a subbundle of E which is isomorphic to the trivial bundle $E_B^1 = \mathbb{R} \times B$. Conclude from Exercise 6.3-4 that $E^1 \oplus (E^1)^\perp = E$.

(ii) Show that a manifold M is parallelizable if and only if TM is isomorphic to a trivial bundle.

(iii) Assume that M is a strong Riemannian manifold, admits a nowhere vanishing vector field and that $TM \oplus E_M^1$ is isomorphic to a trivial bundle. Let N be another manifold of dimension ≥ 1 such that $TN \oplus \hat{E}_N^1$ is trivial. Show that $M \times N$ is parallelizable.

HINT: Use (i) and pull everything back to $M \times N$ by the two projections.

(iv) Show that if dim $N = 0$, the conclusion of (ii) is false.

HINT: It is know that the only odd dimensional spheres with trivial tangent bundle are S^1, S^3 and S^7. Show that TS^{2n-1} has a nowhere vanishing vector field.

(v) Show that $S^{a(1)} \times \cdots \times S^{a(n)}$ is parallelizable provided that $a(i) \geq 1$, $i = 1, \ldots, n$ and at least one $a(i)$ is odd.

HINT: Use (iii) and Exercise 3.4-3.

6.4 The Exterior Derivative, Interior Product, and Lie Derivative

The purpose of this section is to extend the differential of functions to a map

$$\mathbf{d} : \Omega^k(M) \to \Omega^{k+1}(M)$$

defined for any k. This operator turns out to have marvelous algebraic properties. After studying these we shall show how $\mathbf{d}$ is related to the basic operations of div, grad and curl on $\mathbb{R}^3$. Then we develop formulas for the Lie derivative.

The Exterior Derivative. We first develop the exterior derivative $\mathbf{d}$ for finite-dimensional manifolds. The infinite-dimensional case is discussed in Supplement 6.4A.

6.4.1 Theorem. *Let M be an n-dimensional manifold. There is a unique family of mappings $\mathbf{d}^k(U) : \Omega^k(U) \to \Omega^{k+1}(U)$ ($k = 0, 1, 2, \ldots, n$ and U is open in M) which were merely denote by $\mathbf{d}$, called the* ***exterior derivative*** *on M, such that*

(i) $\mathbf{d}$ *is a $\wedge$-**antiderivation**. That is, $\mathbf{d}$ is $\mathbb{R}$ linear and for $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^l(U)$,*

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta \qquad (\textit{product rule});$$

(ii) *If $f \in \mathcal{F}(U)$, then $\mathbf{d}f$ is as defined in Definition* 4.2.5;

(iii) $\mathbf{d}^2 = \mathbf{d} \circ \mathbf{d} = 0$, (*i.e.*, $\mathbf{d}^{k+1}(U) \circ \mathbf{d}^k(U) = 0$);

(iv) $\mathbf{d}$ *is **natural with respect to restrictions;** that is, if $U \subset V \subset M$ are open and $\alpha \in \Omega^k(V)$, then*

$$\mathbf{d}(\alpha|U) = (\mathbf{d}\alpha)|U,$$

that is, or the following diagram commutes:

$$\begin{array}{ccc} \Omega^k(V) & \xrightarrow{\;|U\;} & \Omega^k(U) \\ \Big\downarrow{\scriptstyle \mathbf{d}} & & \Big\downarrow{\scriptstyle \mathbf{d}} \\ \Omega^{k+1}(V) & \xrightarrow[\;|U\;]{} & \Omega^{k+1}(U) \end{array}$$

As usual, condition (iv) *means that $\mathbf{d}$ is a* ***local operator***.

Proof. We first establish uniqueness. Let (U, φ) be a chart, where $\varphi(u) = \left(x^1, \ldots, x^n\right)$, and let

$$\alpha = \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Omega^k(U), \quad i_1 < \cdots < i_k.$$

If $k = 0$, by (ii), the local formula $\mathbf{d}\alpha = \left(\partial\alpha/\partial x^i\right) dx^i$ applied to the coordinate functions x^i, $i = 1, \ldots, n$ shows that the differential of x^i is the one-form dx^i. From (iii), $\mathbf{d}(dx^i) = 0$, so by (i)

$$\mathbf{d}\left(dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) = 0.$$

Thus, again by (i),

$$\mathbf{d}\alpha = \frac{\partial \alpha_{i_1 \cdots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad (\text{sum over } i_1 < \cdots < i_k), \tag{6.4.1}$$

and so $\mathbf{d}$ is uniquely defined on U by properties (i)–(iii), and by (iv) on any open subset of M.

For existence, define on every chart (U, φ) the operator $\mathbf{d}$ by formula (6.4.1). Then (ii) is trivially verified as is $\mathbb{R}$-linearity. If

$$\beta = \beta_{j_1 \cdots j_l} dx^{j_1} \wedge \cdots \wedge dx^{j_l} \in \Omega^l(U),$$

then

$$\begin{aligned}\mathbf{d}(\alpha \wedge \beta) &= \mathbf{d}\left(\alpha_{i_1\cdots i_k}\beta_{j_1\cdots j_l} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots dx^{j_l}\right) \\ &= \left(\frac{\partial \alpha_{i_1\cdots i_k}}{\partial x^i}\beta_{j_1\cdots j_l} + \alpha_{i_1\cdots i_k}\frac{\partial \beta_{j_1\cdots j_l}}{\partial x^i}\right) dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &\quad \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\ &= \frac{\partial \alpha_{i_1\cdots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge \beta_{j_1\ldots j_l} dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\ &\quad + (-1)^k \alpha_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge \frac{\partial \beta_{j_1\cdots j_l}}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\ &= \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta.\end{aligned}$$

and (i) is verified. For (iii), the symmetry of the second partial derivatives shows that

$$\mathbf{d}(\mathbf{d}\alpha) = \frac{\partial^2 \alpha_{i_1\cdots i_k}}{\partial x^i \partial x^j} dx^i \wedge dx^j \wedge dx^{j_1} \wedge \cdots \wedge dx^{i_k} = 0, \quad i_1 < \cdots < i_k.$$

Thus, in every chart (U, φ), equation (6.4.1) defines the operator $\mathbf{d}$ satisfying (i)–(iii). It remains to be shown that these local $\mathbf{d}$'s define an operator $\mathbf{d}$ on any open set and (iv) holds. To do this, it is sufficient to show that this definition is chart independent. Let $\mathbf{d}'$ be the operator given by equation (6.4.1) on a chart (U', φ'), where $U' \cap U \neq \varnothing$. Since $\mathbf{d}'$ also satisfies (i)–(iii), and local uniqueness has already been proved, $\mathbf{d}'\alpha = \mathbf{d}\alpha$ on $U \cap U'$. The theorem thus follows. ■

6.4.2 Corollary. *Let $\omega \in \Omega^k(U)$, where $U \subset \mathbf{E}$ is open. Then*

$$\mathbf{d}\omega(u)\left(v_0, \ldots, v_k\right) = \sum_{i=0}^{k} (-1)^i \, \mathbf{D}\omega(u) \cdot v_i \left(v_0, \ldots, \hat{v}_i, \ldots, v_k\right) \tag{6.4.2}$$

where $\hat{v}_i$ denotes that v_i is deleted. Also, we denote elements (u, v) of TU merely by v for brevity. (Note that $\mathbf{D}\omega(u) \cdot v \in L^k_a(E, \mathbb{R})$ since $\omega : U \to L^k_a(E, \mathbb{R})$.)

Proof. Since we are in the finite dimensional case, we can proceed with a coordinate computation. (An alternative is to check out that $\mathbf{d}$ defined by equation (6.4.2) satisfies (i) to (iv). Checking (i) and (iii) is straightforward but lengthy.) Indeed, if the local coordinates of u are $\left(x^1, \ldots, x^n\right)$,

$$\omega(u) = \omega_{i_1\cdots i_k}(u) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

(sum over $i_1 < \cdots < i_k$), then

$$\mathbf{D}\omega(u) \cdot v_i = \frac{\partial \omega_{i_1\cdots i_k}}{\partial x^j} v_i^j dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

(where the sum is over all j and $i_1 < \cdots < i_k$). From equation (6.4.1),

$$\begin{aligned}\mathbf{d}\omega\left(v_0, \ldots, v_k\right) &= \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\left(v_0, \ldots, v_k\right) \\ &= \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x^j} (\operatorname{sign}\sigma) v_0^{\sigma(j)} v_1^{\sigma(i_1)} \cdots v_k^{\sigma(i_k)}\end{aligned} \tag{6.4.3}$$

(where the sum is over all $i_1 < \cdots < i_k$, j, and σ's satisfying $\sigma(j) < \sigma(i_1) < \cdots < \sigma(i_k)$). The right hand side of equation (6.4.2) is

$$(-1)^i \frac{\partial \omega_{i_1\cdots i_k}}{\partial x^j} v_i^j \left(\operatorname{sign}\eta\right) v_0^{\eta(i_1)} \cdots \hat{v}_i^{\,\eta(i_j)} \cdots v_k^{\eta(i_k)} \tag{6.4.4}$$

(where the sum is over all $i_1 < \cdots < i_k$, j, i, and η's with $\eta(i_1) < \cdots < \eta(i_k)$). Writing σ as a product of a permutation moving j to a designated position and a permutation η, we see that equations (6.4.3) and (6.4.4) coincide. ■

6.4.3 Examples.

A. On $\mathbb{R}^2$, let $\alpha = f(x,y)dx + g(x,y)dy$. Then $\mathbf{d}\alpha = \mathbf{d}f \wedge dx + f\mathbf{d}(dx) + \mathbf{d}g \wedge dy + g\mathbf{d}(dy)$ by linearity and the product rule. Since $\mathbf{d}^2 = 0$,

$$\begin{aligned}\mathbf{d}\alpha &= \mathbf{d}f \wedge dx + \mathbf{d}g \wedge dy \\ &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \wedge dx + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right) \wedge dy.\end{aligned}$$

Since $dx \wedge dx = 0$ and $dy \wedge dy = 0$, this becomes

$$\mathbf{d}\alpha = \frac{\partial f}{\partial y}dy \wedge dx + \frac{\partial g}{\partial x}dx \wedge dy = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy.$$

B. On $\mathbb{R}^3$, let $f(x,y,z)$ be given. Then

$$\mathbf{d}f = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz,$$

so the components of $\mathbf{d}f$ are those of grad f. That is, $(\operatorname{grad} f)^\flat = \mathbf{d}f$, where $^\flat$ is the index lowering operator defined by the standard metric of $\mathbb{R}^3$ (see §5.1).

C. On $\mathbb{R}^3$, let $\mathbf{F}^\flat = F_1(x,y,z)dx + F_2(x,y,z)dy + F_3(x,y,z)dz$. Computing as in Example A yields

$$\begin{aligned}\mathbf{d}\mathbf{F}^\flat = &\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy - \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dx \wedge dz \\ &+ \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz.\end{aligned}$$

Thus associated to each vector field $\mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}$ on $\mathbb{R}^3$ is the one-form $\mathbf{G}^\flat$ and to this the two-form $*(\mathbf{G}^\flat)$ by

$$*\left(\mathbf{G}^\flat\right) = G_3 dx \wedge dy - G_2 dx \wedge dz + G_1 dy \wedge dz,$$

where $*$ is the Hodge operator (see §6.2); it is clear the $\mathbf{d}\mathbf{F}^\flat = *(\operatorname{curl}\mathbf{F})^\flat$.

D. The divergence is obtained from $\mathbf{d}$ by

$$\mathbf{d}{*}\mathbf{F}^\flat = (\operatorname{div}\mathbf{F})dx \wedge dy \wedge dz; \quad \text{that is,} \quad {*}\mathbf{d}{*}\mathbf{F}^\flat = \operatorname{div}\mathbf{F}.$$

Thus associating to a vector field $\mathbf{F}$ on $\mathbb{R}^3$ the one-form $\mathbf{F}^\flat$ and the two-form $\mathbf{d}{*}\mathbf{F}^\flat$, gives rise to the operators $\operatorname{curl}\mathbf{F}$ and $\operatorname{div}\mathbf{F}$. From $\mathbf{d}\mathbf{F}^\flat = *(\operatorname{curl}\mathbf{F})^\flat$ it is apparent that

$$\mathbf{dd}\mathbf{F}^\flat = 0 = \mathbf{d}{*}(\operatorname{curl}\mathbf{F})^\flat = (\operatorname{div}\operatorname{curl}\mathbf{F})dx \wedge dy \wedge dz.$$

That is, $\mathbf{d}^2 = 0$ gives the well-known vector identity $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$. Likewise, $\mathbf{dd}f = 0$ becomes $\mathbf{d}(\operatorname{grad} f)^\flat = 0$; that is, $*(\operatorname{curl}\operatorname{grad} f)^\flat = 0$. So here $\mathbf{d}^2 = 0$ becomes the identity $\operatorname{curl}\operatorname{grad} f = 0$. ♦

We summarize the relationship between the operators in vector calculus and differential forms in the table at the end of this section.

Mappings and the Exterior Derivative. We will now consider the effect of mappings on the exterior derivative operator $\mathbf{d}$. Recall that $\Omega(M)$ is the direct sum of all the $\Omega^k(M)$.

6.4.4 Theorem. *Let $F : M \to N$ be of class C^1. Then $F^* : \Omega(N) \to \Omega(M)$ is a homomorphism of differential algebras; that is,*

(i) $F^*(\psi \wedge \omega) = F^*\psi \wedge F^*\omega$, *and*

(ii) $\mathbf{d}$ *is natural with respect to mappings; that is,*

$$F^*(\mathbf{d}\omega) = \mathbf{d}\left(F^*\omega\right),$$

that is, the following diagram commutes:

$$\begin{array}{ccc} \Omega^k(M) & \xleftarrow{\;F^*\;} & \Omega^k(N) \\ \Big\downarrow \mathbf{d} & & \Big\downarrow \mathbf{d} \\ \Omega^{k+1}(M) & \xleftarrow[\;F^*\;]{} & \Omega^{k+1}(N) \end{array}$$

Proof. Part (i) was established in Proposition 6.3.10. For (ii), we shall show that if $m \in M$, then there is a neighborhood U of $m \in M$ such that $\mathbf{d}(F^*\omega|U) = (F^*\mathbf{d}\omega)|U$, which is sufficient, as F^* and $\mathbf{d}$ are both natural with respect to restriction. Let (V, φ) be a local chart at $F(m)$ and U a neighborhood of $m \in M$ with $F(U) \subset V$. Then for $\omega \in \Omega^k(V)$, we can write

$$\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad (\text{sum over } i_1 < \cdots < i_k)$$

and so $\mathbf{d}\omega = \partial_{i_0}\omega_{i_1\cdots i_k} dx^{i_0} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, where $\partial_{i_0} = \partial/\partial x^{i_0}$ (sum over i_0 and $i_1 < \cdots < i_k$) and by (i)

$$F^*\omega|U = \left(F^*\omega_{i_1\cdots i_k}\right) F^*dx^{i_1} \wedge \cdots \wedge F^*dx^{i_k}.$$

If $\psi \in \Omega^0(N)$ then $\mathbf{d}(F^*\psi) = F^*\mathbf{d}\psi$ by the composite mapping theorem, so by (i) and $\mathbf{d} \circ \mathbf{d} = 0$, we get

$$\mathbf{d}(F^*\omega|U) = F^*\left(\mathbf{d}\omega_{i_1\cdots i_k}\right) \wedge F^*dx^{i_1} \wedge \cdots \wedge F^*dx^{i_k} = F^*(\mathbf{d}\omega)|U. \qquad \blacksquare$$

6.4.5 Corollary. *The operator $\mathbf{d}$ is natural with respect to push-forward by diffeomorphisms. That is, if $F : M \to N$ is a different diffeomorphism, then $F_*\mathbf{d}\omega = \mathbf{d}F_*\omega$, or the following diagram commutes:*

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{\;F_*\;} & \Omega^k(N) \\ \Big\downarrow \mathbf{d} & & \Big\downarrow \mathbf{d} \\ \Omega^{k+1}(M) & \xrightarrow[\;F_*\;]{} & \Omega^{k+1}(N) \end{array}$$

Proof. Since $F_* = \left(F^{-1}\right)^*$, the result follows from Theorem 6.4.4(ii). $\blacksquare$

6.4.6 Corollary. *Let $X \in \mathfrak{X}(M)$. Then $\mathbf{d}$ is natural with respect to $\pounds_X$. That is, for $\omega \in \Omega^k(M)$ we have $\pounds_X\omega \in \Omega^k(M)$ and*

$$\mathbf{d}\pounds_X\omega = \pounds_X\mathbf{d}\omega,$$

that is, the following diagram commutes:

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{\pounds_X} & \Omega^k(M) \\ \mathbf{d}\downarrow & & \downarrow\mathbf{d} \\ \Omega^{k+1}(M) & \xrightarrow[\pounds_X]{} & \Omega^{k+1}(M) \end{array}$$

Proof. Let F_t be the (local) flow of X. Then we know that

$$\pounds_X\omega\,(m) = \frac{d}{dt}\left(F_t^*\omega\right)(m)\bigg|_{t=0}.$$

Since $F_t^*\omega \in \Omega^k(M)$, it follows that $\pounds_X\omega \in \Omega^k(M)$. Now we have $F_t^*\mathbf{d}\omega = \mathbf{d}\left(F_t^*\omega\right)$. Then, since $\mathbf{d}$ is $\mathbb{R}$-linear, it commutes with d/dt and so taking the derivative of this relation at $t = 0$, we get $\pounds_X\mathbf{d}\omega = \mathbf{d}\pounds_X\mathbf{d}\omega$. ■

Interior Products. In Chapter 5, contractions of general tensor fields were studied. For differential forms, contractions play a special role.

6.4.7 Definition. *Let M be a manifold, $X \in \mathfrak{X}(M)$, and $\omega \in \Omega^{k+1}(M)$. Then define $\mathbf{i}_X\omega \in \mathcal{T}_k^0(M)$ by*

$$\mathbf{i}_X\omega(X_1, \ldots, X_k) = \omega(X, X_1, \ldots, X_k).$$

*if $\omega \in \Omega^0(M)$, we put $\mathbf{i}_X\omega = 0$. We call $\mathbf{i}_X\omega$ the **interior product** or **contraction** of X and ω. (Sometimes $X \lrcorner \omega$ is written for $\mathbf{i}_X\omega$.)*

6.4.8 Theorem. *We have $\mathbf{i}_X : \Omega^k(M) \to \Omega^{k-1}(M)$, $k = 1, \ldots, n$, and if $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, and $f \in \Omega^0(M)$, then*

(i) *$\mathbf{i}_X$ is a $\wedge$-**antiderivation;** that is $\mathbf{i}_X$ is $\mathbb{R}$-linear and we have the identity $\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X\beta)$;*

(ii) *$\mathbf{i}_{fX}\alpha = f\mathbf{i}_X\alpha$;*

(iii) *$\mathbf{i}_X\mathbf{d}f = \pounds_X f$;*

(iv) *$\pounds_X(\alpha \wedge \beta) = \pounds_X\alpha \wedge \beta + \alpha \wedge \pounds_X\beta$;*

(v) *$\pounds_X\alpha = \mathbf{i}_X\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_X\alpha$;*

(vi) *$\pounds_{fX}\alpha = f\pounds_X\alpha + \mathbf{d}f \wedge \mathbf{i}_X\alpha$.*

Proof. That $\mathbf{i}_X\alpha \in \Omega^{k-1}(M)$ follows from the definitions. For (i), $\mathbb{R}$-linearity is clear. For the second part of (i), write

$$\mathbf{i}_X(\alpha \wedge \beta)(X_2, X_3, \ldots, X_{k+l}) = (\alpha \wedge \beta)(X, X_2, \ldots, X_{k+l})$$

and

$$\mathbf{i}_X\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{i}_X\beta = \frac{(k+l-1)!}{(k-1)!l!}\mathbf{A}\left(\mathbf{i}_X\alpha \otimes \beta\right) + (-1)^k \frac{(k+l-1)!}{k!(l-1)!}\mathbf{A}(\alpha \otimes \mathbf{i}_X\beta).$$

Now write out the definition of $\mathbf{A}$ in terms of permutations from Definition 6.1.1. The sum over all permutations in the last term can be replaced by the sum over $\sigma\sigma_0$, where σ_0 is the permutation

$$(2, 3, \ldots, k+1, 1, k+2, \ldots k+l) \mapsto (1, 2, 3, \ldots, k+l),$$

whose sign is $(-1)^k$. Hence (i) follows. For (ii), we note that α is $\mathcal{F}(M)$-multilinear, and (iii) is just the definition of $\pounds_X f$.

Part (iv) follows from the fact that $\pounds_X$ is a tensor derivation and commutes with the alternation operator $\mathbf{A}$. (It also follows from the formula for $\pounds_X$ in terms of flows.) For (v) we proceed by induction on k. First note that for $k = 0$, (iv) reduces to (iii). Now assume that (v) holds for k. Then a $(k+1)$-form may be written as $\sum \mathbf{d}f_i \wedge \omega_i$, where ω_i is a k form, in some neighborhood of $m \in M$. But

$$\pounds_X\left(\mathbf{d}f \wedge \omega\right) = \pounds_X\mathbf{d}f \wedge \omega + \mathbf{d}f \wedge \pounds_X\omega$$

by (iv), so

$$\begin{aligned}
&\mathbf{i}_X\mathbf{d}(\mathbf{d}f \wedge \omega) + \mathbf{d}\mathbf{i}_X(\mathbf{d}f \wedge \omega)\\
&\qquad = -\mathbf{i}_X(\mathbf{d}f \wedge \mathbf{d}\omega) + \mathbf{d}(\mathbf{i}_X\mathbf{d}f \wedge \omega - \mathbf{d}f \wedge \mathbf{i}_X\omega)\\
&\qquad = -\mathbf{i}_X\mathbf{d}f \wedge \mathbf{d}\omega + \mathbf{d}f \wedge \mathbf{i}_X\mathbf{d}\omega + \mathbf{d}\mathbf{i}_X\mathbf{d}f \wedge \omega\\
&\qquad\quad + \mathbf{i}_X\mathbf{d}f \wedge \mathbf{d}\omega + \mathbf{d}f \wedge \mathbf{d}\mathbf{i}_X\omega\\
&\qquad = \mathbf{d}f \wedge \pounds_X\omega + \mathbf{d}\pounds_X f \wedge \omega
\end{aligned}$$

by the inductive assumption and (iii). Since $\mathbf{d}\pounds_X f = \pounds_X\mathbf{d}f$, the result follows.

Finally, for (vi) we have

$$\begin{aligned}
\pounds_{fX}\alpha &= \mathbf{i}_{fX}\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_{fX}\alpha = f\mathbf{i}_X\mathbf{d}\alpha + \mathbf{d}\left(f\mathbf{i}_X\alpha\right)\\
&= f\mathbf{i}_X\mathbf{d}\alpha + \mathbf{d}f \wedge \mathbf{i}_X\alpha + f\mathbf{d}\mathbf{i}_X\alpha = f\pounds_X\alpha + \mathbf{d}f \wedge \mathbf{i}_X\alpha.
\end{aligned} \tag{6.4.5}$$

■

Note that proofs of (i), (ii) and (iii) are valid without change on Banach manifolds. Formula (v)

$$\pounds_X\alpha = \mathbf{i}_X\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_X\alpha \tag{6.4.6}$$

(a "magic" formula of Cartan) is particularly useful. It can be used in the following way.

6.4.9 Examples.

A. If α is a k-form such that $\mathbf{d}\alpha = 0$ and X is a vector field such that $\mathbf{d}\mathbf{i}_X\alpha = 0$, then $F_t^*\alpha = \alpha$, where F_t is the flow of X. Indeed,

$$\frac{d}{dt}F_t^*\alpha = F_t^*\pounds_X\alpha = F_t^*(\mathbf{i}_X\mathbf{d}\alpha + \mathbf{d}(\mathbf{i}_X\alpha)) = 0.$$

so $F_t^*\alpha$ is constant in t. Since $F_0 = $ identity, $F_t^*\alpha = \alpha$ for all t.

B. Let $M = \mathbb{R}^3$, suppose $\operatorname{div} X = 0$, and let $\alpha = dx \wedge dy \wedge dz$. Thus $\mathbf{d}\alpha = 0$. Also,

$$\mathbf{i}_X \alpha = \mathbf{i}_X(dx \wedge dy \wedge dz) = X^1 dy \wedge dz - X^2 dx \wedge dz + X^3 dx \wedge dy = *X^\flat.$$

so $\mathbf{d}\mathbf{i}_X \alpha = \mathbf{d}*X^\flat = *(\operatorname{div} X) = 0$. Thus by Example A,

$$F_t^*(dx \wedge dy \wedge dz) = dx \wedge dy \wedge dz.$$

As we shall see in the next section in a more general context, this means that the flow of X is volume preserving. Of course this can be proved directly as well by differentiating the determinant of the Jacobian matrix of F_t in t (see, for example, Chorin and Marsden [1993]). For related applications to fluid mechanics, see §8.2. ♦

Mappings and the Interior Product. The behavior of contractions under mappings is given by the following proposition. (The statement and proof also hold for Banach manifolds.)

6.4.10 Proposition. *Let M and N be manifolds and $F : M \to N$ a C^1 mapping. If $\omega \in \Omega^k(N)$, $X \in \mathfrak{X}(N)$, $Y \in \mathfrak{X}(M)$, and Y is F-related to X, then*

$$\mathbf{i}_Y F^*\omega = F^*\mathbf{i}_X\omega.$$

In particular, if F is a diffeomorphisms, then

$$\mathbf{i}_{F^*X} F^*\omega = F^*\mathbf{i}_X\omega.$$

That is, interior products are natural with respect to diffeomorphisms and the following diagram commutes:

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{F^*} & \Omega^k(M) \\ \Big\downarrow \mathbf{i}_X & & \Big\downarrow \mathbf{i}_{F^*X} \\ \Omega^{k-1}(N) & \xrightarrow[F^*]{} & \Omega^{k-1}(M) \end{array}$$

Similarly for $Y \in \mathfrak{X}(M)$ we have the following commutative diagram:

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{F_*} & \Omega^k(N) \\ \Big\downarrow \mathbf{i}_Y & & \Big\downarrow \mathbf{i}_{F_*Y} \\ \Omega^{k-1}(M) & \xrightarrow[F_*]{} & \Omega^{k-1}(N) \end{array}$$

Proof. Let $v_1, \ldots, v_{k-1} \in T_m(M)$ and $n = F(m)$. Then

$$\begin{aligned} \mathbf{i}_Y F^*\omega(m) \cdot (v_1, \ldots, v_{k-1}) &= F^*\omega(m) \cdot (Y(m), v_1, \ldots, v_{k-1}) \\ &= \omega(n) \cdot ((TF \circ Y)(m), TF(v_1), \ldots, TF(v_{k-1})) \\ &= \omega(n)((X \circ F)(m), TF(v_1), \ldots, TF(v_{k-1})) \\ &= \mathbf{i}_X\omega(n) \cdot (TF(v_1), \ldots, TF(v_{k-1})) \\ &= F^*\mathbf{i}_X\omega(m) \cdot (v_1, \ldots, v_{k-1}). \end{aligned}$$

■

The Lie Derivative and the Exterior Derivative. The next proposition expresses $\mathbf{d}$ in terms of the Lie derivatives (see Palais [1954]).

6.4.11 Proposition. *Let $X_i \in \mathfrak{X}(M)$, $i = 0, \ldots, k$, and $\omega \in \Omega^k(M)$. Then we have*

(i)
$$(\pounds_{X_0}\omega)(X_1, \ldots, X_k) = \pounds_{X_0}(\omega(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \ldots, \pounds_{X_0} X_i, \ldots, X_k)$$

and

(ii)
$$\begin{aligned} &\mathbf{d}\omega(X_0, X_1, \ldots, X_k) \\ &\quad = \sum_{l=0}^{k} (-1)^l \pounds_{X_l}(\omega(X_0, \ldots, \hat{X}_l, \ldots, X_k)) \\ &\quad\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega(\pounds_{X_i}(X_j), X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \end{aligned}$$

where $\hat{X}_i$ denotes that X_i is deleted.

Proof. Part (i) is condition **DO1** in Definition 5.3.1. For (ii) we proceed by induction. For $k = 0$, it is merely $\mathbf{d}\omega(X_0) = \pounds_{X_0}\omega$. Assume the formula for $k-1$. Then if $\omega \in \Omega^k(M)$ we have, by Cartan's formula (6.4.6) and (i)

$$\begin{aligned} &\mathbf{d}\omega(X_0, X_1, \ldots, X_k) \\ &\quad = (\mathbf{i}_{X_0}\mathbf{d}\omega)(X_1, \ldots, X_k) \\ &\quad = (\pounds_{X_0}\omega)(X_1, \ldots, X_k) - (\mathbf{d}(\mathbf{i}_{X_0}\omega))(X_1, \cdots, X_k) \\ &\quad = \pounds_{X_0}(\omega(X_1, \ldots, X_k)) - \sum_{l=1}^{k} \omega(X_1, \ldots, \pounds_{X_0} X_l, \ldots, X_k) \\ &\quad\quad - (\mathbf{d}\mathbf{i}_{X_0}\omega)(X_1, \ldots, X_k) \end{aligned}$$

But $\mathbf{i}_{X_0}\omega \in \Omega^{k-1}(M)$ and we may apply the induction assumption. This gives, after a permutation

$$\begin{aligned} &(\mathbf{d}(\mathbf{i}_{X_0}\omega))(X_1, \ldots, X_k) \\ &\quad = \sum_{l=1}^{k} (-1)^{l-1} \pounds_{X_l}(\omega(X_0, X_1, \ldots, \hat{X}_l, \ldots, X_j)) \\ &\quad\quad - \sum_{1 \le i < j \le k} (-1)^{i+j} \omega(\pounds_{X_i} X_j, X_0, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k). \end{aligned}$$

Substituting this into the foregoing yields the result. ■

Note that the proof of (i) and the first formula in the next corollary holds as well for infinite-dimensional manifolds.

6.4.12 Corollary. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$[\pounds_X, \mathbf{i}_Y] = \mathbf{i}_{[X,Y]} \quad \textit{and} \quad [\pounds_X, \pounds_Y] = \pounds_{[X,Y]}$$

In particular, $\mathbf{i}_X \circ \pounds_X = \pounds_X \circ \mathbf{i}_X$.

Proof. It is sufficient to check the first formula on any k-form $\omega \in \Omega^k(U)$ and any $X_1, \ldots, X_{k-1} \in \mathfrak{X}(U)$ for any open set U of M. Proposition 6.4.11(i) gives

$$\begin{aligned}
&(\mathbf{i}_Y \pounds_X \omega)(X_1, \ldots, X_{k-1}) \\
&\quad = (\pounds_X \omega)(Y, X_1, \ldots, X_{k-1}) \\
&\quad = \pounds_X(\omega(Y, X_1, \ldots, X_{k-1})) \\
&\qquad - \sum_{l=1}^{k-1} \omega(Y, X_1, \ldots [X, X_l], \ldots, X_{k-1}) - \omega([X,Y], X_1, \ldots, X_{k-1}) \\
&\quad = \pounds_X((\mathbf{i}_Y \omega)(X_1, \ldots, X_{k-1})) \\
&\qquad - \sum_{l=1}^{k-1} (\mathbf{i}_Y \omega)(X_1, \ldots, [X, X_l], \ldots, X_{k-1}) - \left(\mathbf{i}_{[X,Y]}\omega\right)(X_1, \ldots, X_{k-1}) \\
&\quad = (\pounds_X \mathbf{i}_Y \omega)(X_1, \ldots X_{k-1}) - (\mathbf{i}_{[X,Y]}\omega)(X_1, \ldots, X_{k-1}).
\end{aligned}$$

One proves $[\pounds_X, \pounds_Y] = \pounds_{[X,Y]}$ using the first relation and Cartan's formula (6.4.6). ■

Supplement 6.4A
The Exterior derivative on Infinite-dimensional Manifolds

Now we discus the exterior derivative on infinite-dimensional manifolds. Theorem 6.4.1 is rather awkward, primarily because we cannot, without a lot of technicalities, pass from, for example, one-forms to two-forms by linear combinations of decomposable two-forms, that is, two-forms of the type $\alpha \wedge \beta$. However, there is a simpler alternative available.

1. Adopt the formula in Proposition 6.4.11(ii) as the definition of $\mathbf{d}$ on any open subset of M. Note that at first it is defined as a multilinear function on vector fields and note that $\pounds_X$ is already defined.

2. In charts, the equation of Proposition 6.4.11(ii) reduces to the local formula (6.4.2). This or a direct computation shows that $\mathbf{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ is well defined, depending only on the point values of the vector fields.

3. One checks the basic properties of $\mathbf{d}$. This can be done in two ways: directly, using the local formula, or using the definition and the following lemma, easily deducible from the Hahn–Banach theorem: *if a k-form ω is zero on any set of k vector fields $X_1, \ldots, X_k \in \mathfrak{X}(U)$ for all open sets U in M, then* $\omega = 0$. This second method is slightly faster if one first proves formula (6.4.6), which in turn implies Corollary 6.4.12.

Proof of formula (6.4.6). Let α be a k-form and $X_1, \ldots, X_k$ be a set of k vector fields defined on some open subset of M. Writing $X_0 = X$, we have

$$\begin{aligned}
&(\mathbf{i}_X \mathbf{d}\alpha + \mathbf{d}\mathbf{i}_X\alpha)(X_1, \ldots, X_k) \\
&\quad = \mathbf{d}\alpha(X, X_1, \ldots, X_k) + \mathbf{d}(\mathbf{i}_X\alpha)(X_1, \ldots, X_k) \\
&\quad = \sum_{l=0}^{k}(-1)^l \pounds_{X_l}(\alpha(X_0, \ldots, \hat{X}_l, \ldots, X_k)) \\
&\qquad + \sum_{0 \le i < j \le k} (-1)^{i+j}\alpha(\pounds_{X_i}X_j, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \\
&\qquad + \sum_{l=1}^{k}(-1)^{l-1} \pounds_{X_l}(\alpha(X_0, X_1, \ldots, \hat{X}_l, \ldots, X_k)) \\
&\qquad - \sum_{1 \le i < j \le k} (-1)^{i+j}\alpha(\pounds_{X_0}X_j, X_0, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \\
&\quad = \pounds_{X_0}(\alpha(X_1, \ldots, X_k)) + \sum_{j=1}^{k}(-1)^j\alpha(\pounds_{X_0}X_j, X_1, \ldots, \hat{X}_j, \ldots, X_k) \\
&\quad = (\pounds_X\alpha)(X_1, \ldots, X_k) \qquad \text{(by Proposition 6.4.11(ii))}.
\end{aligned}$$

■

This and corollary 6.4.12 will allow us to give a proof of the infinite-dimensional version of Corollary 6.4.6 :

$$\pounds_X \circ \mathbf{d} = \mathbf{d} \circ \pounds_X$$

For functions f this formula is proved as follows. By Proposition 6.4.11(ii),

$$\begin{aligned}
(\pounds_X \mathbf{d}f)(Y) &= \pounds_X(\mathbf{d}f(Y)) - \mathbf{d}f([X,Y]) = X[Y[f]] - [X,Y][f] \\
&= Y[X[f]] = \mathbf{d}(X[f])(Y) = (\mathbf{d}\pounds_X f)(Y).
\end{aligned}$$

Inductively, assume the formula holds for $(k-1)$-forms. Then for any k-form α and any vector field Y defined an open subset of M, $\mathbf{d}\pounds_X\mathbf{i}_Y\alpha = \pounds_X\mathbf{d}\mathbf{i}_Y\alpha$. Thus by Corollary 6.4.12,

$$\begin{aligned}
\mathbf{i}_Y\mathbf{d}\pounds_X\alpha &= \pounds_Y\pounds_X\alpha - \mathbf{d}\mathbf{i}_Y\pounds_X\alpha \\
&= \pounds_X\pounds_Y\alpha - \pounds_{[X,Y]}\alpha + \mathbf{d}\mathbf{i}_{[X,Y]}\alpha - \mathbf{d}\pounds_X\mathbf{i}_Y\alpha \\
&= \pounds_X\pounds_Y\alpha - \pounds_X\mathbf{d}\mathbf{i}_Y\alpha - \mathbf{i}_{[X,Y]}\mathbf{d}\alpha \\
&= \pounds_X\mathbf{i}_Y\mathbf{d}\alpha - \mathbf{i}_{[X,Y]}\mathbf{d}\alpha \\
&= \mathbf{i}_Y\pounds_X\mathbf{d}\alpha.
\end{aligned}$$

Hence $\mathbf{d} \circ \pounds_X = \pounds_X \circ \mathbf{d}$.

Next, the remaining properties of $\mathbf{d}$ are checked in the following way. $\mathbb{R}$-linearity and Theorem 6.4.1(iv) are immediate consequences of the definition. For Theorem 6.4.1(ii), note that

$$\mathbf{d}f(X) = \mathbf{i}_X\mathbf{d}f = \pounds_X f - \mathbf{d}\mathbf{i}_X f = \pounds_X f = X[f].$$

To show that $\mathbf{d}^2 = 0$, first observe that

$$\begin{aligned}
\mathbf{i}_X \circ \mathbf{d} \circ \mathbf{d} &= \pounds_X \circ \mathbf{d} - \mathbf{d} \circ \mathbf{i}_X \circ \mathbf{d} \\
&= \mathbf{d} \circ \pounds_X - \mathbf{d} \circ \pounds_X + \mathbf{d} \circ \mathbf{d} \circ \mathbf{i}_X \\
&= \mathbf{d} \circ \mathbf{d} \circ \mathbf{i}_X,
\end{aligned}$$

so that for any k-form α and any vector fields $X_1, \ldots, X_{k+2}$, we have

$$\begin{aligned}(\mathbf{dd}\alpha)(X_1, \ldots, X_{k+2}) &= \mathbf{i}_{X_{k+2}} \ldots \mathbf{i}_{X_1} \mathbf{dd}\alpha = \mathbf{i}_{X_{k+2}} \cdots \mathbf{i}_{X_2} \mathbf{ddi}_{X_1}\alpha \\ &= \cdots = \mathbf{i}_{X_{k+2}} \mathbf{ddi}_{X_{k+1}} \ldots \mathbf{i}_{X_1}\alpha \\ &= \mathbf{i}_{X_{k+2}} \mathbf{ddi}_{X_{k+1}}(\alpha(X_1, \ldots, X_k)) \\ &= 0.\end{aligned}$$

The antiderivation property of $\mathbf{d}$ is proved by induction using equation (6.4.6) and the antiderivation property for the interior products. Finally, the formula $F^* \circ \mathbf{d} = \mathbf{d} \circ F^*$ for a map F follows by definition and the properties

$$\begin{aligned}F^*(\pounds_X \omega) &= \pounds_X(F^*\omega), \\ (F^*\omega)(X_1, \ldots, X_k) &= F^*(\omega(X_1', \ldots, X_k')), \\ F^*[X, Y] &= [X', Y']\end{aligned}$$

if $X_i \sim_F X'$, $i = 1, \ldots, k$, $X \sim_F X'$, and $Y \sim_F Y'$. Thus, with the preceding procedure, $\mathbf{d}$ is defined on Banach manifolds and satisfies all the key properties that it does in the finite-dimensional case. These key properties are summarized at the end of this section.

Vector Valued Forms. For vector-valued forms, we adopt, as in the preceding supplement, Palais' formula from Proposition 6.4.11(ii) as the definition of $\mathbf{d}$ on an open subset of M. Note again that this definition uses the fact that $\pounds_X$ is defined for vector-valued tensors, and again one has to prove that the local formula in Corollary 6.4.2 holds. Then all properties in the table at the end of this section are verified in the same manner as previously.

For vector-valued forms we have an additional formula on $\Omega^k(M;\, F)$

$$\mathbf{d} \circ A = A \circ \mathbf{d}$$

for any $A \in L(F, F')$. If F is finite dimensional, the definition and properties of $\mathbf{d}$ become quite obvious; one notices that if

$$\omega = \sum_{j=1}^{n} \omega_j f_j \in \Omega^k(M; F),$$

where $\omega_j \in \Omega^k(M)$ and $f_1, \ldots, f_n$ is a basis of F, then $\mathbf{d}\omega$ is given by

$$\mathbf{d}\omega = \sum_{j=1}^{n} \mathbf{d}\omega_j f_j$$

and this formula can be taken as the definition of $\mathbf{d}$ in this case. This method does not work for vector-***bundle*** valued forms. Additional structure on the bundle is required to be able to lift $\pounds_X$.

Closed and Exact Forms and the Poincaré Lemma. The Poincaré lemma is a generalization and unification of two well-known facts in vector calculus:

1. if $\operatorname{curl} \mathbf{F} = 0$,then locally $\mathbf{F} = \nabla f$;
2. if $\operatorname{div} \mathbf{F} = 0$, then locally $\mathbf{F} = \operatorname{curl} \mathbf{G}$.

6.4.13 Definition. *We call $\omega \in \Omega^k(M)$ **closed** if $\mathbf{d}\omega = 0$, and **exact** if there is an $\alpha \in \Omega^{k-1}(M)$ such that $\omega = \mathbf{d}\alpha$.*

6.4.14 Theorem. *The following hold:*

(i) *Every exact form is closed.*

(ii) **Poincaré Lemma.** *If ω is closed, then for each $m \in M$, there is a neighborhood U for which $\omega|U \in \Omega^k(U)$ is exact.*

Proof. Part (i) is clear since $\mathbf{d} \circ \mathbf{d} = 0$. Using a local chart it is sufficient to consider the case $\omega \in \Omega^k(U)$, where $U \subset E$ is a disk about $0 \in E$, to prove (ii). On U we construct an $\mathbb{R}$-linear mapping $\mathbf{H} : \Omega^k(U) \to \Omega^{k-1}(U)$ such that $\mathbf{d} \circ \mathbf{H} + \mathbf{H} \circ \mathbf{d}$ is the identity on $\Omega^k(U)$. This will give the result, for $\mathbf{d}\omega = 0$ implies $\mathbf{d}(\mathbf{H}\omega) = \omega$. For $e_1, \ldots, e_k \in E$, define

$$\mathbf{H}\omega(u)(e_1, \ldots, e_{k-1}) = \int_0^1 t^{k-1}\omega(tu)(u, e_1, \ldots, e_{k-1})dt.$$

By Corollary 6.4.2,

$$\begin{aligned}
\mathbf{dH}\omega(u) \cdot (e_1, \ldots, e_k) &= \sum_{i=1}^k (-1)^{i+1}\mathbf{DH}\omega(u) \cdot e_i(e_1, \ldots, \hat{e}_i, \ldots, e_k) \\
&= \sum_{i=1}^k (-1)^{i+1} \int_0^1 t^{k-1}\omega(tu)(e_i, e_1, \ldots, \hat{e}_i, \ldots, e_k)dt \\
&\quad + \sum_{i=1}^k (-1)^{i+1} \int_0^1 t^k \mathbf{D}\omega(tu) \cdot e_i(u, e_1, \ldots, \hat{e}_i, \ldots, e_k)dt.
\end{aligned}$$

(The interchange of $\mathbf{D}$ and the integral is permissible, as ω is smooth and bounded as a function of $t \in [0, 1]$.) However, we also have, by Corollary 6.4.2,

$$\begin{aligned}
\mathbf{Hd}\omega(u) \cdot (e_1, \ldots, e_k) &= \int_0^t t^k \mathbf{d}\omega(tu)(u, e_1, \ldots, e_k)dt \\
&= \int_0^t t^k \mathbf{D}\omega(tu) \cdot u(e_1, \ldots, e_k)dt \\
&\quad + \sum_{i=1}^k (-1)^i \int_0^1 t^k \mathbf{D}\omega(tu) \cdot e_i(u, e_1, \ldots, \hat{e}_i, \ldots, e_k)dt.
\end{aligned}$$

Hence

$$\begin{aligned}
[\mathbf{dH}\omega(u) + \mathbf{HD}\omega(u)](e_1, \ldots, e_k) &= \int_0^1 kt^{k-1}\omega(tu) \cdot (e_1, \ldots, e_k)dt \\
&\quad + \int_0^1 t^k \mathbf{D}\omega(tu) \cdot u(e_1, \ldots, e_k)dt \\
&= \int_0^1 \frac{d}{dt}[t^k\omega(tu) \cdot (e_1, \ldots, e_k)]dt \\
&= \omega(u) \cdot (e_1, \ldots, e_k).
\end{aligned}$$

■

There is another proof of the Poncaré lemma based on the Lie transform method 5.4.7. It will help the reader master the proof of Darboux' Theorem in §8.1 and is similar in spirit to the proof of Frobenius' Theorem 4.4.3.

Alternative Proof of the Poincaré Lemma. Again let U be a ball about 0 in E. Let, for $t > 0$, $F_t(u) = tu$. Thus, F_t is a diffeomorphism and, starting at $t = 1$, is generated by the time-dependent vector field

$$X_t(u) = \frac{u}{t};$$

that is, $F_1(u) = u$ and $dF_t(u)/dt = X_t(F_t(u))$. Therefore, since ω is closed, Cartan's magic formula gives

$$\frac{d}{dt} F_t^* \omega = F_t^* \pounds_{X_t} \omega = \mathbf{d}(F_t^* \mathbf{i}_{X_t} \omega).$$

For $0 < t_0 \leq 1$, we get

$$\omega - F_{t_0}^* \omega = \mathbf{d} \int_{t_0}^1 F_t^* \mathbf{i}_{X_t} \omega \, dt.$$

Letting $t_0 \to 0$, we get $\omega = \mathbf{d}\beta$, where

$$\beta = \int_0^1 F_t^* \mathbf{i}_{X_t} \omega \, dt.$$

Explicitly,

$$\beta_u(e_1, \ldots, e_{k-1}) = \int_0^1 t^{k-1} \omega_{tu}(u, e_1, \ldots, e_{k-1}) \, dt.$$

(Note that this formula for β agrees with that in the previous proof.) ■

Cohomology. It is not true that closed forms are always exact (for example, on $\mathbb{R}^2 \backslash \{(0,\ 0)\}$ or on a sphere—see Exercise 6.4-4). In fact, the quotient groups of closed forms by exact forms (called the ***deRham cohomology groups*** of M) are important algebraic objects attached to a manifold; they are discussed further in §7.6. Below we shall prove that on smoothly contractible manifolds, closed forms are always exact.

6.4.15 Definition. *Let $t \geq 1$. Two C^r maps $f, g : M \to N$ are said to be (properly) C^r-**homotopic**, if there exists an $\epsilon > 0$ and a C^r (proper) mapping $F :]-\epsilon, 1 + \epsilon[\times M \to N$ such that $F(0, m) = f(m)$, and $F(1, m) = g(m)$ for all $m \in M$. The manifold M is called C^r-**contractible** if there exists a point $m_0 \in M$ and C^r-homotopy F of the constant map $m \mapsto m_0$ with the identity map of M; F is called a C^r-**contraction** of M to m_0*

The following theorem represents a verification of the ***homotopy axiom for the deRham cohomology***.

6.4.16 Theorem. *Let $f, g : M \to N$ be two (properly) C^r-homotopic maps and $\alpha \in \Omega^k(N)$ a closed k-form (with compact support) on N. Then $g^*\alpha - f^*\alpha \in \Omega^k(M)$ is an exact k-form on M (with compact support).*

The proof is based on the following.

6.4.17 Lemma (Deformation Lemma). *For a C^r-manifold M, let the C^r mapping*

$$i_t : M \to]-\epsilon, 1 + \epsilon[\times M$$

be given by $i_t(m) = (t, m)$. Define

$$\mathbf{H} : \Omega^{k+1}(]-\epsilon, 1 + \epsilon[\times M) \to \Omega^k(M)$$

by

$$\mathbf{H}\alpha = \int_0^1 i_s^*(\mathbf{i}_{\partial/\partial t} \alpha) ds.$$

Then $\mathbf{d} \circ \mathbf{H} + \mathbf{H} \circ \mathbf{d} = i_1^* - i_0^*$.

Proof. Since the flow of the vector field $\partial/\partial t \in \mathfrak{X}(]-\epsilon, 1+\epsilon[\times M)$ is given by $F_\lambda(s,m) = (s+\lambda, m)$, that is, $i_{s+\lambda} = F_\lambda \circ i_s$, for any form $\beta \in \Omega^l(]-\epsilon, 1+\epsilon[\times M)$ we get

$$i_s^* \pounds_{\partial/\partial t}\beta = i_s^* \left.\frac{d}{d\lambda}\right|_{\lambda=0} F_\lambda^*\beta = \left.\frac{d}{d\lambda}\right|_{\lambda=0} i_s^* F_\lambda^* \beta = \left.\frac{d}{d\lambda}\right|_{\lambda=0} i_{s+\lambda}^*\beta = \frac{d}{ds} i_s^*\beta.$$

Therefore, since the integrand in the formula for $\mathbf{H}$ is smooth, $\mathbf{d}$ and the integral sign commute, so that by Cartan's formula (6.4.6) and the above formula we get

$$\begin{aligned}\mathbf{d}(\mathbf{H}\alpha) + \mathbf{H}(\mathbf{d}\alpha) &= \int_0^1 i_s^*(\mathbf{d}\mathbf{i}_{\partial/\partial t} + \mathbf{i}_{\partial/\partial t}\mathbf{d})\alpha\, ds = \int_0^1 i_s^* \pounds_{\partial/\partial t}\alpha\, ds \\ &= \int_0^1 \frac{d}{ds} i_s^*\alpha\, ds = i_1^*\alpha - i_0^*\alpha.\end{aligned}$$ ■

Proof of Theorem 6.4.16. Define $G = \mathbf{H} \circ F^*$, where $\mathbf{H}$ is given in the deformation lemma 6.4.17 and F is the (proper) homotopy between f and g. Since F^* commutes with $\mathbf{d}$ we get $\mathbf{d} \circ G + G \circ \mathbf{d} = g^* - f^*$, so that if the form $\alpha \in \Omega^k(N)$ is closed (and has compact support), $(g^* - f^*)(\alpha) = \mathbf{d}(G\alpha)$ (and $G\alpha$ has compact support). ■

6.4.18 Lemma (Poincaré Lemma for Contractible Manifolds)**.**
Any closed form on a smoothly contractible manifold is exact.

Proof. Apply the previous theorem with $g =$ identity on M and $f(m) = m_0$. ■

The naturality of the exterior derivative has been investigated by Palais [1959]. He proves the following result. Let M be a connected paracompact n-manifold and assume that $A : \Omega^p(M) \to \Omega^q(M)$ is a linear operator commuting with pull-back, that is, $A \circ \varphi^* = \varphi^* \circ A$ for any diffeomorphism $\varphi : M \to M$. Then

$$A = \begin{cases} 0, & \text{if } 0 \le p \le n,\ 0 < q < n,\ q \ne p,\ p+1; \\ a(\text{Identity}), & \text{if } 0 < q = p \le n; \\ b\mathbf{d}, & \text{if } 0 \le p \le n-1,\ q = p+1, \end{cases}$$

for some real constants a, b. If M is compact, then in addition we have

$$A = \begin{cases} 0, & \text{if } q = 0,\ 0 < p < n; \\ c(\text{Identity}), & \text{if } p = q = 0; \\ 0, & \text{if } q = 0,\ p = n,\ M \text{ is non-orientable or orientable} \\ & \quad \text{and reversible}; \\ d\int_M, & \text{if } q = 0,\ p = n,\ M \text{ is orientable and non-reversible}, \end{cases}$$

for some real constants c, d. (Orientability and reversibility will be defined in the next section whereas integration will be the subject of Chapter 7.)

Supplement 6.4B

Differential Ideals and Pfaffian Systems

This box discusses a reformation of the Frobenius theorem in terms of differential ideals in the spirit of E. Cartan. Recall that a subbundle $E \subset TM$ is called ***involutive*** if for all pairs (X, Y) of local sections of E

defined on some open subset of M, the bracket $[X, Y]$ is also a local section of E. The subbundle E is called ***integrable*** if at every point $m \in M$ there is a local submanifold N of M such that $T_m N = E_m$. Frobenius' theorem states that E is integrable iff it is involutive (see §4.4).

Before starting the general theory let us show by a simple example how forms and involutive subbundles are interconnected. Let $\omega \in \Omega^2(M)$ and assume that $E_\omega = \{ v \in TM \mid \mathbf{i}_v \omega = 0 \}$ is a subbundle of TM. If X and Y are two sections of E_ω then

$$\mathbf{i}_{[X,Y]}\omega = \pounds_X \mathbf{i}_Y \omega - \mathbf{i}_Y \pounds_X \omega = -\mathbf{i}_Y \mathbf{d} \mathbf{i}_X \omega - \mathbf{i}_Y \mathbf{i}_X \mathbf{d}\omega = \mathbf{i}_X \mathbf{i}_Y \mathbf{d}\omega.$$

For any subbundle E, the k-***annihilator*** of E is defined by

$$E^0(k) = \left\{ \alpha \in \textstyle\bigwedge_m^k(M) \;\middle|\; \begin{array}{l} \alpha(m)(v_1, \ldots, v_k) = 0 \\ \quad \text{for all } v_1, \ldots, v_k \in E_m,\ m \in M \end{array} \right\}.$$

This is a subbundle of the bundle $\bigwedge^k(M)$ of k-forms. Denote by $\Gamma(U, E)$ the C^∞ sections of E over the open set U of M and notice that

$$I(E) = \oplus_{0 \le k < \infty} \textstyle\bigwedge(M, E^0(k))$$

is an ***ideal*** of $\omega(M)$; that is, if $\omega_1, \omega_2, \in I(E)$ and $\rho \in \Omega(M)$, then $\omega_1 + \omega_2 \in I(E)$ and $\rho \wedge \omega_1 \in I(E)$.

6.4.19 Proposition. *The subbundle E of TM is involutive if for all open subsets U of M and all $\omega \in \Gamma(U, E^0(1))$, we have $\mathbf{d}\omega \in \Gamma(U, E^0(2))$. If E is involutive, then $\omega \in \Gamma(U, E^0(k))$ implies $\mathbf{d}\omega \in \Gamma(U, E^0(k+1))$.*

Proof. For any $\alpha \in \Gamma(U, E^0(1))$ and $X, Y \in \Gamma(U, E)$, Proposition 6.4.11(ii) yields

$$\mathbf{d}\omega(X, Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X, Y]) = -\alpha([X, Y]).$$

Thus, E is involutive iff $\mathbf{d}\alpha(X, Y) = 0$; that is, $\mathbf{d}\alpha \in \Gamma(U, E^0(2))$. ■

The Frobenius theorem in terms of differential forms takes the following form.

6.4.20 Corollary. *The subbundle $E \subset TM$ is integrable if, for all open subsets U of M, $\omega \in \Gamma(U, E^0(1))$ implies $\mathbf{d}\omega \in \Gamma(U, E^0(2))$.*

The following considerations are strictly finite dimensional. They can be generalized to infinite-dimensional manifolds under suitable splitting assumptions. We restrict ourselves to the finite-dimensional situation due to their importance in applications and for simplicity of presentation.

6.4.21 Definition. *Let M be an n-manifold and $I \subset \Omega(M)$ be an ideal. We say that I is **locally generated by** $n - k$ **independent one-forms**, if every point of M has a neighborhood U and $n - k$ pointwise linearly independent one-forms $\omega_1, \ldots, \omega_{n-k} \in \Omega^1(U)$ such that:*

(i) *if $\omega \in I$, then $\omega | U = \sum_{i=1}^{n-k} \Theta_i \wedge \omega_i$ for some $\Theta_i \in \Omega(M)$;*

(ii) *if $\omega \in \Omega(M)$ and M is covered by open sets U such that for each U in this cover, $\omega | U = \sum_{i=1}^{n-k} \Theta_i \wedge \omega_i$ for some $\Theta_i \in \Omega(M)$, then $\omega \in I$.*

*The ideal $I \subset \omega(M)$ is called a **differential ideal** if $\mathbf{d}I \subset I$.*

Finitely generated ideals of $\Omega(M)$ are characterized by being of the form $I(E)$. More precisely, we have the following.

6.4.22 Proposition. *Let I be an ideal of $\omega(M)$ and let $n = \dim(M)$. the ideal I is locally generated by $n - k$ linearly independent one-forms iff there exists a subbundle $E \subset TM$ with k-dimensional fiber such that $I = I(E)$. Moreover, the bundle E is uniquely determined by I.*

Proof. If E has k-dimensional fiber, let $X_{n-l+1}, \ldots, X_n$ be a local basis of $\Gamma(U, E)$. Complete this to a basis of $\mathfrak{X}(U)$ and let $\omega_i \in \Omega^1(U)$ be the dual basis. Then clearly $\omega_1, \ldots, x_{n-k}$ are linearly independent and locally generate $I(E)$.

Conversely, let $\omega_1, \ldots, \omega_{n-k}$ generate I over U and define

$$E_m = \{ v \in T_m M \mid \omega_i(m)(v) = 0, 1, \ldots, n-k \}.$$

E_m is clearly independent of the generators of I over U so that $E = \cap_{m \in M} E_m$ is a subbundle of TM. It is straightforward to check that $I = I(E)$. Finally, E is unique since $E \neq E'$ implies $I(E) \neq I(E')$ by construction. ■

Different ideals are characterized among finitely generated ones by the following.

6.4.23 Proposition. *Let I be an ideal of $\Omega(M)$ locally generated by $n-k$ linearly independent forms $\omega_1, \ldots, \omega_{w-k} \in \Omega^1(U)$, $n = \dim(M)$, and let $\omega_1 \wedge \cdots \wedge \omega_{w-k} = \omega \in \Omega^{n-k}(U)$. Then the following are equivalent*:

(i) *I is a differential ideal*;

(ii) $\mathbf{d}\omega = \sum_{j=1}^{n-k} \omega^{ij} \wedge \omega_j$ *for some $\omega_{ij} \in \Omega^l(U)$ and for every U, as in the hypothesis*;

(iii) $\mathbf{d}\omega_i \wedge \omega = 0$ *for all open sets U, as in the hypothesis*;

(iv) *there exists $\Theta \in \Omega^1(U)$ such that $\mathbf{d}\omega = \Theta \wedge \omega$ for all open sets U, as in the hypothesis.*

Proof. That conditions (i) and (ii) are equivalent and (ii) implies (iv) follows from the definitions. Condition (iv) means that

$$\sum_{i=1}^{n-k} (-1)^i \mathbf{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_i \wedge \cdots \wedge \omega_{n-k} = \Theta \wedge \omega_1 \wedge \cdots \wedge \omega_{n-k},$$

so that multiplying by ω_i we get (iii). Finally, we show that (iii) implies (ii). Let $\omega_1, \ldots, \omega_n \in \Omega^1(U)$ be a basis that $\omega_1, \ldots, \omega_{n-k}$ generates for I over U. Then

$$\mathbf{d}\omega_i = \sum_{j<l} \alpha_{ijl} \omega_j \wedge \omega_l, \quad \text{where } \alpha_{ijl} \in \mathcal{F}(U).$$

But

$$0 = \mathbf{d}\omega_i \wedge \omega = \sum_{n-k<j<l} \alpha_{ijl} \omega_j \wedge \omega_l \wedge \omega_1 \wedge \cdots \wedge \omega_{n-k}$$

and thus $\alpha_{ijl} = 0$ for $n-k < j < l$. Hence

$$\mathbf{d}\omega_i = \sum_{j=1}^{n-k} \left(- \sum_{l=j+1}^{n} \alpha_{ijl} \omega_l \right) \wedge \omega_j.$$ ■

Assembling the preceding results, we get the following version of the Frobenius theorem.

6.4.24 Theorem. *Let M be an n-manifold and $E \subset TM$ be a subbundle with k-dimensional fiber, and $I(E)$ the associated ideal. The following are equivalent*:

(i) *E is integrable*;

(ii) *E is involutive;*

(iii) *$I(E)$ is a differential ideal locally generated by $n-k$ linearly independent one-forms $\omega_1, \ldots, \omega_{n-k} \in \Omega^1(U)$;*

(iv) *for every point of M there exists an open set U and $\omega_1, \ldots, \omega_{n-k} \in \Omega^1(U)$ generating $I(E)$ such that*

$$\mathbf{d}\omega_i = \sum_{j=1}^{n-k} \omega_{ij} \wedge \omega_j \quad \textit{for some } \omega_{ij} \in \Omega^1(U);$$

(v) *same as* (iv) *but where the ω_i satisfy:*

$$\mathbf{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_{n-k} = 0;$$

(vi) *same as* (iv) *but with the condition on ω_i being: there exists $\Theta \in \Omega^1(U)$ such that $\mathbf{d}\omega = \Theta \wedge \omega$, where $\omega = \omega_1 \wedge \cdots \wedge \omega_{n-k}$.*

6.4.25 Examples.

A. In classical texts (such as Cartan [1945] and Flanders [1963]), a system of equations

$$\omega_1 = 0, \ldots, \omega_{n-k} = 0 \quad \text{where } \omega_i \in \Omega^1(U) \text{ and } U \subset \mathbb{R}^n$$

is called ***Pfaffian system.*** A solution to this system is a k-dimensional submanifold N of U given by $x^i = x^i(u^1, \ldots, u^k)$ such that if one substitutes these values of x^i in the system, the result is identically zero. Geometrically, this means that $\omega_1, \ldots, \omega_{n-k}$ annihilate TN. Thus, finding solutions of the Pfaffian system reverts to finding integral manifolds of the subbundle

$$E = \{\, v \in TU \mid \omega(v) = 0,\ i = 1, \ldots, n-k \,\}$$

for which Frobenius' theorem is applicable; thus we must have

$$\mathbf{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_{n-k} = 0.$$

This condition is equivalent to the existence of smooth functions a_{ij}, b_j on U such that

$$\omega_i = \sum_{j=1}^{n-k} a_{ij} \mathbf{d} b_j.$$

To see this, recall that by the Frobenius theorem there are local coordinates $b_1, \ldots, b_n$ on U such that the integral of E are given by $b_1 =$ constant, so that $\mathbf{d}b_i$, $i = 1, \ldots, n-k$ annihilate the tangent spaces to these submanifolds. Thus the ideal $\mathbf{I}$ generated by $\mathbf{d}b_1, \ldots, \mathbf{d}b_{n-k}$; that is, $\omega_i = \sum_{j=1,\ldots,n-k} a_{ij}\mathbf{d}b_j$ for some smooth functions a_{ij} on U.

B. Let us analyze the case of one Pfaffian equation in $\mathbb{R}^2$. Let

$$\omega = P(x,y)dx + Q(x,y)dy \in \Omega^1(\mathbb{R}^2)$$

using standard (x,y) coordinates. We seek a solution to $\omega = 0$. This is equivalent to $dy/dx = -P(x,y)/Q(x,y)$, so existence and uniqueness of solutions for ordinary differential equations assures the local existence of a function $f(x,y)$ such that $f(x,y) =$ constant give the integral curves $y(x)$. In other words, $f(x,y) =$ constant is an integral manifold of $\omega = 0$. The same result could have been obtained by means of the Frobenius theorem. Since $\mathbf{d}\omega \wedge \omega \in \Omega^3(\mathbb{R}^2)$, we get $\mathbf{d}\omega \wedge \omega = 0$, so integral manifolds exist and are unique. In texts on differential equations, this problem is often solved with the aid of integrating factors. More precisely, if ω

is not (locally) exact, can a function f and function g, called an ***integrating factor***, be found, such that $g\omega = \mathbf{d}f$? The answer is "yes" by Theorem 6.4.24(iii) for choosing f as above, $E = \ker \mathbf{d}f$ locally. Thus, g is found by solving the partial differential equation,

$$\frac{\partial(gP)}{\partial y} = \frac{\partial(gQ)}{\partial x}.$$

This always has a solution and the connection between g and f is given by

$$g = \frac{1}{P}\frac{\partial f}{\partial x} = \frac{1}{Q}\frac{\partial f}{\partial y},$$

$f(x, y) =$ constant being the solution of $\omega = 0$.

C. Let us analyze a Pfaffian equation $\omega = 0$ in $\mathbb{R}^n$. As in Example B, we would like to be able to write $g\omega = \mathbf{d}f$ with $\mathbf{d}f \neq 0$ on $U \subset \mathbb{R}^r$, for then $f(x^1, \ldots, x^n) =$ constant gives the $(n-1)$-dimensional integral manifolds; that is, the bundle defined by ω integrable. Conversely, if the bundle defined by ω is integrable, then by Example B, $g\omega = \mathbf{d}f$. Integrability is (by the Frobenius theorem) equivalent to $\mathbf{d}\omega \wedge \omega = 0$, which, as we have seen in Example B, is always verified for $n = 2$. For $n \geq 3$, however, this is a genuine condition. If $n = 3$, let

$$\omega = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz.$$

Then

$$\mathbf{d}\omega \wedge \omega = \Big[(P\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}) + Q(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}) + R(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\Big]\, dx \wedge dy \wedge dz;$$

thus, $\omega = 0$ is integrable iff the term in the square brackets vanishes.

D. The Frobenius' theorem is often used in overdetermined systems of partial differential equations to answer the question of existence and uniqueness of solutions. Consider for instance the following system of Mayer [1872] in $\mathbb{R}^{p+q} = \{(x^1, \ldots, x^p, y^1, \ldots, y^q)\}$:

$$\frac{dy^\alpha}{dx^i} = A_i^\alpha(x^1, \ldots, x^p, y^1, \ldots, y^q), \quad i = 1, \ldots, p,\ \alpha = 1, \ldots, q.$$

We ask whether there is a solution $y = f(x, c)$ for any choice of initial conditions c such that $f(0, c) = c$. The system is equivalent to the following Pfaffian system:

$$\omega^\alpha = dy^\alpha - A_i^\alpha dx^i = 0.$$

Since the existence of a solution is equivalent to the existence of p-dimensional integral manifolds, Frobenius' theorem asserts that the existence and uniqueness is equivalent to

$$\mathbf{d}\omega^\alpha = \sum_{\beta=1,\ldots,q} \omega^{\alpha\beta} \wedge \omega^\beta$$

for some one-forms $\omega^{\alpha\beta}$. A straightforward computation shows that

$$\mathbf{d}\omega^\alpha = C_{jk}^\alpha dx^j \wedge dx^k + \frac{\partial A_i^\alpha}{\partial y^\beta} dx^i \wedge \omega^\beta,$$

where

$$C^\alpha_{jk} = \frac{\partial A^\alpha_j}{\partial x^k} - \frac{\partial A^\alpha_k}{\partial x^j} + \frac{\partial A^\alpha}{\partial y^\beta} A^\beta_k - \frac{\partial A^\alpha_k}{\partial y^\beta} A^\beta_j.$$

Since $dx^1, \ldots, dx^p, \omega^1, \ldots, \omega^q$ are a basis of $\Omega^1(\mathbb{R}^{p+q})$, we see that

$$\mathbf{d}\omega^\alpha = \sum_{\beta=1,\ldots,q} \omega^{\alpha\beta} \wedge \omega^b \quad \text{for some one-forms } \omega^{\alpha\beta} \text{ iff } C^\alpha_{jk} = 0.$$

Thus the Mayer system is integrable iff $C^\alpha_{jk} = 0$. ♦

In §8.4 and §8.5 we shall give some applications of Frobenius' theorem to problems in constraints and control theory. Many of these applications may alternatively be understood in terms of Pfaffian systems; see, for example, Hermann [1977] (Chapter 18).

VECTOR CALCULUS AND DIFFERENTIAL FORMS

1. **Sharp and Flat** (Using standard coordinates in $\mathbb{R}^3$)
   (a) $v^\flat = v^1 dx + v^2 dy + v^3 dz =$ one-form corresponding to the vector $v = v^1 e_1 + v^2 e_2 + v^3 e_3$
   (b) $\alpha^\# = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 =$ vector corresponding to the one-form $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$
2. **Hodge Star Operator** (equation (6.2.6)),
   (a) $*1 = dx \wedge dy \wedge dz$
   (b) $*dx = dy \wedge dz,\ *(dx \wedge dz) = -dy,\ *(dx \wedge dy) = dz$
   (c) $*(dy \wedge dz) = dx,\ *(dx \wedge dz) = -dy,\ *(dx \wedge dy) = dz$
   (d) $*(dx \wedge dy \wedge dz) = 1$
3. **Cross Product and Dot Product**
   (a) $\mathbf{v} \times \mathbf{w} = \left[* \left(\mathbf{v}^\flat \wedge \mathbf{w}^\flat\right)\right]^\#$
   (b) $(\mathbf{v} \cdot \mathbf{w}) dx \wedge dy \wedge dz = \mathbf{v}^\flat \wedge * \left(\mathbf{w}^\flat\right)$
4. **Gradient**

$$\nabla f = \operatorname{grad} f = (\mathbf{d}f)^\#$$

5. **Divergence**

$$\nabla \cdot \mathbf{F} = \operatorname{div} \mathbf{F} = *\mathbf{d}\left(*\mathbf{F}^\flat\right)$$

6. **Curl**

$$\nabla \times \mathbf{F} = \operatorname{curl} \mathbf{F} = \left[*\left(\mathbf{d}\mathbf{F}^\flat\right)\right]^\#$$

IDENTITIES FOR VECTOR FIELDS AND FORMS

1. Vector fields on M with the bracket $[X, Y]$ form a Lie algebra; that is, $[X, Y]$ is real bilinear, skew symmetric, and Jacobi's identity holds:

$$[[X,Y],Z]+[[Z,X],Y]+[[Y,Z],X]=0.$$

Locally,

$$[X,Y]=\mathbf{D}Y\cdot X-\mathbf{D}X\cdot Y$$

and on functions, $[X,Y][f]=X[Y[f]]-Y[X[f]]$.

2. For diffeomorphisms φ, ψ, we have

$$\varphi_*[X,Y]=[\varphi_*X,\psi_*Y] \quad \text{and} \quad (\varphi\circ\psi)_*X=\varphi_*\psi_*X.$$

3. The forms on a manifold are a real associative algebra with $\wedge$ as multiplication. Furthermore,

$$\alpha\wedge\beta=(-1)^{kl}\beta\wedge\alpha$$

for k- and l-forms α and β, respectively.

4. For maps, φ, ψ, we have

$$\varphi^*(\alpha\wedge\beta)=\varphi^*\alpha\wedge\varphi^*\beta, \quad (\varphi\circ\psi)^*\alpha=\psi^*\varphi^*\alpha.$$

5. $\mathbf{d}$ is a real linear map on forms and

$$\mathbf{dd}\alpha=0, \quad \mathbf{d}(\alpha\wedge\beta)=\mathbf{d}\alpha\wedge\beta+(-1)^k\alpha\wedge\mathbf{d}\beta \quad \text{for } \alpha \text{ a } k\text{-form}.$$

6. For α a k-form and $X_0, \ldots, X_k$ vector fields:

$$\begin{aligned}\mathbf{d}\alpha&(X_0,\ldots,X_k)\\ &=\sum_{i=0}^{k}(-1)^i X_i[\alpha(X_0,\ldots,\hat{X}_i,\ldots,X_k)]\\ &\quad+\sum_{0\le i<j\le k}(-1)^{i+j}\alpha([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k).\end{aligned}$$

If M is finite dimensional and $\alpha=\alpha_{i_1\ldots i_k}dx^{i_1}\wedge\cdots\wedge\ldots dx^{i_k}$, $i_1<\cdots<i_k$, then

$$\begin{aligned}(\mathbf{d}\omega)_{j_1\ldots j_{k+1}}&=\sum_{p=1}^{k}(-1)^{p-1}\frac{\partial}{\partial x^{j_k}}\alpha_{j_1\ldots j_{p-1}j_{p+1}\ldots j_{k+1}}\\ &\qquad+(-1)^k\frac{\partial}{\partial x^{j_{k+1}}}\alpha_{j_1\ldots j_k}, \quad \text{for } j_1<\cdots<j_{k+1}.\end{aligned}$$

Locally,

$$\mathbf{d}\omega(x)(v_0,\ldots,v_k)=\sum_{i=0}^{k}(-1)^i\mathbf{D}\omega(x)\cdot v_i(v_0,\ldots,\hat{v}_i,\ldots,v_k)$$

7. For a map φ, $\varphi^*\mathbf{d}\alpha = \mathbf{d}\varphi^*\alpha$.

8. (**Poincaré lemma**.) If $\mathbf{d}\alpha = 0$, then α is locally exact; that is, there is a neighborhood U about each point on which $\alpha = \mathbf{d}\beta$ for some form β defined on U. The same result holds globally on a contractible manifold.

9. $\mathbf{i}_X\alpha$ is real bilinear in X, α and for $h : M \to \mathbb{R}$, $\mathbf{i}_{hX}\alpha = h\mathbf{i}_X\alpha = \mathbf{i}_X h\alpha$. Also $\mathbf{i}_X\mathbf{i}_X\alpha = 0$, and

$$\mathbf{i}_X(\alpha \wedge \beta) = \mathbf{i}_X\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{i}_X\beta$$

for α a k-form.

10. For a diffeomorphism φ, we have

$$\varphi^*\mathbf{i}_X\alpha = \mathbf{i}_{\varphi^*X}\varphi^*\alpha;$$

if $f : M \to N$ is a mapping and Y is f-related to X, then

$$\mathbf{i}_Y f^*\alpha = f^*\mathbf{i}_X\alpha.$$

11. $\pounds_X\alpha$ is real bilinear in X, α and

$$\pounds_X(\alpha \wedge \beta) = \pounds_X\alpha \wedge \beta + \alpha \wedge \pounds_X\beta.$$

12. (**Cartan's Magic Formula**.) $\pounds_X\alpha = \mathbf{d}\mathbf{i}_X\alpha + \mathbf{i}_X\mathbf{d}\alpha$.

13. For a diffeomorphism φ,

$$\varphi^*\pounds_X\alpha = \pounds_{\varphi^*X}\varphi^*\alpha;$$

if $f : M \to N$ is a mapping and Y is f-related to X, then

$$\pounds_Y f^*\alpha = f^*\pounds_X\alpha.$$

14.
$$\begin{aligned}(\pounds_X\alpha)(X_1, \ldots, X_k) &= X[\alpha(X_1, \ldots, X_k)] \\ &\quad - \sum_{i=1}^{k} \alpha(X_1, \ldots, [X, X_i], \ldots, X_k).\end{aligned}$$

Locally,

$$\begin{aligned}(\pounds_X\alpha)_x \cdot (v_1, \ldots, v_k) &= \mathbf{D}\alpha_X \cdot X(x) \cdot (v_1, \ldots, v_k) \\ &\quad + \sum_{k=1}^{n} \alpha_x(v_1, \ldots, \mathbf{D}X_x \cdot v_i, \ldots, v_k).\end{aligned}$$

15. The following identities hold:
   (a) $\pounds_{fX}\alpha = f\pounds_X\alpha + \mathbf{d}f \wedge \mathbf{i}_X\alpha$.
   (b) $\pounds_{[X,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$.
   (c) $\mathbf{i}_{[X,Y]}\alpha = \pounds_X\mathbf{i}_Y\alpha - \mathbf{i}_Y\pounds_X\alpha$.
   (d) $\pounds_X\mathbf{d}\alpha = \mathbf{d}\pounds_X\alpha$.
   (e) $\pounds_X\mathbf{i}_X\alpha = \mathbf{i}_X\pounds_X\alpha$.

16. If M is a finite dimensional manifold, $X = X^l \partial/\partial x^l$ and

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad i_1 < \cdots < i_k,$$

the following local formulas hold:

$$\begin{aligned}
\mathbf{d}\alpha &= \frac{\partial \alpha_{i_1 \dots i_k}}{\partial x^l} dx^l \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \\
\mathbf{i}_X \alpha &= X^l \alpha_{l i_2 \dots i_k} dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \\
\pounds_X \alpha &= X^l \frac{\partial \alpha_{i_1 \dots i_k}}{\partial x^l} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\
&\quad + \alpha_{i_1 i_2 \dots i_k} \frac{\partial X^{i_1}}{\partial x^l} dx^l \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \\
&\quad + \alpha_{i_2 i_2 \dots i_k} \frac{\partial X^{i_2}}{\partial x^l} dx^{i_1} \wedge dx^l \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_k} + \dots
\end{aligned}$$

Exercises

◇ **6.4-1.** Compute the exterior derivative of the following differential forms on $\mathbb{R}^3$:

$$\begin{aligned}
\alpha &= x^3 dx + y^3 dx \wedge dy + xyz dx \wedge dz; \\
\beta &= 3d^x dx \wedge dy + 9\cos(xy) dx \wedge dy \wedge dz.
\end{aligned}$$

◇ **6.4-2.** Using Examples 6.4.3 and the properties of $\mathbf{d}$ and $*$, prove the following formulas in $\mathbb{R}^3$ for $f, g : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{F}, \mathbf{G} \in \mathfrak{X}(\mathbb{R}^3)$:

(i) $\operatorname{grad}(fg) = (\operatorname{grad} g) f + f(\operatorname{grad} g)$.

(ii) curl $(f\mathbf{F}) = (\operatorname{grad} f) \times \mathbf{F} + f(\operatorname{grad} \mathbf{F})$.

(iii) div $(f\mathbf{F}) = \operatorname{grad}(f) \cdot \mathbf{F} + f \operatorname{div} \mathbf{F}$.

(iv) div $(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$.

(v) $\pounds_F \mathbf{G} = (\mathbf{F} \cdot \nabla)\mathbf{G} - (\mathbf{G} \cdot \nabla)\mathbf{F} = \mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F} - \operatorname{curl}(\mathbf{F} \times \mathbf{G})$.

(vi) $\operatorname{curl}(\mathbf{F} \times \mathbf{G}) = (\operatorname{div} \mathbf{G})\mathbf{F} - (\operatorname{div} \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$.

(vii) $\operatorname{curl}(\operatorname{div} \mathbf{F}) = \operatorname{grad}(\operatorname{div} \mathbf{F}) - \Delta \mathbf{F}$, where

$$(\Delta \mathbf{F})^i = \frac{\partial^2 F^i}{\partial x^2} + \frac{\partial^2 F^i}{\partial y^2} + \frac{\partial^2 F^i}{\partial z^2}$$

is the usual Laplacian.

(viii) $\nabla(\mathbf{F} \cdot \mathbf{F}) = 2(\mathbf{F} \cdot \nabla)\mathbf{F} + 2\mathbf{F} \times \operatorname{curl} \mathbf{F}$.

◇ **6.4-3.** Let $\varphi : S^1 \times \mathbb{R}_+ \to \mathbb{R}^2$ be the polar coordinate mapping defined by $\varphi(\theta, r) = (r\cos\theta, r\sin\theta)$. Compute $\varphi^*(dx \wedge dy)$ from the definitions and verify that it equals $\mathbf{d}(\varphi^* x) \wedge \mathbf{d}(\varphi^* y)$.

◇ **6.4-4.** On S^1 find a closed one-form α that is not exact.
Hint: On $\mathbb{R}^2 \backslash \{0\}$ consider $\alpha = (ydx - xdy)/(x^2 + y^2)$.

◇ **6.4-5.** Show that the following properties uniquely characterize $\mathbf{i}_X$ on finite-dimensional manifolds

(i) $\mathbf{i}_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is a $\wedge$ antiderivation;

(ii) $\mathbf{i}_X f = 0$ for $f \in \mathcal{F}(M)$;

(iii) $\mathbf{i}_X \omega = \omega(X)$ for $\omega \in \Omega^1(M)$;

(iv) $\mathbf{i}_X$ is natural with respect to restrictions.

Use these properties to show that $\mathbf{i}_{[X,Y]} = \pounds_X \mathbf{i}_Y - \mathbf{i}_Y \pounds_X$. Finally, show $\mathbf{i}_X \circ \mathbf{i}_X = 0$.

◇ **6.4-6.** Show that a derivation mapping $\Omega^k(M)$ to $\Omega^{k+1}(M)$ for all $k \geq 0$ is zero (note that $\mathbf{d}$ and $\mathbf{i}_X$ are *anti*derivations).

◇ **6.4-7.** Let $s : T^2M \to T^2M$ be the canonical involution of the second tangent bundle (see Exercise 3.4-4).

(i) If X is a vector field on M, show that $s \circ TX$ is a vector field on TM.

(ii) If F_t is the flow on X, prove that TF_t is a flow on TM generated by $s \circ TX$.

(iii) If μ is a one-form on M, $\mu' : TM \to \mathbb{R}$ is the corresponding function, and $w \in T^2M$, then show that

$$\mathbf{d}\mu'(sw) = \mathbf{d}\mu(\tau_{TM}(w), T\tau_M(w)) + \mathbf{d}\mu'(w).$$

◇ **6.4-8.** Prove the following ***relative Poincaré lemma.*** Let ω be a closed k-form on a manifold M and let $N \subset M$ be a closed submanifold. Assume that the pull-back of ω to N is zero. Then there is a $(k-1)$-form α on a neighborhood N such that $\mathbf{d}\alpha = \omega$ and α vanishes on N. If ω vanishes on N, then α can be chosen so that all its first partial derivatives vanish on M.
Hint: Let φ_t be a homotopy of a neighborhood of N to N and construct an $\mathbf{H}$ operator as in the Poncaré lemma using φ_t.

◇ **6.4-9** (Angular variables)**.** Let S^1 denote the circle identified as $S^1 = \mathbb{R}/(2\pi) = \{\, x \in \mathbb{C} \mid |z| = 1 \,\}$. Let $\gamma : \mathbb{R} \to S^1$; $x \mapsto e^{ix}$ be the exponential map. Show that γ induces a isomorphism $TS^1 = S^1 \times \mathbb{R}$. Let M be a manifold and let ω be an "angular variable," that is a smooth map $\omega : M \to S^1$. Define $\mathbf{d}\omega$, a one-form on M by taking the $\mathbb{R}$-projection of $T\omega$. Show that (i) if $\omega : M \to S^1$, then $\mathbf{dd}\omega = 0$; and (ii) if $f : M \to N$ is smooth, then $f^*(\mathbf{d}\omega) = \mathbf{d}(f^*\omega)$, where $f^*\omega = \omega \circ f$.

◇ **6.4-10.** Prove the identity

$$\pounds_X \mathbf{i}_Y - \pounds_Y \mathbf{i}_X - \mathbf{i}_{[X,Y]} = [\mathbf{d}, \mathbf{i}_X \circ \mathbf{i}_Y].$$

◇ **6.4-11.** (i) Let $X = (X^1, X^2, 0)$ be a vector field defined on the plane $S = \{\, (x, y, 0) \mid x, y \in \mathbb{R} \,\}$ in $\mathbb{R}^3$. Show that there exists $Y \in \mathfrak{X}(\mathbb{R}^3)$ such that $X = \operatorname{curl} Y$ on S.

Hint: Let

$$Y(x, y, z) = (zX^2(x, y), -zX^1(x, y), 0).$$

(ii) Let S be a closed surface on $\mathbb{R}^3$ and $X \in \mathfrak{X}(S)$. Show that there exists $Y \in \mathfrak{X}(\mathbb{R}^3)$ such that $X = $ curl Y on S.

Hint: By Theorem 5.5.9 extend X to $\tilde{X} \in \mathfrak{X}(\mathbb{R}^3)$ and put $\omega = *\tilde{X}^\flat$. Locally find α such that $\mathbf{d}\alpha = \omega$ by (i). Use a partition of unity (φ_i) to write $\omega = \sum \varphi_i \omega$ and let $\mathbf{d}\alpha_i = \varphi_i \omega, \alpha = \sum \alpha_i$.

(iii) Generalize this to forms on a closed submanifold of a manifold admitting C^k-partitions of unity.

◇ **6.4-12.** Let M be a manifold and $\alpha \in \Omega^k(M)$. If $\tau_M : TM \to M$ denotes the tangent bundle projection, let $\alpha' = \tau^*\alpha \in \Omega^k(TM)$. A k-form Γ on TM for which there is an $\alpha \in \Omega^k(m)$ such that $\alpha' = \Gamma$ is called ***basic.*** A vector field $x \in \mathfrak{X}(TM)$ is said to be ***vertical*** if $T\tau_M \circ X = 0$. Show that $\Gamma \in \Omega^k(TM)$ is basic if and only if $\mathbf{i}_X\Gamma = 0$, and $\pounds_X\Gamma = 0$ for any vertical vector field χ on TM. Conclude that if Γ is closed it is basic if and only if $\mathbf{i}_X\Gamma = 0$ for every vertical vector field X on TM.
HINT: Since X and the zero vector field on M are τ_M-related, if Γ is vertical, the two identities follow. Conversely, if F_t is the flow of X, then $F_t^*\Gamma = \Gamma$. Define $\alpha \in \Omega^k(TM)$ by

$$\alpha(m)(v_1, \ldots, v_k) = \Gamma(u)(V_1, \ldots, V_k),$$

where $\tau(u) = m$, $T_m\tau_M(V_i) = v_i$, $i = 1, \ldots, k$. Show that this definition is independent of the choices of $u, V_1, \ldots, V_k$ in the following way. Let $\tau(u') = m$, $T_m\tau_M(V_i') = v_i$, $i = 1, \ldots, k$, $w = u - u'$. Consider the local flow F_t in a vector bundle chart of TM containing T_mM which occurs only in the fibers and which on T_mM itself is translation by tw. The vector field it generates is vertical so that $T_t^*\Gamma = \Gamma$ and $F_1(u) = u'$. Let $T_{u'}F_1(V_i') = V_i'' \in T_u(TM)$ and show $T_u\tau(V_i'') = v_i$ since $\tau \circ \varphi_t = \tau; thatis, V_i'' - V_i$ is a vertical vector. Now use the fact that $V_i'' - V_i$ contracts with Γ to give zero to prove inductively that $\Gamma(u)(V_i, \ldots, V_k) = \Gamma(u')(V_i', \ldots, V_k')$.

◇ **6.4-13.** Show that on $\mathbb{R}^4$, the ideal generated by

$$\omega_1 = x^2dx^1 + x^3dx^4, \quad \omega_2 = x^3dx^2 + x^2dx^3$$

is a differential ideal. Find its integral manifolds.

6.5 Orientation, Volume Elements and the Codifferential

Orientation and Volume Manifolds. This section globalizes the setting of §6.2 from linear spaces to manifolds. All manifolds in this section are finite dimensional.[1]

6.5.1 Definition. *A **volume form** on an n-manifold M is an n-form $\mu \in \Omega^n(M)$ such that $\mu(m) \neq 0$ for all $m \in M$; M is called **orientable** if there exists some volume form on M. The pair (M, μ) is called a **volume manifold.***

Thus, μ assigns an orientation, as defined in equation (6.2.5), to each fiber of TM. For example, $\mathbb{R}^3$ has the same standard volume form $\mu = dx \wedge dy \wedge dz$.

6.5.2 Proposition. *Let M be an n-manifold.*

(i) *M is orientable iff there is an element $\mu \in \Omega^n(M)$ such that every other $\nu \in \Omega^n(M)$ may be written $\nu = f\mu$ for some $f \in \mathcal{F}(M)$.*

(ii) *If M is orientable then it has an atlas $\{(U_i, \varphi_i)\}$, where $\varphi_i : U_i \to U_i' \subset \mathbb{R}^n$, such that the Jacobian determinant of the overlap maps is positive (the Jacobian determinant is the determinant of the derivative, a linear map from $\mathbb{R}^n$ into $\mathbb{R}^n$). The converse is true if M is paracompact.*

Proof. For (i) assume first that M is orientable, with a volume form μ. Let ν be any other element of $\Omega^n(M)$. Now each fiber of $\bigwedge^n(M)$ is one-dimensional, so we may define a map $f : M \to \mathbb{R}$ by

$$\nu'(m) = f(m)\mu'(m) \quad \text{where } \mu(m) = \mu'(m)dx^1 \wedge \cdots \wedge dx^n$$

and similarly for ν'. Since, $\nu'(m) \neq 0$ for all $m \in M$, $F(m) = \nu'(m)/\mu'(m)$ is of class C^∞. Conversely, if $\Omega^n(M)$ is generated by ν, then $\nu'(m) \neq 0$ for all $m \in M$ since each fiber is one-dimensional.

[1]For infinite-dimensional analogues of orientability, see Elworthy and Tromba [1970b].

To prove (ii), let $\{(U_i, \varphi_i)\}$ be an atlas with $\varphi_i(U_i) = U_i' \subset \mathbb{R}^n$. We may assume that all U_i' are connected by taking restrictions if necessary. Now $\varphi_i^*\mu = f_i dx^1 \wedge \cdots \wedge dx^n = f_i\mu_0$, where μ_0 is the standard volume form on $\mathbb{R}^n$. By means of a reflection if necessary, we may assume that $f_i(u') > 0$ ($f_i \neq 0$ since ν is a volume form). However, a continuous real-valued function on a connected space that is nowhere zero is always > 0 or always < 0. Hence, for overlap maps we have

$$\begin{aligned}(\varphi_i \circ \varphi_j^{-1})_* dx^1 \wedge \cdots \wedge dx^n &= \varphi_{i*} \circ \varphi_{j*} dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{f_i}{f_j \circ \varphi_j \circ \varphi_i^{-1}} dx^1 \wedge \cdots \wedge dx^n.\end{aligned}$$

But

$$\psi^*(u)(\alpha^1 \wedge \cdots \wedge \alpha^n) = \mathbf{D}\psi(u)^* \cdot \alpha^1 \wedge \mathbf{D}\psi(u)^* \cdot \alpha^2 \wedge \cdots \wedge \mathbf{D}\psi(u)^* \cdot \alpha^n.$$

where $\mathbf{D}\psi(u)^* \cdot \alpha^1(e) = \alpha^1(\mathbf{D}\psi(u) \cdot e)$. Hence, by the definition of determinant,

$$\det(\mathbf{D}(\varphi_j \circ \varphi_i^{-1})) = \frac{f_i(u)}{f_j[(\varphi_j \circ \varphi_i^{-1})(u)]} > 0.$$

(We leave as an exercise the fact that the canonical isomorphism $L(E, E) \approx L(E^*, E^*)$ used before does not affect determinants.)

For the converse of (ii), suppose $\{(V_\alpha, \pi_\alpha)\}$ is an atlas with the given property, and let $\{(U_i, \varphi_i, g_i)\}$ a subordinate partition of unity. Let

$$\mu_i = \varphi_i^*(dx^1 \wedge \cdots \wedge dx^n) \in \Omega^n(U_i)$$

and let $\tilde{\mu_i}(m) = g_i(m)\mu_i(m)$ if $m \in U_i$ and $\tilde{\mu_i} = 0$ if $\notin U_i$. Since $\operatorname{supp}(g_i) \subset U_i$, we have $\tilde{\mu_i} \in \Omega^n(M)$. Let $\mu = \sum_i \tilde{\mu_i}$. Since this sum is finite in some neighborhood point, it is clear from local representatives that $\mu \in \Omega^n(M)$. To show that μ is a volume form on M, notice that on $U_i \cap U_j \neq \varnothing$ we have

$$\begin{aligned}\mu_j &= \varphi_j^*(dx^1 \wedge \cdots \wedge dx^n) = \varphi_i^*(\varphi_j \circ \varphi_i^{-1})^*(dx^1 \wedge \cdots \wedge dx^n) \\ &= [\det \mathbf{D}(\varphi_j \circ \varphi_i^{-1}) \circ \varphi_i] \circ \varphi_i^*(dx^1 \wedge \cdots \wedge dx^n) \\ &= [\det \mathbf{D}(\varphi_j \circ \varphi_i^{-1}) \circ \varphi_i]\mu_i = a_{ji}\mu_i\end{aligned}$$

where $a_{ji} \in \mathcal{F}(U_i \cap U_j)$, $a_{ji} > 0$ and there is no implied sum. By local finiteness of the covering $\{U_i\}$, a given point $m \in M$ belongs only to a finite number of open sets, say $U_{i_0}, U_{i_1}, \ldots, U_{i_N}$. Thus,

$$\mu(M) = \sum_{k=0}^{N} \mu_{i_k}(m) = \left\{\sum_{k=1}^{N} (1 + a_{i_k i_0}(m))\right\} \mu_{i_0}(m) \neq 0$$

since $\mu_{i_0}(m) \neq 0$ and each $a_{i_k i_0}(m) > 0$. It follows that $\mu(m) \neq 0$ for each $m \in M$. ■

Thus, if (M, μ) is a volume manifold we get a map from $\Omega^n(M)$ to $\mathcal{F}(M)$; namely, for each $\nu \in \Omega^n(M)$, there is a unique $f \in \mathcal{F}(M)$ such that $\nu = f\mu$.

6.5.3 Definition. *Let M be an orientable manifold. Two volume forms μ_1 and μ_2 on M are called* ***equivalent*** *if there is an $f \in \mathcal{F}(M)$ with $f(m) > 0$ for all $m \in M$ such that $\mu_1 = f\mu_2$. (This is clearly an equivalence relation.) An* ***orientation*** *of M is an equivalence class $[\mu]$ of volume forms on M. An* ***oriented manifold*** *$(M, [\mu])$, is an orientable manifold $\mathcal{M}$ together with an orientation $[\mu]$ on M.*

If $[\mu]$ is an orientation of M, then $[-\mu]$, (which is clearly another orientation) is called the ***reverse orientation.***

The next proposition tells us when $[\mu]$ and $[-\mu]$ are the only two orientations.

6.5.4 Proposition. *Let M be an orientable manifold. Then M is connected iff M has exactly two orientations.*

Proof. Suppose M is connected, and μ, ν are two volume forms with $\nu = f\mu$. Since M is connected, and $f(m) \neq 0$ for all $m \in M$, $f(m) > 0$ for all m or else $f(m) < 0$ for all m. Thus ν is equivalent to μ or $-\mu$. Conversely, if M is not connected, let U (not equal to either $\varnothing$ or M), be a subset that is both open and closed. If ν is a volume form on M, define ν by letting $\nu(m) = \nu(m)$ if $m \in U$ and $\nu(m) = -\nu(m)$ if $m \notin U$. Obviously, χ is a volume form on M, and $\nu \notin [\nu] \cup [-\nu]$. ■

A simple example of a nonorientable manifold is the Möbius band (see Figure 6.5.1 and Exercise 6.5-12), For other examples, see Exercises 6.5-11 and 6.5-13.

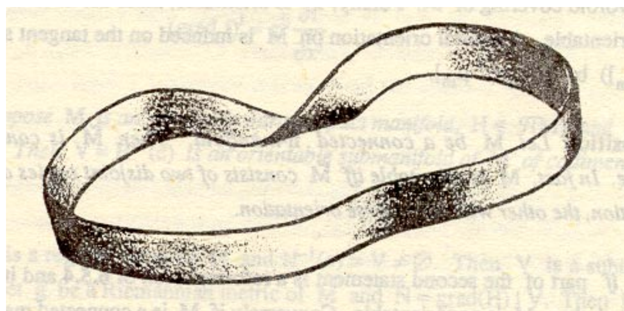

FIGURE 6.5.1.

6.5.5 Proposition. *The equivalence relation in Definition* 6.5.3 *is natural with respect to mappings and diffeomorphisms. That is, if $f : M \to N$ is of class C^∞, μ_N and ν_N are equivalent volume forms on N, and $f^*(\mu_N)$ is a volume form on M, then $f^*(\nu_N)$ is an equivalent volume form. If f is a diffeomorphism and μ_M and ν_M are equivalent volume forms on M, then $f_*(\mu_M)$ and $f_*(\nu_M)$ are equivalent volume forms on N.*

Proof. This follows from the fact that $f^*(g\omega) = (g \circ f)f^*\omega$, which implies $f^*(g\omega) = (g \circ f^{-1})f^*\omega$ when f is a diffeomorphism. ■

6.5.6 Definition. *Let M be an orientable manifold with orientation $[\mu]$. A chart (U, φ) with $\varphi(U) = U' \subset \mathbb{R}^n$ is called **positively oriented** if $\varphi_*(\mu|U)$ is equivalent to the standard volume form $dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(U')$.*

From Proposition 6.5.5 we see that this definition does not depend on the choice of the representative from $[\mu]$.

If M is orientable, we can find an atlas in which every chart has positive orientation by choosing an atlas of connected charts and, if a chart has negative orientation, by composing it with a reflection. Thus, in Proposition 6.5.2(ii) the atlas consists of positively oriented charts.

Orientable Double Covering. If M is not orientable, there is an orientable manifold $\tilde{M}$ and a two-to-one C^∞ surjective local diffeomorphism $\pi : \tilde{M} \to M$. The manifold $\tilde{M}$ is called the ***orientable double covering*** and is useful for reducing certain facts to the orientable case. The double covering is constructed as follows. Let

$$\tilde{M} = \{\, (m, [\mu_m]) \mid m \in M,\ [\mu_m] \text{ an orientation of } T_mM \,\}.$$

Define a chart at $(m, [\mu_m])$ in the following way. Fix an orientation $[\omega]$ of $\mathbb{R}^n$ and an orientation reversing isomorphism A of $\mathbb{R}^n$, for example, the isomorphism given by $A(e_1) = -e_1$, $A(e_i) = e_i$, $i = 2, \ldots, n$ where

$[e_i, \ldots, e_n]$ is the standard basis of $\mathbb{R}^n$. If $\varphi : U \to U' \subset \mathbb{R}^n$ is a chart of M at m, setting

$$U^{\pm} = \{ (u, [\mu_u]) \mid u \in U,\ [\varphi_*(\mu_u)] = [\pm\omega] \},$$

and defining $\varphi^{\pm} \to U'$ by

$$\varphi^{+}(u, [\mu_u]) = \varphi(u), \quad \varphi^{-}(u, [\mu_u]) = (A \circ \varphi)(u),$$

we get charts $(U^{\pm}, \varphi^{\pm})$ of $\tilde{M}$. It is straightforward to check that the family $\{(U^{\pm}, \varphi^{\pm})\}$ constructed in this way forms an atlas, thus making $\tilde{M}$ into a differentiable n-manifold. Define $\pi : \tilde{M} \to M$ by $\pi(m, [\mu_m]) = m$. In local charts, π is the identity mapping, so that π is a surjective local diffeomorphism. Moreover

$$\pi^{-1}(m) = \{(m, [\mu_m]), (m, [-\mu_m])\},$$

so that π is a twofold covering of M. Finally, $\tilde{M}$ is orientable, since the atlas formed by the charts $(U^{\pm}, \varphi^{\pm})$ is orientable. A natural orientation on M is induced on the tangent space to $\tilde{M}$ at the point $(m, [\mu_m])$ by $[(T_m\pi)^*\mu_m]$.

6.5.7 Proposition. *Let m be a connected n-manifold. Then $\tilde{M}$ is connected iff M is nonorientable. In fact, M is orientable iff $\tilde{M}$ consists of two disjoint copies of M, one with the given orientation, the other with the reverse orientation.*

Proof. The *if* part of the second statement is a reformulation of Proposition 6.5.4 and it also proves that if $\tilde{M}$ is connected, then M is nonorientable. Conversely if M is a connected manifold and if $\tilde{M}$ is disconnected, let C be a connected component of $\tilde{M}$. Then since π is a local diffeomorphism, $\pi(C)$ is open in M. We shall prove that $\pi(C)$ is closed. Indeed, if $m \in \mathrm{cl}(\pi(C))$, let $\tilde{m}_1, \tilde{m}_2 \in \tilde{M}$ be such that $\pi(\tilde{m}_1) = \pi(\tilde{m}_2) = m$. If there exists neighborhoods $\tilde{U}_1, \tilde{U}_2$, of $\tilde{m}_1, \tilde{m}_2$ such that $\tilde{U}_1 \cap C = \varnothing$ and $\tilde{U} \cap C = \varnothing$, then shrinking $\tilde{U}_1$ and $\tilde{U}_2$ if necessary, the open neighborhoods $\pi(\tilde{U}_1)$ and $\pi(\tilde{U}_2)$ of m have empty intersection with $\pi(C)$, contradicting the fact that $m \in \mathrm{cl}(\pi(C))$. Thus at least one of $\tilde{m}_1, \tilde{m}_2$ is in $\mathrm{cl}(C) = C$; that is, $m \in \pi(C)$ and hence $\pi(C)$ is closed. Since M is connected, $\pi(C) = M$. But π is double covering of M so that $\tilde{M}$ can have at most two components, each of them being diffeomorphic to M. Hence M is orientable, the orientation being induced from one of the connected components via π. ■

Conditions for Orientability. Another criterion of orientability is the following.

6.5.8 Proposition. *Suppose M is an orientable n-manifold and V is a submanifold of codimension k with trivial normal bundle. That is, there are C^∞ maps $N_i : V \to TM$, $i = 1, \ldots, k$ such that $N_i(v) \in T_v(M)$, and $N_i(v)$ span a subspace W_v, such that $T_vM = T_vV \oplus W_v$ for all $v \in V$. Then V is orientable.*

Proof. Let μ be a volume form on M. Consider the restriction $\mu|V : V \to \Gamma^n(M)$. Let us first note that $\mu|\Gamma$ is a smooth mapping of manifolds. This follows by using charts with the submanifold property, where the local representation is a restriction to a subspace. Now define $\mu_0 : V \to \Gamma^{n-k}(V)$ as follows. For $X_1, \ldots, X_{n-k} \in \mathfrak{X}(V)$, put

$$\mu_0(v)(X_1(v), \ldots, X_{n-k}(v)) = \mu(v)(N_1(v), \ldots, N_k(v), X_1(v), \ldots, X_{n-k}(v)).$$

It is clear that $\mu_0(v) \neq 0$ for all v. It remains only to show that μ_0 is smooth, but this follows from the fact that $\mu|V$ is smooth. ■

If g is a Riemannian metric, then $g^\flat : TM \to T^*M$ denotes the index-lowering operator and we write $g^\# = (g^\flat)^{-1}$. For $f \in \mathcal{F}(M)$,

$$\mathrm{grad}\, f = g^\#(\mathbf{d}f)$$

is called the ***gradient*** of f. Thus, $\mathrm{grad}\, f \in \mathfrak{X}(M)$. In local coordinates, if $[g_{ij}] = [g(\partial/\partial x^i, \partial/\partial x^j)]$ and $[g^{ij}]$ is the inverse matrix, then

$$(\mathrm{grad}\, f)^i = g^{ij}\frac{\partial f}{\partial x^j}. \tag{6.5.1}$$

6.5.9 Corollary. *Suppose M is an orientable paracompact manifold, $H \in \mathcal{F}(M)$ and $c \in \mathbb{R}$ is a regular value of H. Then $V = H^{-1}(c)$ is an orientable submanifold of M of codimension one, if it is nonempty.*

Proof. Suppose c is a regular value of H and $H^{-1}(c) = V \neq \varnothing$. Then $N(v) \notin T_vV$ for $v \in V$, because F_vV is the kernel of

$$\mathbf{d}H(v)[N(v)] = g(N, N)(v) > 0$$

as $\mathbf{d}H(v) \neq 0$ by hypothesis. Then Proposition 6.5.8 applies, and so V is orientable. ■

Thus if we interpret V as the "energy surface," we see that it is an oriented submanifold for "almost all" energy values by Sard's theorem.

Orientation Preserving Maps. The notion of orientation preserving maps between oriented manifolds can now be defined.

6.5.10 Definition. *Let M and N be two orientable n-manifolds with volume forms μ_M and μ_N, respectively. Then we call a C^∞ map $f : M \to N$* ***volume preserving*** *(with respect to μ_M and μ_N) if $f^*\mu_N = \mu_M$,* ***orientation preserving*** *if $f^*(\mu_N) \in [\mu_M]$, and* ***orientation reversing*** *if $f^*(\mu_N) \in [-\mu_M]$. An orientable manifold admitting (respectively, not admitting) an orientation reversing diffeomorphism is called* ***reversible*** *(respectively,* ***non-reversible****).*

From Proposition 6.5.5, $[f^*\mu_N]$ depends only on $[\mu_N]$. Thus the first part of the definition explicitly depends on μ_M and μ_N, while the last four parts depend only on the orientations $[\mu_M]$ and $[\mu_N]$. Furthermore, we see from Proposition 6.5.5 that if f is volume preserving with respect to μ_M and μ_N, then f is volume preserving with respect to $h\mu_M$ and $g\mu_N$ iff $h = g \circ f$. It is also clear that if f is volume preserving with respect to μ_M and μ_N, then f is orientation preserving with respect to $[\mu_M]$ and $[\mu_N]$.

6.5.11 Proposition. *Let M and N be n-manifolds with volume forms μ_M and μ_N, respectively. Suppose $f : M \to N$ is of class C^∞. Then $f^*(\mu_N)$ is a volume form iff f is a local diffeomorphism; that is, for each $m \in M$, there is a neighborhood V of m such that $f|V : V \to f(V)$ is a diffeomorphism. If M is connected, then f is a local diffeomorphism iff f is orientation preserving or orientation reversing.*

Proof. If f is a local diffeomorphism, then clearly $F^*(\mu_N)(m) \neq 0$, by Proposition 6.2.3(ii). Conversely, if $f^*(\mu_M)$ is a volume form, then the determinant of the derivative of the local representative is not zero, and hence the derivative is an isomorphism. The result then follows by the inverse function theorem. The second statement follows at once from the first and Proposition 6.5.4. ■

Jacobian Determinant. Next we consider the global analog of the determinant.

6.5.12 Definition. *Suppose M and N are n-manifolds with volume forms μ_M and μ_N, respectively. If $f : M \to N$ is of class C^∞, the unique C^∞ function $J(\mu_M, \mu_N)f \in \mathcal{F}(M)$ such that $f^*\mu_N = (J(\mu_M, \mu_N)f)\mu_M$ is called the* ***Jacobian determinant*** *of f (with respect to μ_M and μ_N). If $f : M \to M$ we write $J_\mu f = J(\mu, \mu)f$.*

Note that $J(\mu_M, \mu_N)f(m) = \det(T_mf)$, the determinant being taken with respect to the volume forms $\mu_M(m)$ on T_mM and $\mu_N(f(m))$ on $T_{f(m)}N$. The basic properties of determinants that were developed in §6.2 also hold in the global case, as follows. First, we have the following consequences of Proposition 6.5.11.

6.5.13 Proposition. *The C^k map $f : M \to N$, $k \geq 1$, is a local C^k diffeomorphism iff $J(\mu_M, \mu_N)f(m) \neq 0$ for all $m \in M$.*

Second, we have consequences of the definition and properties of pull-back.

6.5.14 Proposition. *Let (M, μ) be a volume manifold.*

(i) *If $f : M \to M$ and $g : M \to M$ are of class C^k, $k \geq 1$, then*

$$J_\mu(f \circ g) = [(J_\mu f) \circ g][J_\mu g].$$

(ii) *If $h : M \to M$ is the identity, then $J_\mu h = 1$.*

(iii) *If $f : M \to M$ is a diffeomorphism, then*

$$J_\mu(f^{-1}) = \frac{1}{[(J_\mu f) \circ f^{-1}]}.$$

Proof. For (i),

$$\begin{aligned}\mathbf{J}_\mu(f \circ g)\mu &= (f \circ g)^*\mu = g^* f^* \mu = g^*(J_\mu f)\mu \\ &= ((J_\mu f) \circ g) g^* \mu = ((J_\mu f) \circ g)(J_\mu g)\mu.\end{aligned}$$

Part (ii) follows since h^* is the identity. For (iii) we have

$$J_\mu(f \circ f^{-1}) = 1 = ((J\mu f) \circ f^{-1})(J_\mu f^{-1}).$$

■

6.5.15 Proposition. *Let $(M, [\mu_M])$ and $(N, [\mu_N])$ be oriented manifolds and $f : M \to N$ be a map of class C^k, $k \geq 1$. Then f is orientation preserving iff $J(\mu_M, \mu_N)f(m) > 0$ for all $m \in M$, and orientation reversing if $J(\mu_M, \mu_N)f(m) < 0$ for all $m \in M$. Also, f is volume preserving with respect to μ_M and μ_N iff $J(\mu_M, \mu_N)f = 1$.*

This proposition follows from the definitions. Note that the first two assertions depend only on the orientations $[\mu_M]$ and $[\mu_N]$, since

$$J(h\mu_M, g\mu_N)f = \left(\frac{g \circ f}{h}\right) J(\mu_M, \mu_N)f,$$

which the reader can easily check. Here $g \in \mathcal{F}(N)$, $h \in \mathcal{F}(M)$, $g(n) \neq 0$, and $h(m) \neq 0$ for all $n \in N$, $m \in M$.

Divergence. We have seen that in $\mathbb{R}^3$ the divergence of a vector field is expressible in terms of the standard volume element $\mu = dx \wedge dy \wedge dz$ by the use of the metric in $\mathbb{R}^3$ (see Example 6.4.3D). There is, however, a second characterization of the divergence that does not require a metric but only a volume form μ, namely

$$\pounds_F \mu = (\operatorname{div} F)\mu,$$

as a simple computation shows. This can now be generalized.

6.5.16 Definition. *Let (M, μ) be a volume manifold; that is, M is an orientable manifold with a volume form μ. Let X be a vector field on M. The unique function $\operatorname{div}_\mu X \in \mathfrak{X}(M)$, such that*

$$\pounds_X \mu = (\operatorname{div}_\mu X)\mu$$

*is called the **divergence** of X. We say X is **incompressible** or **divergence free** (with respect to μ) if $\operatorname{div}_\mu X = 0$.*

6.5.17 Proposition. *Let (M, μ) be a volume manifold and X a vector field on M.*

(i) *If $f \in \mathcal{F}(M)$ and $f(m) \neq 0$ for all $m \in M$, then*

$$\operatorname{div}_{f\mu} X = \operatorname{div}_\mu X + \frac{\pounds_X f}{f}.$$

(ii) *For* $g \in \mathcal{F}(M)$, $\operatorname{div}_\mu gX = g \operatorname{div}_\mu X + \pounds_X g$.

Proof. Since $\pounds_X$ is a derivation,

$$\pounds_X(f\mu) = (\pounds_X f)\mu + f\pounds_X\mu.$$

As $f\mu$ is a volume form,

$$(\operatorname{div}_{f\mu} X)(f\mu) = (\pounds_X f)\mu + f(\operatorname{div}_\mu X)\mu.$$

Then (i) follows. For (ii),

$$\pounds_{gX}\mu = g\pounds_X\mu + \mathbf{d}g \wedge \mathbf{i}_X\mu,$$

and from the antiderivation property of $\mathbf{i}_X$,

$$\mathbf{d}g \wedge v_X\mu = -\mathbf{i}_X(\mathbf{d}g \wedge \mu) + \mathbf{i}_X\mathbf{d}g \wedge \mu.$$

But $\mathbf{d}g \wedge \mu \in \Omega^{n+1}(M)$, and hence $\mathbf{d}g \wedge \mu = 0$. Also, $\mathbf{i}_X\mathbf{d}g = \pounds_X g$, so

$$\pounds_{gX}\mu = g\pounds_X\mu + (\pounds_X g)\mu.$$

The result follows from this. ■

6.5.18 Proposition. *Let (M, μ) be a volume manifold and X a vector field on M. Then X is incompressible (with respect to μ) iff the flow of X is volume preserving; that is, the local diffeomorphism $F_t : U \to V$ is volume preserving with respect to $\mu|U$ and $\mu|V$.*

Proof. Since X is incompressible, $\pounds_X\mu = 0$, and so μ is constant along the flow of X; $\mu(m) = (F_t^*\mu)(m)$. Thus F_t is volume preserving. Conversely, if $(F_t^*\mu)(m) = \mu(m)$, then $\pounds_X\mu = 0$. ■

6.5.19 Corollary. *Let (M, μ) be a volume manifold and X a vector field with flow F_t on M. Then X is incompressible iff $J_\mu F_t = 1$ for all $t \in \mathbb{R}$.*

One-Densities. The above developments regarding the Jacobian and divergence can also be carried out for one-densities. If $|\mu_M|, |\mu_N|$ are one-densities and $f : M \to N$ is C^∞, we shall write

$$f^*|\mu_N| = J(|\mu_M|, |\mu_N|, f)|\mu_M|,$$

where the pull back is defined as for forms. Then Propositions 6.5.13 and 6.5.14 go through for one-densities. The Lie derivative of a one-density is defined by

$$\pounds_X|\mu| = \left.\frac{d}{dt}\right|_{t=0} F_t^*|\mu|$$

and one defines the divergence of X with respect to $|\mu|$ as in Definition 6.5.16. Then it is easy to check that Proposition 6.5.17–Corollary 6.5.19 have analogues for one-densities.

Riemannian Volume Forms. We shall now globalize the concepts pertaining to Riemannian volume forms and densities, as well as the Hodge star operator discussed in §6.2.

6.5.20 Proposition. *Let (M, g) be a pseudo-Riemannian manifold of signature s; that is, $g(m)$ has signature s for all $m \in M$.*

(i) *If M is orientable, then there exists a unique volume element $\mu = \mu(g)$ on M, called the **g-volume** (or **pseudo-Riemannian volume of g**), such that μ equals one on all positively oriented orthonormal bases on the tangent spaces to M. If $X_1, \ldots, X_n$ is such a basis in an open set U of M and if $\zeta^1, \ldots, \zeta^n$ is the dual basis, then $\mu = \zeta^1 \wedge \cdots \wedge \zeta^n$. More generally, if $v_1, \ldots, v_n \in T_xM$ are positively oriented, then*

$$\mu(x)(v_1, \ldots, v_n) = |\det[g(x)(v_i, v_j)]|^{1/2}.$$

(ii) *For every* $\alpha \in \mathbb{R}$ *there exists a unique* α*-density* $|\mu|^\alpha$*, called the* $\mathbf{g}$-α-***density*** *(or the* ***pseudo-Riemannian*** α-***density of*** $\boldsymbol{g}$*), such that* $|\mu|^\alpha$ *equals* 1 *on all orthonormal bases of the tangent spaces to* M*. If* $X_i, \ldots, X_n$ *is such as a basis in an open set* U *of* M *with dual basis* $\zeta^1, \ldots, \zeta^n$*, then* $|\mu|^\alpha = |\zeta^1 \wedge \cdots \wedge \zeta^n|^\alpha$*. More generally, if* $v_1, \ldots, v_n \in T_xM$*, then*

$$|\mu|^\alpha(x)(v_1, \ldots, v_n) = |\det[g(x)(v_i, v_j)]|^{\alpha/2}.$$

This is a consequence of Proposition 6.2.10 and the fact that μ and $|\mu|^\alpha$ are smooth. Also note that in an oriented chart $(x^1, \ldots, x^n)$ on M, we have

$$\mu = |\det[g_{ij}]|^{1/2} dx^1 \wedge \cdots \wedge dx^n.$$

As in the vector space situation, g induces a pseudo-Riemannian metric on $\Gamma^k(M)$ by

$$\langle \alpha, \beta \rangle_x = \alpha_{i_1 \ldots i_k} \beta^{i_1 \ldots i_k}, \tag{6.5.2}$$

where the sum is over $i_1 < \cdots < i_k$ and where, $\alpha, \beta \in \bigwedge^k(M)_x$ and $\beta^{i_1 \cdots i_k}$ are the components of the associated contravariant tensor to β. As in Proposition 6.2.11, if $X_1, \ldots X_n$ is an orthonormal basis in $U \subset M$ with dual basis $\zeta^1, \ldots \zeta^n$, then the elements $\zeta^{i_1} \wedge \cdots \wedge \zeta^{i_k}$, where $i_1 < \cdots < i_k$ form an orthonormal basis of $\Gamma^k(U)$.

The Hodge Star Operator. On an orientable pseudo-Riemannian manifold with pseudo-Riemannian volume form μ, the Hodge operator is defined pointwise by

$$* : \Omega^k(M) \to \Omega^{n-k}(M), \quad (*\alpha)(x) = *\alpha(x);$$

that is, $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, \mu$ for $\alpha, \beta \in \Omega^k(M)$. The properties in Propositions 6.2.12 and 6.2.13 carry over since they hold pointwise. One can check that if α is C^r then so is $*\alpha$.

The Codifferential. The exterior derivative and the Hodge star operator enable us to introduce the following linear operator δ. (The reason for the strange-looking factor (-1) in the definition is so a later integration by parts formula, proved in Corollary 7.2.13, will come out simple.)

6.5.21 Definition. *The* ***codifferential*** $\delta : \Omega^{k+1}(M) \to \Omega^k(M)$ *is defined by* $\delta(\Omega^0(M)) = 0$ *and on* $k+1$ *forms* β *by*

$$\delta\beta = (-1)^{nk+1+\mathrm{Ind}(g)} * \mathbf{d} * \beta.$$

Notice that since $\mathbf{d}^2 = 0$ and $**$ is a multiple of the identity, $\delta^2 = 0$.

For example, in $\mathbb{R}^3$, let $\alpha = a\mathbf{d}y \wedge \mathbf{d}z - b\mathbf{d}x \wedge \mathbf{d}z + c\mathbf{d}x \wedge \mathbf{d}y$. Then

$$*\alpha = a\mathbf{d}x + b\mathbf{d}y + c\mathbf{d}z,$$

so

$$\mathbf{d}*\alpha = (b_x - a_y)\mathbf{d}x \wedge \mathbf{d}y + (c_x - a_z)\mathbf{d}x \wedge \mathbf{d}z + (c_y - b_z)\mathbf{d}y \wedge \mathbf{d}z$$

and

$$*\mathbf{d}*\alpha = (b_x - a_y)\mathbf{d}z - (c_c - a_z)\mathbf{d}y + (c_y - b_z)\mathbf{d}x.$$

Thus, as $nk + 1 + \mathrm{Ind}\,(g) = 4$ is even,

$$\mathbf{d}\alpha = (c_\gamma - b_z)\mathbf{d}x + (a_z - c_x)\mathbf{d}y + (b_x - a_y)\mathbf{d}z.$$

The formula for $\delta\beta$ in coordinates is given by

$$(\delta\beta)_{i_1\ldots i_k} = \frac{1}{k+1}|\det[g_{ij}]|^{-1/2} g_{i_1 r_1}\ldots g_{i_k r_k} \times \frac{\partial}{\partial x^l}\left(\sum_{p=1}^{k+1}(-1)^p g^{r_1 j_1}\cdots g^{r_{p-1}j_{p\neq 1}} g^{l j_r} g^{r_p i_{p\neq 1}} \beta_{j_1\ldots j_{k\neq 1}} |\det[g_{ij}]|^{1/2}\right)$$

or as a contravariant tensor

$$(\delta\beta)^{r_1\ldots r_k} = \frac{1}{k+1}|\det[g_{ij}]|^{-1/2} \times \frac{\partial}{\partial x^l}\left(\sum_{p=1}^{k=1}(-1)^p \beta^{r_1\ldots r_{p-1} l r_p \ldots r_k} |\det[g_{ij}]|^{1/2}\right),$$

where

$$\beta = \beta_{r_1\ldots r_k r_{k+1}} dx^{r_1}\wedge\cdots\wedge dx^{r_{k+1}},$$

(sum over $r_1 < \cdots < r_{k+1}$) is the usual coordinate expression for β. The coordinate formula is messy to prove directly. However it follows fairly readily from integration by parts in local coordinates and that fact that δ is the adjoint of $\mathbf{d}$, a fact that will be proved in Chapter 7 (see Corollary 7.2.13 and Exercise 7.5-7).

We now express the divergence of a vector field $X \in \mathfrak{X}(M)$ in terms of δ. We define the divergence $\operatorname{div}_g(X)$ with respect to a pseudo-Riemannian metric g to be the divergence of X with respect to the pseudo-Riemannian volume $\mu = \mu(g)$; that is, $\pounds_X \mu = \operatorname{div}_g(X)\mu$. To compute the divergence, we prepare a lemma.

6.5.22 Lemma. $\mathbf{i}_X\mu = *X^\flat$.

Proof. Let $v_2, \ldots, v_n \in T_xM$ be orthonormal and orthogonal to $X(n)$. From Proposition 6.2.10, we have

$$\begin{aligned}\mathbf{i}_X\mu(v_2,\ldots,v_n) &= \mu(X(x), v_2, \ldots, v_n)\\ &= \sqrt{g(X(x),X(x))}.\end{aligned}$$

On the other hand, we claim that

$$*X^\flat = \sqrt{g(X(x),X(x))}\, v_2^\flat\wedge\cdots\wedge v_n^\flat$$

Indeed, this may be verified using the definition in Proposition 6.2.12 with x a 1-form and $\beta = X^\flat$. Using this formula for $*X^\flat$, we get

$$\begin{aligned}*X^\flat(x)(v_2,\ldots,v_n) &= \sqrt{g(X(x),X(n))}\\ &= \mathbf{i}_X\mu(v_2,\ldots,v_n).\end{aligned}$$

Equality on such $v_2, \ldots, v_n$ implies equality, as is readily seen. ■

This may be seen in coordinates using the formula (6.2.12) of §6.2.

6.5.23 Proposition. *Let g be a pseudo-Riemannian metric on the orientable n-manifold M. Then*

$$\operatorname{div}_g(X) = -\delta X^\flat \tag{6.5.3}$$

In (positively oriented) local coordinates

$$\operatorname{div}_g(X) = |\det[g^{ij}]|^{-1/2}\frac{\partial}{\partial x^k}\left(|\det[g_{ij}]|^{1/2}X^k\right) \tag{6.5.4}$$

Proof. Let

$$\pounds_X \mu = \mathbf{d}\mathbf{i}_X \mu = \mathbf{d}{*}X^\flat$$

by the lemma. But then

$$(\mathrm{div}_g X)\mu = -{*}\delta X^\flat = -(\delta X^\flat){*}1$$

by the definition of δ and the formula for $**$. Since $*1 = \mu$, we get equation (6.5.3). To prove formula (6.5.4), write $\mu = |\det[g_{ij}]|^{1/2} dx^1 \wedge \cdots \wedge dx^n$ and compute $\pounds_X \mu = \mathbf{d}\mathbf{i}_X \mu$ in these coordinates. We have

$$\mathbf{i}_X \mu = \det[g_{ij}]^{1/2} X^k (-1)^k dx^1 \wedge \cdots \wedge dx^k \wedge \cdots \wedge dx^n$$

and so

$$\begin{aligned} \mathbf{d}\mathbf{i}_X \mu &= \left(\frac{\partial}{\partial x^k} |\det[g_{ij}]|^{1/2} X^k \right) dx^1 \wedge \cdots \wedge dx^n. \\ &= \frac{1}{2} \frac{1}{|\det[g_{ij}]|^{1/2}} \left(|\det[g_{ij}]|^{\partial/\partial x^k} X_k \right) \mu. \end{aligned}$$

■

6.5.24 Definition. *The* **Laplace–Beltrami operator** *on functions on a orientable pseudo-Riemannian manifold is defined by* $\nabla^2 = \mathrm{div} \circ \mathrm{grad}$.

Thus, in a positively oriented chart, equation (6.5.4) gives

$$\nabla^2 f = |\det g_{ij}|^{-1/2} \frac{\partial}{\partial x^k} \left(g^{lk} |\det g_{ij}|^{1/2} \frac{\partial f}{\partial x^l} \right). \tag{6.5.5}$$

Exercises

◇ **6.5-1.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with positive Jacobian and $f(0) = 0$. Prove that there is a continuous curve f_t of diffeomorphisms joining f to the identity.
Hint: First join f to $\mathbf{D}f(0)$ by $g_t(x) = f(tx)/t$.

◇ **6.5-2.** If t is a tensor density of M, that is, $t = t' \otimes \mu$, where μ is a volume form, show that

$$\pounds_X t = (\pounds_X t') \otimes \mu + (\mathrm{div}_\mu X) t' \otimes \mu.$$

◇ **6.5-3.** A map $A : E \to E$ is said to be *derived from a variational principle* if there is a function $L : E \to \mathbb{R}$ such that

$$\mathbf{d}L(x)\dot{v} = \langle A(x), v \rangle,$$

where $\langle\, , \rangle$ is an inner product on E. Prove *Vainberg's theorem: A comes from a variational principle if and only if $\mathbf{D}A(x)$ is a symmetric linear operator*. do this by applying the Poincaré lemma to the one-form $\alpha(x)\dot{v} = \langle A(x), v \rangle$ (see Marsden and Hughes [1983]).

◇ **6.5-4.** Show in three different ways that the sphere S^n is orientable by using Proposition 6.5.2 and the two charts given in Figure 3.1.2, by constructing an explicit n-form, and by using Corollary 6.5.9.

◇ **6.5-5.** Use formula (6.5.5) to show that in polar coordinates (r, θ) in $\mathbb{R}^2$,

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2} + \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r} \frac{\partial f}{\partial r}$$

and that in spherical coordinates (ρ, θ, ϕ) in $\mathbb{R}^3$,

$$\nabla^2 f = \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial f}{\partial \mu} \right) + \frac{1}{1 - \mu^2} \frac{\partial^2 f}{\partial \theta^2} + \rho \frac{\partial^2 f}{\partial \rho^2}$$

where $\mu = \cos \phi$.

◇ **6.5-6.** Let (M, μ) be a volume manifold. Prove the identity

$$\operatorname{div}_\mu [X, Y] = X[\operatorname{div}_\mu Y] - Y[\operatorname{div}_\mu X].$$

◇ **6.5-7.** Let $f : M \to N$ be a diffeomorphism of connected oriented manifolds with boundary. Assuming that $T_m f : T_m M \to T_{f(m)} N$ is orientation preserving for some $m \in \operatorname{Int}(M)$, show that $J(f) > 0$ on M; that is, f is orientation preserving.

◇ **6.5-8.** Let g be a pseudo-Riemannian metric on M and define $g_\lambda = \lambda g$ for $\lambda > 0$. Let $*_\lambda$ be the Hodge-star operator defined by g_λ and set $*_1 = *$. Show that if $\lambda \in \Omega^k(M)$, then

$$*_\lambda \alpha = \lambda^{(n/2)-k} * \alpha.$$

◇ **6.5-9.** In $\mathbb{R}^3$ equipped with the standard Euclidean metric, show that for any vector field F and any function f we have: $\operatorname{div} F = -\delta F^\flat$, $\operatorname{curl} F = (\delta * F^\flat)^\#$, and $\operatorname{grad} f = -(*\delta * f)^\#$.

◇ **6.5-10.** Show that if M and N are orientable, then so is $M \times N$.

◇ **6.5-11.** (i) Let $\sigma : M \to M$ be an involution of M, that is, $\sigma \circ \sigma =$ identity, and assume that the equivalence relation defined by σ is regular, that is, there exists a surjective submersion $\pi : M \to N$ such that $\pi^{-1}(n) = \{m, \sigma(m)\}$, where $\pi(m) = n$. Let

$$\Omega_\pm(M) = \{\, \alpha \in \Omega(M) \mid \sigma^*\alpha = \pm\alpha \,\}$$

be the ± 1 eigenspaces of σ^*. Show that $\pi^* : \Omega(N) \to \Omega_+(M)$ is an isomorphism.

Hint: To show that range $\pi^* = \Omega_+(M)$, note that $\pi \circ \sigma = \pi$ implies range$(\pi^*) \subset \Omega_+(M)$. For the converse conclusion, note that $T_m\pi$ is an isomorphism, so a form at $\pi(m)$ uniquely determines a form at m. Show that this resulting form is smooth by working in a chart on M diffeomorphic to a chart on N.

(ii) Show that $\mathbb{RP}^n$ is orientable if n is odd and is not orientable if n is even.

Hint: In (i) take $M = S^n \subset \mathbb{R}^{n+1}$, $\sigma(x) = x$, and $N = \mathbb{RP}^n$. Let ω be a volume element on S^n induced by a volume element of $\mathbb{R}^{n+1}$. Show that $\sigma^*\omega = (-1)^{n+1}\omega$. Now apply (i) to orient $\mathbb{RP}^n$ for n odd. If n is even, let ν be an n-form on $\mathbb{RP}^n$; then $\pi^*\nu = f\omega$. Show that $f(x) = -f(-x)$ so f must vanish at a point of S^n.

◇ **6.5-12.** In Example 3.4.10C, the ***Möbius band*** $\mathbb{M}$ was defined as the quotient of $\mathbb{R}^2$ by the equivalence relation $(x, y) \sim (x + k, (-1)^k y)$ for any $k \in \mathbb{Z}$.

(i) Show that this equivalence relation is regular. Show that $\mathbb{M}$ is a non-compact, connected, two-manifold.

(ii) Define the map $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ by $\sigma(x, y) = (x + 1, -y)$. Show that $\pi \circ \sigma = \sigma$, where $\sigma : \mathbb{R}^2 \to \mathbb{M}$, is the canonical projection. If $\nu \in \Omega^2(\mathbb{M})$ define $f \in \mathcal{F}(\mathbb{R}^2)$ by $\pi^*\nu = f\omega$, where ω is an area form on $\mathbb{R}^2$. Show that $f(x + 1, -y) = -f(x, y)$.

(iii) Conclude that f must vanish at a point of $\mathbb{R}^2$, and that this implies $\mathbb{M}$ is not orientable.

◇ **6.5-13.** The ***Klein bottle*** $\mathbb{K}$ is defined as the quotient of $\mathbb{R}^2$ by the equivalence relation defined by

$$(x, y) \sim (x + n, (-1)^n y + m)$$

for any $n, m \in \mathbb{Z}$.

(i) Show that this equivalence relation is regular. Show that $\mathbb{K}$ is a compact, connected, smooth, two-manifold.

(ii) Use Exercise 6.5-12(ii), (iii) to show that $\mathbb{K}$ is non-orientable.

◇ **6.5-14** (Orientation in vector bundles). Let $\pi : E \to B$ be a vector bundle with finite dimensional fiber modeled on a vector space E, and assume B is connected. The vector bundle is said to be ***orientable*** if the line bundle $L = E^* \wedge \cdots \wedge E^*$ ($\dim(E)$ times) has a global nowhere vanishing section. An ***orientation*** of E is an equivalence class of global nowhere vanishing sections of L under the equivalence relation: $\sigma_1 \sim \sigma_2$ iff there exists $f \in \mathcal{F}(B)$, $f > 0$ such that $\sigma_2 = f\sigma_1$.

(i) Prove that E is orientable iff L is a trivial line bundle. Show that E admits exactly two orientations. Show that an orientation $[\sigma]$ of E induces an orientation in each fiber of E.

(ii) Show that a manifold M is orientable iff its tangent bundle is an oriented vector bundle.

(iii) Let E, F be vector bundles over the same base. Show that if two of E, F, and $E \oplus F$ are orientable, so is the third.

(iv) Let E, F be vector bundles (over possibly different bases). Show that $E \times F$ is orientable if and only if E and F are both orientable. Conclude that if M, N are finite dimensional manifolds, then $M \times N$ is orientable if and only if M and N are both orientable.

(v) Show that $E \oplus E^*$ is an orientable vector bundle if E is any vector bundle.

Hint: Consider the section

$$\Omega(x)((e_1, \alpha_1), (e_2, \alpha_2)) = \langle \alpha_2, e_1 \rangle - \langle \alpha_1, e_2 \rangle$$

of $(E \oplus E^*) \wedge (E \oplus E^*)$.

(vi) Choose an orientation of the vector space E and assume B admits partitions of unity. Show that the vector bundle atlas all of whose change of coordinate maps have positive determinant relative to the orientation of E, when restricted to the fiber.

Hint: If E is oriented and $\psi : \pi^{-1}(U) \to U' \times E$ is a vector bundle chart with U' open in the model space of B, and U is connected in B, define $\phi : \pi^{-1}(U) \to U' \times F$ by $\phi(e) = \psi(e)$ if the linear map $\psi_b : \pi^{-1}(b) \to F$ is orientation preserving and $\phi(e) = (\alpha \circ \psi)(e)$, where $\alpha : F \to F$ is an orientation reversing isomorphism of F, if ψ_b is orientation reversing. For the converse, choose a volume form ω on F and define on a vector bundle chart (V, ϕ) of E, with U connected in B, $\pi^{-1}(U) = V$, $\omega(U) : U \to L|U$ by

$$\omega(U)(b)(e_1, \ldots, e_r) = \omega(\phi_0(b))(\phi_b(e_1), \ldots, \phi_b(e_r)),$$

where $r = \dim F$, $b \in B$, $e_i \in \pi^{-1}(b)$, $i = 1, \ldots, r$, and $\phi_0 : U \to U'$ is the induced chart on B. Show that if $b \in U_1 \cap U_2$, then

$$\omega(U_1)(b) = \det_\omega \left(\phi^1_b \circ (\phi^2_b)^{-1}\right) \omega(U_2)$$

where $(\pi^{-1}(U_i), \phi_i)$ are vector bundle charts, U_i connected, $i = 1, 2$. Next, glue the $\omega(U)$'s together using a partition of unity.

(vii) Use (iv) to show that if E and F are oriented, then there exists a vector bundle atlas on E such that all $\phi_b : \pi^{-1}(b) \to F$ are orientation preserving isomorphisms. Such an atlas is called ***positively oriented***.

(viii) Let E, F, B be finite dimensional, E oriented by σ and B oriented by ω. Show that $\pi^*\omega \wedge \sigma$ is a volume form on E. Conclude that an orientation of B and an orientation of F uniquely determine an orientation of E as a manifold. This orientation is called the ***local product orientation***.

(ix) Show that any vector bundle $\pi : E \to B$ with finite dimensional fiber has an oriented double cover $\tilde{\pi} : \tilde{E} \to \tilde{B}$, where

$$\tilde{B} = \{ (b, [\mu_b]) \mid b \in B,\ [\mu_b] \text{ is an orientation of } \pi^{-1}(b) \},$$

$p : \tilde{B} \to B$ is the map $p(b, [\mu_b]) = b$, and $\tilde{E} = p^* E$. Find the vector bundle charts of $\tilde{E}$ and show that the fiber at $(b, [\mu_b])$ is oriented by $[\tilde{p}^*(\mu_b)]$, where $\tilde{p} : \tilde{E} \to E$ is the mapping induced by p on the pull-back bundle $\tilde{E}$. If $E = TB$, what is $\tilde{E}$?

◇ **6.5-15.** Let M be a compact manifold and $\mathcal{M}$ the space of Riemannian metrics on M. Let $\mathcal{T}$ be a space of tensor fields on M of a fixed type. A mapping $\Phi : \mathcal{M} \to \mathcal{T}$ is called ***covariant*** if for every diffeomorphism $\phi : M \to M$, we have $\Phi(\phi^* g) = \phi^* \Phi(g)$.

(i) Show that covariant maps satisfy the identity

$$\mathbf{D}\Phi(g) \cdot \pounds_X g = \pounds_X \Phi(g)$$

for every vector field X. (Assume Φ is differentiable and $\mathcal{M}, \mathcal{T}$ are given suitable Banach space topologies.)

(ii) Show that if M is oriented, then the map $g \mapsto \mu(g)$, the volume element of g, is covariant. Is the identity in (i) anything interesting?

◇ **6.5-16.** Let X be a vector field density on the oriented n-manifold M; that is, $X = F \otimes \mu$, where F is a vector field and μ is a density. Use Exercise 6.5-2 to define $\operatorname{div} X$ and to show it makes intrinsic sense.

◇ **6.5-17.** Show that an orientable line bundle over a base admitting partitions of unity is trivial.
HINT: Since the bundle is orientable there exist local charts which when restricted to each fiber give positive functions. Regard these as local sections and then glue.

◇ **6.5-18.** Let $\pi : E \to S^1$ be a vector bundle with n-dimensional fiber. If E is orientable show that it is isomorphic to a trivial bundle over S^1. Show that if $\rho : F \to S^1$ is a non-orientable vector bundle with n-dimensional fiber and if E is non-orientable, then E and F are isomorphic. Conclude that there are exactly two isomorphic classes of vector bundles with n-dimensional fiber over S^1. Construct a representative for the class corresponding to the non-orientable case.
HINT: Construct a non-orientable vector bundle like the Möbius band: the equivalence relation has a factor $(-1)^{2n-1}$ if the dimension of the fiber is $2n-1$ or $2n$. To prove non-orientability, proceed as in Exercise 6.5-12.

◇ **6.5-19.** Let $\{e_1, \ldots, e_{n+1}\}$ be the standard bases of $\mathbb{R}^{n+1}$ and

$$\Omega_{n+1} = e_1 \wedge \cdots \wedge e_{n+1}$$

be the induced volume form. On S^n define $\omega_n \in \Omega^n(S^n)$ by

$$\omega_n(s)(v_1, \ldots, v_n) = \Omega_{n+1}(s, v_1, \ldots, v_n)$$

for $s \in S^n$, $v_1, \ldots, v_n \in T_s S^n$.

(i) Use Proposition 6.5.8 to show that ω_n is a volume form on S^n; ω_n is called the ***standard volume form*** on S^n.

(ii) Let $f : \mathbb{R}_+ \times \mathbb{R}^{n+1} \backslash \{0\}$ be given by $f(t, s) = ts$, where $\mathbb{R}_+$ is defined to be the set $\{ t \in \mathbb{R} \mid t > 0 \}$. Show that if $\mathbb{R}_+$ is oriented by dt, S^n by ω_n, and $\mathbb{R}^{n+1}$ by Ω_{n+1}, then $(Jf)(t, s) = t^n$. Conclude that f is orientation preserving.