4

Vector Fields and Dynamical Systems

This chapter studies vector fields and the dynamical systems they determine. The ensuing chapters will study the related topics of tensors and differential forms. A basic operation introduced in this chapter is the Lie derivative of a function or a vector field. It is introduced in two different ways, algebraically as a type of directional derivative and dynamically as a rate of change along a flow. The *Lie derivative formula* asserts the equivalence of these two definitions. The Lie derivative is a basic operation used extensively in differential geometry, general relativity, Hamiltonian mechanics, and continuum mechanics.

4.1 Vector Fields and Flows

This section introduces vector fields and the flows they determine. This topic puts together and globalizes two basic ideas we learn in undergraduate calculus: the study of vector fields on the one hand and differential equations on the other.

4.1.1 Definition. *Let M be a manifold. A* ***vector field*** *on M is a section of the tangent bundle TM of M. The set of all C^r vector fields on M is denoted by $\mathfrak{X}^r(M)$ and the C^∞ vector fields by $\mathfrak{X}^\infty(M)$ or $\mathfrak{X}(M)$.*

Thus, a vector field X on a manifold M is a mapping $X : M \to TM$ such that $X(m) \in T_mM$ for all $m \in M$. In other words, a vector field assigns to each point of M a vector based (i.e., bound) at that point.

4.1.2 Example. Consider the force field determined by Newton's law of gravitation. Here the manifold is $\mathbb{R}^3$ minus the origin and the vector field is

$$\mathbf{F}(x, y, x) = -\frac{mMG}{r^3}\mathbf{r},$$

where m is the mass of a test body, M is the mass of the central body, G is the constant of gravitation, $\mathbf{r}$ is the vector from the origin to (x, y, z), and $r = (x^2 + y^2 + z^2)^{1/2}$; see Figure 4.1.1. ♦

The study of dynamical systems, also called flows, may be motivated as follows. Consider a physical system that is capable of assuming various "states" described by points in a set S. For example, S might be $\mathbb{R}^3 \times \mathbb{R}^3$ and a state might be the position and momentum $(\mathbf{q}, \mathbf{p})$ of a particle. As time passes, the state evolves. If the state is $s_0 \in S$ at time λ and this changes to s at a later time t, we set

$$F_{t,\lambda}(s_0) = s$$

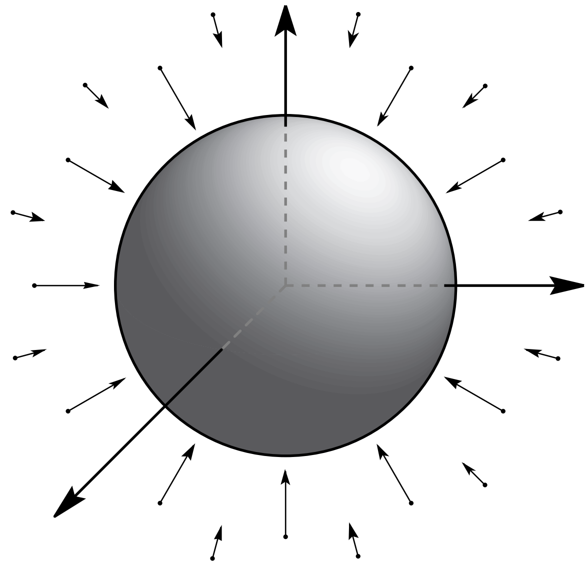


Figure 4.1.1. The gravitational vector field

and call $F_{t,\lambda}$ the ***evolution operator***; it maps a state at time λ to what the state would be at time t; that is, after time $t - \lambda$ has elapsed. "Determinism" is expressed by the law

$$F_{\tau,t} \circ F_{t,\lambda} = F_{\tau,\lambda}, \quad F_{t,t} = \text{ identity},$$

sometimes called the ***Chapman–Kolmogorov law***.

The evolution laws are called ***time independent*** when $F_{t,\lambda}$ depends only on $t - \lambda$; that is,

$$F_{t,\lambda} = F_{s,\mu} \quad \text{if} \quad t - \lambda = s - \mu.$$

Setting $F_t = F_{t,0}$, the preceding law becomes the ***group property***:

$$F_t \circ F_\tau = F_{t+\tau}, \quad F_0 = \text{identity}\,.$$

We call such an F_t a ***flow*** and $F_{t,\lambda}$ a ***time-dependent flow***, or as before, an evolution operator. If the system is nonreversible, that is, defined only for $t \geq \lambda$, we speak of a ***semi-flow***.

It is usually not $F_{t,\lambda}$ that is given, but rather the ***laws of motion***. In other words, differential equations are given that we must solve to find the flow. These equations of motion have the form

$$\frac{ds}{dt} = X(s), \quad s(0) = s_0$$

where X is a (possibly time-dependent) vector field on S.

4.1.3 Example. The motion of a particle of mass m under the influence of the gravitational force field in Example 4.1.2 is determined by Newton's second law:

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F};$$

that is, by the ordinary differentatial equations

$$m\frac{d^2x}{dt^2} = -\frac{mMGx}{r^3};$$
$$m\frac{d^2y}{dt^2} = -\frac{mMGy}{r^3};$$
$$m\frac{d^2z}{dt^2} = -\frac{mMGz}{r^3}.$$

Letting $\mathbf{q} = (x, y, z)$ denote the position and $\mathbf{p} = m(d\mathbf{r}/dt)$ the momentum, these equations become

$$\frac{d\mathbf{q}}{dt} = \frac{\mathbf{p}}{m}; \quad \frac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{q}).$$

The phase space here is the manifold $(\mathbb{R}^3 \backslash \{\mathbf{0}\}) \times \mathbb{R}^3$, that is, the cotangent manifold of $\mathbb{R}^3 \backslash \{\mathbf{0}\}$. The right-hand side of the preceding equations define a vector field on this six-dimensional manifold by

$$X(\mathbf{q}, \mathbf{p}) = \left((\mathbf{q}, \mathbf{p}), \left(\frac{\mathbf{p}}{m}, \mathbf{F}(\mathbf{q}) \right) \right).$$

In courses on mechanics or differential equations, it is shown how to integrate these equations explicitly, producing trajectories, which are planar conic sections. These trajectories comprise the flow of the vector field. ♦

Let us now turn to the elaboration of these ideas when a vector field X is given on a manifold M. If $M = U$ is an open subset of a Banach space $\mathbf{E}$, then a vector field on U is a map $X : U \to U \times \mathbf{E}$ of the form $X(x) = (x, V(x))$. We call V the ***principal part*** of X. However, having a separate notation for the principal part turns out to be an unnecessary burden. By abuse of notation, in linear spaces we shall write a vector field simply as a map $X : U \to \mathbf{E}$ and shall mean the vector field $x \mapsto (x, X(x))$. When it is necessary to be careful with the distinction, we shall be.

If M is a manifold and $\varphi : U \subset M \to V \subset \mathbf{E}$ is a local coordinate chart for M, then a vector field X on M induces a vector field X on $\mathbf{E}$ called the ***local representative*** of X by the formula $X(x) = T\varphi \cdot X(\varphi^{-1}(x))$. If $\mathbf{E} = \mathbb{R}^n$ we can identify the principal part of the vector field X with an n-component vector function $(X^1(x), \ldots, X^n(x))$. Thus we sometimes just say "the vector field X whose local representative is $(X^i) = (X^1, \ldots, X^n)$."

Recall that a *curve* c at a point m of a manifold M is a C^1-map from an open interval I of $\mathbb{R}$ into M such that $0 \in I$ and $c(0) = m$. For such a curve we may assign a tangent vector at each point $c(t)$, $t \in I$, by $c'(t) = T_t c(1)$.

4.1.4 Definition. *Let M be a manifold and $X \in \mathfrak{X}(M)$. An **integral curve** of X at $m \in M$ is a curve c at m such that $c'(t) = X(c(t))$ for each $t \in I$.*

In case $M = U \subset \mathbf{E}$, a curve $c(t)$ is an integral curve of $X : U \to \mathbf{E}$ when

$$c'(t) = X(c(t)),$$

where $c' = dc/dt$. If X is a vector field on a manifold M and X denotes the principal part of its local representative in a chart φ, a curve c on M is an integral curve of X when

$$\frac{dc}{dt}(t) = X(c(t)),$$

where $c = \varphi \circ c$ is the ***local representative*** of the curve c. If M is an n-manifold and the local representatives of X and c are $(X^1, \ldots, X^n)$ and $(c^1, \ldots, c^n)$ respectively, then c is an integral curve of X when the following system of ordinary differential equations is satisfied

$$\begin{aligned} \frac{dc^1}{dt}(t) &= X^1(c^1(t), \ldots, c^n(t)); \\ &\vdots \\ \frac{dc^n}{dt}(t) &= X^n(c^1(t), \ldots, c^n(t)). \end{aligned}$$

The reader should chase through the definitions to verify this assertion.

These equations are autonomous, corresponding to the fact that X is time independent. If X were time dependent, time t would appear explicitly on the right-hand side. As we saw in Example 4.1.3, the preceding system of equations includes equations of higher order (by their usual reduction to first-order systems) and the Hamilton equations of motion as special cases.

4.1.5 Theorem (Local Existence, Uniqueness, and Smoothness). *Let* $\mathbf{E}$ *be a Banach space,* $U \subset \mathbf{E}$ *be open, and suppose* $X : U \subset \mathbf{E} \to \mathbf{E}$ *is of class* C^k, $k \geq 1$. *Then*

1. *For each* $x_0 \in U$, *there is a curve* $c : I \to U$ *at* x_0 *such that* $c'(t) = X(c(t))$ *for all* $t \in I$.

2. *Any two such curves are equal on the intersection of their domains.*

3. *There is a neighborhood* U_0 *of the point* $x_0 \in U$, *a real number* $a > 0$, *and a* C^k *mapping* $F : U_0 \times I \to \mathbf{E}$, *where* I *is the open interval* $]-a, a[$, *such that the curve* $c_u : I \to \mathbf{E}$, *defined by* $c_u(t) = F(u, t)$ *is a curve at* $u \in \mathbf{E}$ *satisfying the differential equations* $c_u'(t) = X(c_u(t))$ *for all* $t \in I$.

4.1.6 Lemma. *Let* $\mathbf{E}$ *be a Banach space,* $U \subset \mathbf{E}$ *an open set, and* $X : U \subset \mathbf{E} \to \mathbf{E}$ *a Lipschitz map; that is, there is a constant* $K > 0$ *such that*

$$\|X(x) - X(y)\| \leq K\|x - y\|$$

for all $x, y \in U$. *Let* $x_0 \in U$ *and suppose the closed ball of radius* b,

$$B_b(x_0) = \{\, x \in \mathbf{E} \mid \|x - x_0\| \leq b \,\}$$

lies in U, *and* $\|X(x)\| \leq M$ *for all* $x \in B_b(x_0)$. *Let* $t_0 \in \mathbb{R}$ *and let* $\alpha = \min(1/K, b/M)$. *Then there is a unique* C^1 *curve* $x(t)$, $t \in [t_0 - \alpha, t_0 + \alpha]$ *such that* $x(t) \in B_b(x_0)$ *and*

$$x'(t) = X(x(t)), \quad x(t_0) = x_0.$$

Proof. The conditions $x'(t) = X(x(t))$, $x(t_0) = x_0$ are equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} X(x(s))\, ds$$

Put $x_0(t) = x_0$ and define inductively

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} X(x_n(s))\, ds$$

Clearly $x_n(t) \in B_b(x_0)$ for all n and $t \in [t_0 - \alpha, t_0 + \alpha]$ by definition of α. We also find by induction that

$$\|x_{n+1}(t) - x_n(t)\| \leq \frac{MK^n}{(n+1)!}\, |t - t_0|^{n+1}.$$

Thus $x_n(t)$ converges uniformly to a continuous curve $x(t)$. Clearly $x(t)$ satisfies the integral equation and thus is the solution we sought.

For uniqueness, let $y(t)$ be another solution. By induction we find that $\|x_n(t) - y(t)\| \leq MK^n|t - t_0|^{n+1}/(n+1)!$; thus, letting $n \to \infty$ gives $x(t) = y(t)$. ■

The same argument holds if X depends explicitly on t or on a parameter ρ, is jointly continuous in (t, ρ, x), and is Lipschitz in x uniformly in t and ρ. Since $x_n(t)$ is continuous in (x_0, t_0, ρ) so is $x(t)$, being a uniform limit of continuous functions; thus the integral curve is jointly continuous in (x_0, t_0, ρ).

4.1.7 Proposition (Gronwall's Inequality). *Let* $f, g : [a, b[\to \mathbb{R}$ *be continuous and nonnegative. Suppose that for all* t *satisfying* $a \leq t < b$,

$$f(t) \leq A + \int_a^t f(s)\, g(s)\, ds, \qquad \text{for a constant } A \geq 0.$$

Then

$$f(t) \leq A \exp\left(\int_a^t g(s)\, ds \right) \qquad \text{for all } t \in [a, b[.$$

Proof. First suppose $A > 0$. Let

$$h(t) = A + \int_a^t f(s)\, g(s)\, ds;$$

thus $h(t) > 0$. Then $h'(t) = f(t)g(t) \leq h(t)g(t)$. Thus $h'(t)/h(t) \leq g(t)$. Integration gives

$$h(t) \leq A \exp\left(\int_a^t g(s)\, ds\right).$$

This gives the result for $A > 0$. If $A = 0$, then we get the result by replacing A by $\varepsilon > 0$ for every $\varepsilon > 0$; thus h and hence f is zero. ■

4.1.8 Lemma. *Let X be as in Lemma 4.1.6. Let $F_t(x_0)$ denote the solution (= integral curve) of $x'(t) = X(x(t))$, $x(0) = x_0$. Then there is a neighborhood V of x_0 and a number $\varepsilon > 0$ such that for every $y \in V$ there is a unique integral curve $x(t) = F_t(y)$ satisfying $x'(t) = X(x(t))$ for all $t \in [-\varepsilon, \varepsilon]$, and $x(0) = y$. Moreover,*

$$\|F_t(x) - F_t(y)\| \leq e^{K|t|}\|x - y\|.$$

Proof. Choose $V = B_{b/2}(x_0)$ and $\varepsilon = \min(l/K, b/2M)$. Fix an arbitrary $y \in V$. Then $B_{b/2}(y) \subset B_b(x_0)$ and hence $\|X(z)\| \leq M$ for all $z \in B_{b/2}(y)$. By Theorem 4.1.5 with x_0 replaced by y, b by $b/2$, and t_0 by 0, there exists an integral curve $x(t)$ of $x'(t) = X(x(t))$ for $t \in [-\varepsilon, \varepsilon]$ and satisfying $x(0) = y$. This proves the first part. For the second, let $f(t) = \|F_t(x) - F_t(y)\|$. Clearly

$$f(t) = \left\|\int_0^t [X(F_s(x)) - X(F_s(y))]\, ds + x - y\right\| \leq \|x - y\| + K\int_0^t f(s)\, ds,$$

so the result follows from Gronwall's inequality. ■

This result shows that $F_t(x)$ depends in a continuous, indeed Lipschitz, manner on the initial condition x and is jointly continuous in (t, x). Again, the same result holds if X depends explicitly on t and on a parameter ρ is jointly continuous in (t, ρ, x), and is Lipschitz in x uniformly in t and ρ; $(F_{t,\lambda})^\rho(x)$ is the unique integral curve $x(t)$ satisfying $x'(t) = X(x(t), t, \rho)$ and $x(\lambda) = x$. By the remark following Lemma 4.1.6, $(F_{t,\lambda})^\rho(x)$ is jointly continuous in the variables (λ, t, ρ, x), and is Lipschitz in x, uniformly in (λ, t, ρ). The next result shows that F_t is C^k if X is, and completes the proof of Theorem 4.1.5. For the next lemma, recall that a C^1-function is locally Lipschitz.

4.1.9 Lemma. *Let X in Lemma 4.1.6 be of class C^k, $l \leq k \leq \infty$, and let $F_t(x)$ be defined as before. Then locally in (t, x), $F_t(x)$ is of class C^k in x and is C^{k+1} in the t-variable.*

Proof. We define $\psi(t, x) \in L(\mathbf{E}, \mathbf{E})$, the set of continuous linear maps of $\mathbf{E}$ to $\mathbf{E}$, to be the solution of the "linearized" or "first variation" equations:

$$\frac{d}{dt}\psi(t, x) = \mathbf{D}X(F_t(x)) \circ \psi(t, x), \quad \text{with } \psi(0, x) = \text{ identity},$$

where $\mathbf{D}X(y) : \mathbf{E} \to \mathbf{E}$ is the derivative of X taken at the point y. Since the vector field $\psi \mapsto \mathbf{D}X(F_t(x)) \circ \psi$ on $L(\mathbf{E}, \mathbf{E})$ (depending explicitly on t and on the parameter x) is Lipschitz in ψ, uniformly in (t, x) in a neighborhood of every (t_0, x_0), by the remark following Lemma 4.1.8 it follows that $\psi(t, x)$ is continuous in (t, x) (using the norm topology on $L(\mathbf{E}, \mathbf{E})$).

We claim that $\mathbf{D}F_t(x) = \psi(t,x)$. To show this, fix t, set $\theta(s,h) = F_s(x+h) - F_s(x)$, and write

$$\begin{aligned}\theta(t,h) - \psi(t,x)\cdot h &= \int_0^t \{X(F_s(x+h)) - X(F_s(x))\}\, ds \\ &\quad - \int_0^t [\mathbf{D}X(F_s(x)) \circ \psi(s,x)]\cdot h\, ds \\ &= \int_0^t \mathbf{D}X(F_s(x))\cdot[\theta(s,h) - \psi(s,x)\cdot h]\, ds \\ &\quad + \int_0^t \{X(F_s(x+h)) - X(F_s(x)) \\ &\quad - \mathbf{D}X(F_s(x))\cdot[F_s(x+h) - F_s(x)]\}\, ds.\end{aligned}$$

Since X is of class C^1, given $\varepsilon > 0$, there is a $\delta > 0$ such that $\|h\| < \delta$ implies the second term is dominated in norm by

$$\int_0^t \varepsilon \|F_s(x+h) - F_s(x)\|\, ds,$$

which is, in turn, smaller than $A\varepsilon\|h\|$ for a positive constant A by Lemma 4.1.8. By Gronwall's inequality we obtain $\|\theta(t,h) - \psi(t,x)\cdot h\| \leq$ (constant) $\varepsilon\|h\|$. It follows that $\mathbf{D}F_t(x)\cdot h = \psi(t,x)\cdot h$. Thus both partial derivatives of $F_t(x)$ exist and are continuous; therefore $F_t(x)$ is of class C^1.

We prove $F_t(x)$ is C^k by induction on k. Begin with the equation defining F_t :

$$\frac{d}{dt}F_t(x) = X(F_t(x))$$

so

$$\frac{d}{dt}\frac{d}{dt}F_t(x) = \mathbf{D}X(F_t(x))\cdot X(F_t(x))$$

and

$$\frac{d}{dt}\mathbf{D}F_t(x) = \mathbf{D}X(F_t(x))\cdot \mathbf{D}F_t(x).$$

Since the right-hand sides are C^{k-1}, so are the solutions by induction. Thus F itself is C^k. ■

Again there is an analogous result for the evolution operator $(F_{t,\lambda})^\rho(x)$ for a time-dependent vector field $X(x,t,\rho)$, which depends on extra parameters ρ in a Banach space P. If X is C^k, then $(F_{t,\lambda})^\rho(x)$ is C^k in all variables and is C^{k+1} in t and λ. The variable ρ can be easily dealt with by suspending X to a new vector field obtained by appending the trivial differential equation $\rho' = 0$; this defines a vector field on $\mathbf{E}\times P$ and Theorem 4.1.5 may be applied to it. The flow on $\mathbf{E}\times P$ is just $F_t(x,\rho) = (F_t^\rho(x), \rho)$.

For another more "modern" proof of Theorem 4.1.5 see Supplement 4.1C. That alternative proof has a technical advantage: it works easily for other types of differentiability assumptions on X or on F_t, such as Hölder or Sobolev differentiability; this result is due to Ebin and Marsden [1970].

The mapping F gives a locally unique integral curve c_u for each $u \in U_0$, and for each $t \in I$, $F_t = F|(U_0 \times \{t\})$ maps U_0 to some other set. It is convenient to think of each point u being allowed to "flow for time t" along the integral curve c_u (see Figure 4.1.2 and our opening motivation). This is a picture of a U_0 "flowing," and the system (U_0, a, F) is a local flow of X, or ***flow box***. The analogous situation on a manifold is given by the following.

4.1.10 Definition. *Let M be a manifold and X a C^r vector field on M, $r \geq 1$. A flow box of X at $m \in M$ is a triple (U_0, a, F), where*

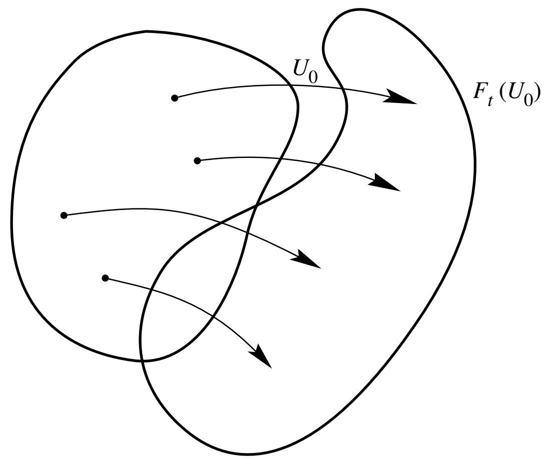


FIGURE 4.1.2. Picturing a flow

(i) *$U_0 \subset M$ is open, $m \in U_0$, and $a \in \mathbb{R}$, where $a > 0$ or $a = +\infty$;*

(ii) *$F : U_0 \times I_a \to M$ is of class C^r, where $I_a =]-a, a[$;*

(iii) *for each $u \in U_0$, $c_u : I_a \to M$ defined by $c_u(t) = F(u,t)$ is an integral curve of X at the point u;*

(iv) *if $F_t : U_0 \to M$ is defined by $F_t(u) = F(u,t)$, then for $t \in I_a$, $F_t(U_0)$ is open, and F_t is a C^r diffeomorphism onto its image.*

Before proving the existence of a flow box, it is convenient first to establish the following, which concerns uniqueness.

4.1.11 Proposition (Global Uniqueness)**.** *Suppose c_1 and c_2 are two integral curves of X at $m \in M$. Then $c_1 = c_2$ on the intersection of their domains.*

Proof. This does not follow at once from Theorem 4.1.5 for c_1 and c_2 may lie in different charts. (Indeed, if the manifold is not Hausdorff, Exercise 4.1-13 shows that this proposition is false.) Suppose $c_1 : I_1 \to M$ and $c_2 : I_2 \to M$. Let $I = I_1 \cap I_2$, and let $K = \{ t \in I \mid c_1(t) = c_2(t) \}$; K is closed since M is Hausdorff. We will now show that K is open. From Theorem 4.1.5, K contains some neighborhood of 0. For $t \in K$ consider c_1^t and c_2^t, where $c^t(s) = c(t+s)$. Then c_1^t and c_2^t are integral curves at $c_1(t) = c_2(t)$. Again, by Theorem 4.1.5 they agree on some neighborhood of 0. Thus some neigborhood of t lies in K, and so K is open. Since I is connnected, $K = I$. ■

4.1.12 Proposition. *Suppose (U_0, a, F) is a triple satisfying* (i)*,* (ii)*, and* (iii) *of Definition* 4.1.10*. Then for t, s and $t+s \in I_a$,*

$$F_{t+s} = F_t \circ F_s = F_s \circ F_t \quad \textit{and} \quad F_0 \textit{ is the identity map,}$$

whenever the compositions above are defined. Moreover, if $U_t = F_t(U_0)$ and $U_t \cap U_0 \neq \varnothing$, then $F_t|U_{-t} \cap U_0 : U_{-t} \cap U_0 \to U_0 \cap U_t$ is a diffeomorphism and its inverse is $F_{-t}|U_0 \cap U_t$.

Proof. $F_{t+s}(u) = c_u(t+s)$, where c_u is the integral curve defined by F at u. But $d(t) = F_t(F_s(u)) = F_t(c_u(s))$ is the integral curve through $c_u(s)$ and $f(t) = c_u(t+s)$ is also an integral curve at $c_u(s)$. Hence by global uniqueness Proposition 4.1.11 we have $F_t(F_s(u)) = c_u(t+s) = F_{t+s}(u)$. To show that $F_{t+s} = F_s \circ F_t$, observe that $F_{t+s} = F_{s+t} = F_s \circ F_t$. Since $c_u(t)$ is a curve at u, $c_u(0) = u$, so F_0 is the identity. Finally, the last statement is a consequence of $F_t \circ F_{-t} = F_{-t} \circ F_t =$ identity. Note, however, that $F_t(U_0) \cap U_0 = \varnothing$ can occur.

■

4.1.13 Proposition (Existence and Uniqueness of Flow Boxes). *Let X be a C^r vector field on a manifold M. For each $m \in M$ there is a flow box of X at m. Suppose $(U_0, a, F), (U_0', a', F')$ are two flow boxes at $m \in M$. Then F and F' are equal on $(U_0 \cap U_0') \times (I_a \cap I_{a'})$.*

Proof. (**Uniqueness**). Again we emphasize that this does not follow at once from Theorem 4.1.5, since U_0 and U_0' need not be chart domains. However, for each point $u \in U_0 \cap U_0'$ we have $F|\{u\} \times I = F'|\{u\} \times I$, where $I = I_a \cap I_{a'}$. This follows from Proposition 4.1.11 and Definition 4.1.10(iii). Hence $F = F'$ on the set $(U_0 \cap U_0') \times I$.

(**Existence**). Let (U, φ) be a chart in M with $m \in U$. It is enough to establish the result in $\varphi(U)$ by means of the local representation. Thus let (U_0', a, F') be a flow box of X, the local representative of X, at $\varphi(m)$ as given by Theorem 4.1.5, with

$$U_0' \subset U' = \varphi(U) \quad \text{and} \quad F'(U_0' \times I_a) \subset U', \quad U_0 = \varphi^{-1}(U_0')$$

and let

$$F : U_0 \times I_a \to M; \quad (u, t) \mapsto \varphi^{-1}(F'(\varphi(u), t)).$$

Since F is continuous, there is a $b \in]0, a[\subset \mathbb{R}$ and $V_0 \subset U_0$ open, with $m \in V_0$, such that $F(V_0 \times I_b) \subset U_0$. We contend that (V_0, b, F) is a flow box at m (where F is understood as the restriction of F to $V_0 \times I_b$). Parts (i) and (ii) of Definition 4.1.10 follow by construction and (iii) is a consequence of the remarks following Definition 4.1.4 on the local representation. To prove (iv), note that for $t \in I_b$, F_t has a C^r inverse, namely, F_{-t} as $V_t \cap U_0 = V_t$. It follows that $F_t(V_0)$ is open. And, since F_t and F_{-t} are both of class C^r, F_t is a C^r diffeomorphism. ■

As usual, there is an analogous result for time- (or parameter-) dependent vector fields. The following result shows that near a point m satisfying $X(m) \neq 0$, the flow can be transformed by a change of variables so that the integral curves become straight lines.

4.1.14 Theorem (Straightening Out Theorem). *Let X be a vector field on a manifold M and suppose at $m \in M$, $X(m) \neq 0$. Then there is a local chart (U, φ) with $m \in U$ such that*

(i) *$\varphi(U) = V \times I \subset G \times \mathbb{R} = \mathbf{E}$, $V \subset G$, open, and $I =]-a, a[\subset \mathbb{R}$, $a > 0$;*

(ii) *$\varphi^{-1}|\{v\} \times I : I \to M$ is an integral curve of X at $\varphi^{-1}(v, 0)$, for all $v \in V$;*

(iii) *the local representative X has the form $X(y, t) = (y, t; 0, 1)$.*

Proof. Since the result is local, by taking any initial coordinate chart, it suffices to prove the result in $\mathbf{E}$. We can arrange things so that we are working near $0 \in \mathbf{E}$ and $X(0) = (0, 1) \in \mathbf{E} = G \oplus \mathbb{R}$ where $\mathbf{G}$ is a complement to the span of $X(0)$. Letting (U_0, b, F) be a flow box for X at 0 where $U_0 = V_0 \times]-\varepsilon, \varepsilon[$ and V_0 is open in $\mathbf{G}$, define

$$f_0 : V_0 \times I_b \to \mathbf{E} \quad \text{by} \quad f_0(y, t) = F_t(y, 0).$$

But

$$\mathbf{D} f_0(0, 0) = \text{Identity}$$

since

$$\left. \frac{\partial F_t(0, 0)}{\partial t} \right|_{t=0} = X(0) = (0, 1) \quad \text{and} \quad F_0 = \text{Identity}.$$

By the inverse mapping theorem there are open neighborhoods $V \times I_a \subset V_0 \times I_b$ and $U = f_0(V \times I_a)$ of $(0, 0)$ such that $f = f_0 | V \times I_a : V \times I_a \to U$ is a diffeomorphism. Then $f^{-1} : U \to V \times I_a$ can serve as chart for (i). Notice that $c = f|\{y\} \times I : I \to U$ is the integral curve of X through $(y, 0)$ for all $y \in V$, thus proving (ii). Finally, the expression of the vector field X in this local chart given by f^{-1} is $\mathbf{D}f^{-1}(y,t) \cdot X(f(y,t)) = \mathbf{D}f^{-1}(c(t)) \cdot c'(t) = (f^{-1} \circ c)'(t) = (0,1)$, since $(f^{-1} \circ c)(t) = (y,t)$, thus proving (iii). ■

In §4.3 we shall see that singular points, where the vector field vanishes, are of great interest in dynamics. The straightening out theorem does not claim anything about these points. Instead, one needs to appeal to more sophisticated normal form theorems; see Guckenheimer and Holmes [1983].

Now we turn our attention from local flows to global considerations. These ideas center on considering the flow of a vector field as a whole, extended as far as possible in the t-variable.

4.1.15 Definition. *Given a manifold M and a vector field X on M, let $\mathcal{D}_X \subset M \times \mathbb{R}$ be the set of $(m,t) \in M \times \mathbb{R}$ such that there is an integral curve $c : I \to M$ of X at m with $t \in I$. The vector field X is **complete** if $\mathcal{D}_X = M \times \mathbb{R}$. A point $m \in M$ is called σ-**complete**, where $\sigma = +, -$, or $\pm$, if $\mathcal{D}_X \cap (\{m\} \times \mathbb{R})$ contains all (m,t) for $t > 0$, < 0, or $t \in \mathbb{R}$, respectively. Let $T^+(m)$ (resp., $T^-(m)$) denote the sup (resp., inf) of the times of existence of the integral curves through m; $T^+(m)$ (resp., $T^-(m)$) is called the **positive** (resp., **negative**) **lifetime of** m.*

Thus, X is complete iff each integral curve can be extended so that its domain becomes $]-\infty, \infty[$; that is, $T^+(m) = \infty$ and $T^-(m) = -\infty$ for all $m \in M$.

4.1.16 Examples.

A. For $M = \mathbb{R}^2$, let X be the constant vector field, whose principal part is $(0,1)$. Then X is complete since the integral curve of X through (x,y) is $t \mapsto (x, y+t)$.

B. On $M = \mathbb{R}^2 \backslash \{0\}$, the same vector field is not complete since the integral curve of X through $(0,-1)$ cannot be extended beyond $t = 1$; in fact as $t \to 1$ this integral curve tends to the point $(0,0)$. Thus $T^+(0,-1) = 1$, while $T^-(0,-1) = -\infty$.

C. On $\mathbb{R}$ consider the vector field $X(x) = 1 + x^2$. This is not complete since the integral curve c with $c(0) = 0$ is $c(t) = \tan^{-1} t$ and thus it cannot be continuously extended beyond $-\pi/2$ and $\pi/2$; that is, $T^{\pm}(0) = \pm\pi/2$. ♦

4.1.17 Proposition. *Let M be a manifold and $X \in \mathfrak{X}^r(M)$, $r \geq 1$. Then*

(i) $\mathcal{D}_X \supset M \times \{0\}$;

(ii) $\mathcal{D}_X$ *is open in* $M \times \mathbb{R}$;

(iii) *there is a unique C^r mapping $F_X : \mathcal{D}_X \to M$ such that the mapping $t \mapsto F_X(m,t)$ is an integral curve at m for all $m \in M$;*

(iv) *for $(m,t) \in \mathcal{D}_X$, $(F_X(m,t), s) \in \mathcal{D}_X$ iff $(m, t+s) \in \mathcal{D}_X$; in this case*

$$F_X(m, t+s) = F_X(F_X(m,t), s).$$

Proof. Parts (i) and (ii) follow from the flow box existence theorem. In (iii), we get a unique map $F_X : \mathcal{D}_X \to M$ by the global uniqueness and local existence of integral curves: $(m,t) \in \mathcal{D}_X$ when the integral curve $m(s)$ through m exists for $s \in [0,t]$. We set $F_X(m,t) = m(t)$. To show F_X is C^r, note that in a neighborhood of a fixed m_0 and for small t, it is C^r by local smoothness. To show F_X is globally C^r, first note that (iv) holds by global uniqueness. Then in a neighborhood of the compact set $\{\, m(s) \mid s \in [0,t] \,\}$ we can write F_X as a composition of finitely many C^r maps by taking short enough time steps so the local flows are smooth. ■

4.1.18 Definition. *Let M be a manifold and $X \in \mathfrak{X}^r(M)$, $r \geq 1$. Then the mapping F_X is called the* ***integral*** *of X, and the curve $t \mapsto F_X(m,t)$ is called the* ***maximal integral curve*** *of X at m. In case X is complete, F_X is called the* ***flow*** *of X.*

Thus, if X is complete with flow F, then the set $\{ F_t \mid t \in \mathbb{R} \}$ is a group of diffeomorphisms on M, sometimes called a ***one-parameter group of diffeomorphisms***. Since $F_n = (F_1)^n$ (the nth power), the notation F^t is sometimes convenient and is used where we use F_t. For incomplete flows, (iv) says that $F_t \circ F_s = F_{t+s}$ wherever it is defined. Note that $F_t(m)$ is defined for $t \in \,]T^-(m), T^+(m)[$. The reader should write out similar definitions for the time-dependent case and note that the lifetimes depend on the starting time t_0.

4.1.19 Proposition. *Let X be C^r, where $r \geq 1$. Let $c(t)$ be a maximal integral curve of X such that for every finite open interval $]a,b[$ in the domain $]T^-(c(0)), T^+(c(0))[$ of c, $c(]a,b[)$ lies in a compact subset of M. Then c is defined for all $t \in \mathbb{R}$.*

Proof. It suffices to show that $a \in I$, $b \in I$, where I is the interval of definition of c. Let $t_n \in \,]a,b[$, $t_n \to b$. By compactness we can assume some subsequence $c(t_{n(k)})$ converges, say, to a point x in M. Since the domain of the flow is open, it contains a neighborhood of $(x,0)$. Thus, there are $\varepsilon > 0$ and $\tau > 0$ such that integral curves starting at points (such as $c(t_{n(k)})$ for large k) closer than ε to x persist for a time longer than τ. This serves to extend c to a time greater than b, so $b \in I$ since c is maximal. Similarly, $a \in I$. ■

The ***support*** of a vector field X defined on a manifold M is defined to be the closure of the set $\{ m \in M \mid X(m) \neq 0 \}$.

4.1.20 Corollary. *A C^r vector field with compact support on a manifold M is complete. In particular, a C^r vector field on a compact manifold is complete.*

Completeness corresponds to well-defined dynamics persisting eternally. In some circumstances (shock waves in fluids and solids, singularities in general relativity, etc.) one has to live with incompleteness or overcome it in some other way. Because of its importance we give two additional criteria. In the first result we use the notation $X[f] = \mathbf{d}f \cdot X$ for the derivative of f in the direction X. Here $f : \mathbf{E} \to \mathbb{R}$ and $\mathbf{d}f$ stands for the derivative map. In standard coordinates on $\mathbb{R}^n$,

$$\mathbf{d}f(x) = \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n} \right) \qquad \text{and} \qquad X[f] = \sum_{i=1}^{n} X^i \frac{\partial f}{\partial x^i}.$$

4.1.21 Proposition. *Suppose X is a C^k vector field on the Banach space $\mathbf{E}$, $k \geq 1$, and $f : \mathbf{E} \to \mathbb{R}$ is a C^1* ***proper map****; that is, if $\{x_n\}$ is any sequence in $\mathbf{E}$ such that $f(x_n) \to a$, then there is a convergent subseqence $\{x_{n(i)}\}$. Suppose there are constants $K, L \geq 0$ such that*

$$|X[f](m)| \leq K|f(m)| + L \quad \textit{for all } m \in \mathbf{E}.$$

Then the flow of X is complete.

Proof. From the chain rule we have $(\partial/\partial t) f(F_t(m)) = X[f](F_t(m))$, so that

$$f(F_t(m)) - f(m) = \int_0^t X[f](F_\tau(m))\, d\tau$$

Applying the hypothesis and Gronwall's inequality we see that $|f(F_t(m))|$ is bounded and hence relatively compact on any finite t-interval, so as f is proper, a repetition of the argument in the proof of Proposition 4.1.19 applies. ■

Note that the same result holds if we replace "properness" by "inverse images of compact sets are bounded" and assume X has a uniform existence time on each bounded set. This version is useful in some infinite dimensional examples.

4.1.22 Proposition. *Let X be a C^r vector field on the Banach space $\mathbf{E}$, $r \geq 1$. Let σ be any integral curve of X. Assume $\|X(\sigma(t))\|$ is bounded on finite t-intervals. Then $\sigma(t)$ exists for all $t \in \mathbb{R}$.*

Proof. Suppose $\|X(\sigma(t))\| < A$ for $t \in]a, b[$ and let $t_n \to b$. For $t_n < t_m$ we have

$$\|\sigma(t_n) - \sigma(t_m)\| \leq \int_{t_n}^{t_m} \|\sigma'(t)\| \, dt = \int_{t_n}^{t_m} \|X(\sigma(t))\| \, dt \leq A|t_m - t_n|.$$

Hence $\sigma(t_n)$ is a Cauchy sequence and therefore, converges. Now argue as in Proposition 4.1.19. ■

4.1.23 Examples.

A. Let X be a C^r vector field, $r \geq 1$, on the manifold M admitting a ***first integral***, that is, a function $f : M \to \mathbb{R}$ such that $X[f] = 0$. If all level sets $f^{-1}(r)$, $r \in \mathbb{R}$ are compact, X is complete. Indeed, each integral curve lies on a level set of f so that the result follows by Proposition 4.1.19.

B. Newton's equations for a moving particle of mass m in a potential field in $\mathbb{R}^n$ are given by $\ddot{\mathbf{q}}(t) = -(1/m)\nabla V(\mathbf{q}(t))$, for $V : \mathbb{R}^n \to \mathbb{R}$ a smooth function. We shall prove that *if there are constants $a, b \in \mathbb{R}$, $b \geq 0$ such that $(1/m)V(\mathbf{q}) \geq a - b\|\mathbf{q}\|^2$, then every solution exists for all time*. To show this, rewrite the second order equations as a first order system $\dot{\mathbf{q}} = (1/m)\mathbf{p}$, $\dot{\mathbf{p}} = -\nabla V(\mathbf{q})$ and note that the energy $E(\mathbf{q}, \mathbf{p}) = (1/2m)\|\mathbf{p}\|^2 + V(\mathbf{q})$ is a first integral. Thus, for any solution $(\mathbf{q}(t), \mathbf{p}(t))$ we have $\beta = E(\mathbf{q}(t), \mathbf{p}(t)) = E(\mathbf{q}(0), \mathbf{p}(0)) \geq V(\mathbf{q}(0))$. We can assume $\beta > V(\mathbf{q}(0))$, that is, $\mathbf{p}(0) \neq 0$, for if $\mathbf{p}(t) \equiv 0$, then the conclusion is trivially satisifed; thus there exists a t_0 for which $\mathbf{p}(t_0) \neq 0$ and by time translation we can assume that $t_0 = 0$. Thus we have

$$\begin{aligned}
\|\mathbf{q}(t)\| &\leq \|\mathbf{q}(t) - \mathbf{q}(0)\| + \|\mathbf{q}(0)\| \leq \|\mathbf{q}(0)\| + \int_0^t \|\dot{\mathbf{q}}(s)\| \, ds \\
&= \|\mathbf{q}(0)\| + \int_0^t \sqrt{2\left[\beta - \frac{1}{m}V(\mathbf{q}(s))\right]} \, ds \\
&\leq \|\mathbf{q}(0)\| + \int_0^t \sqrt{2(\beta - a + b\|\mathbf{q}(s)\|^2)} \, ds
\end{aligned}$$

or in differential form

$$\frac{d}{dt}\|\mathbf{q}(t)\| \leq \sqrt{2(\beta - a + b\|\mathbf{q}(t)\|^2)}$$

whence

$$t \leq \int_{\|\mathbf{q}(0)\|}^{\|\mathbf{q}(t)\|} \frac{du}{\sqrt{2(\beta - a + bu^2)}} \tag{4.1.1}$$

Now let $r(t)$ be the solution of the differential equation

$$\frac{d^2 r(t)}{dt^2} = -\frac{d}{dr}(a - br^2)(t) = 2br(t),$$

which, as a second order equation with constant coefficients, has solutions for all time for any initial conditions. Choose

$$r(0) = \|\mathbf{q}(0)\|, \quad [\dot{r}(0)]^2 = 2(\beta - a + b\|\mathbf{q}(0)\|^2)$$

and let $r(t)$ be the corresponding solution. Since

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}(t)^2 + a - br(t)^2\right) = 0,$$

it follows that $(1/2)\dot{r}(t)^2 + a - br(t)2 = (1/2)\dot{r}(0)^2 + a - br(0)^2 = \beta$, that is,

$$\frac{dr(t)}{dt} = \sqrt{2(\beta - a + br(t)^2)}$$

whence

$$t = \int_{\| \mathrm{q}(0) \|}^{r(t)} \frac{du}{\sqrt{2(\beta - \alpha + \beta u^2}} \tag{4.1.2}$$

Comparing the two expressions (4.1.1) and (4.1.2) and taking into account that the integrand is > 0, it follows that for any finite time interval for which $\mathbf{q}(t)$ is defined, we have $\|\mathbf{q}(t)\| \leq r(t)$, that is, $\mathbf{q}(t)$ remains in a compact set for finite t-intervals. But then $\dot{\mathbf{q}}(t)$ also lies in a compact set since

$$\|\dot{\mathbf{q}}(t)\| \leq 2(\beta - a + b\|\mathbf{q}(s)\|^2).$$

Thus by Proposition 4.1.19, the solution curve $(\mathbf{q}(t), \mathbf{p}(t))$ is defined for any $t \geq 0$. However, since $(\mathbf{q}(-t), \mathbf{p}(-t))$ is the value at t of the integral curve with initial conditions $(-\mathbf{q}(0), -\mathbf{p}(0))$, it follows that the solution also exists for all $t \leq 0$. (This example will be generalized in §8.1 to any Lagrangian system on a complete Riemannian manifold whose energy function is kinetic energy of the metric plus a potential, with the potential obeying an inequality of the sort here).

The following counterexample shows that the condition $V(\mathbf{q}) \geq a - b\|\mathbf{q}\|^2$ cannot be relaxed much further. Take $n = 1$ and

$$V(q) = -\tfrac{1}{8}\varepsilon^2 q^{2+(4/\varepsilon)}, \quad \varepsilon > 0.$$

Then the equation $\ddot{q} = \varepsilon(\varepsilon + 2)q^{1+(4/\varepsilon)}/4$ has the solution $q(t) = 1/(t-1)^{\varepsilon/2}$, which cannot be extended beyond $t = 1$.

C. Let $\mathbf{E}$ be a Banach space. Suppose

$$A(x) = A \cdot x + B(x),$$

where A is a bounded linear operator of $\mathbf{E}$ to $\mathbf{E}$ and B is ***sublinear***; that is, $B : \mathbf{E} \to \mathbf{E}$ is C^r with $r \geq 1$ and satisfies $\|B(x)\| \leq K\|x\| + L$ for constants K and L. We shall show that X is complete by using Proposition 4.1.22. (In $\mathbb{R}^n$, Proposition 4.1.21 can also be used with $f(x) = \|x\|^2$.) Let $x(t)$ be an integral curve of X on the bounded interval $[0, T]$. Then

$$x(t) = x(0) + \int_0^t (A \cdot x(s) + B(x(s)))\, ds$$

Hence

$$\|x(t)\| \leq \|x(0)\| \int_0^t (\|A\| + K)\|x(s)\|\, ds + Lt.$$

By Gronwall's inequality,

$$\|x(t)\| \leq (LT + \|x(0)\|)e^{(\|A\|+K)t}.$$

Hence $x(t)$ and so $X(x(t))$ remain bounded on bounded t-intervals. ♦

4.1.24 Proposition. *Let X be a C^r vector field on the manifold M, $r \geq 1$, $m_0 \in M$, and $T^+(m_0)(T^-(m_0))$ the positive (negative) lifetime of m_0. Then for each $\varepsilon > 0$, there exists a neighborhood V of m_0 such that for all $m \in V$, $T^+(m) > T^+(m_0) - \varepsilon$ (respectively, $T^-(m_0) < T^-(m_0) + \varepsilon$). [One says $T^+(m_0)$ is a lower semi-continuous function of m.]*

Proof. Cover the segment $\{ F_t(m_0) \mid t \in [0, T^+(m_0) - \varepsilon] \}$ with a finite number of neighborhoods $U_0, \ldots, U_n$, each in a chart domain and such that $\varphi_i(U_i)$ is diffeomorphic to the open ball $B_{b(i)/2}(0)$ in $\mathbf{E}$ given in the proof of Lemma 4.1.8, where φ_i is the chart map. Let $m_i \in U_i$ be such that $\varphi_i(m_i) = 0$ and $t(i)$ such that $F_{t(i)}(m_0) = m_i$, $i = 0, \ldots, n$, $t(0) = 0$, $t(n) = T^+(m_0) - \varepsilon$. By Lemma 4.1.8 the time of existence of all integral curves starting in U_i is uniformly at least $\alpha(i) > 0$. Pick points $p_i \in U_i \cap U_{i+1}$ and let $s(i)$ be such that

$$F_{s(i)}(m_0) = p_i,\ i = 0, \ldots, n-1, \quad s(0) = 0, \quad p_0 = m_0,$$
$$s(i) < s(i+1), \quad s(i+1) - t(i) < \alpha(i), \quad t(i+1) - s(i+1) < \alpha(i+1),$$
$$s(i+1) - s(i) < \min(\alpha(i), \alpha(i+1));$$

see Figure 4.1.3.

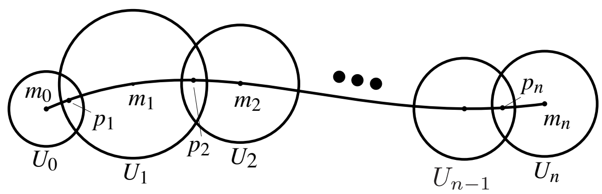


FIGURE 4.1.3. A chain of charts

Let $W_1 = U_0 \cap U_1$. Since $s(2) - s(1) < \alpha(1)$ and any integral curve starting in $W_1 \subset U_1$ exists for time at least $\alpha(1)$, the domain of $F_{s(2)-s(1)}$ includes W_1 and hence $F_{s(2)-s(1)}(W_1)$ makes sense. Define the open set $W_2 = F_{s(2)-s(1)}(W_1) \cap U_1 \cap U_2$ and use $s(3) - s(2) < \alpha(2)$, $W_2 \subset U_2$ to conclude that the domain of $F_{s(3)-s(2)}$ contains U_2. Define the open set $W_3 = F_{s(3)-s(2)}(W_2) \cap U_2 \cap U_3$ and inductively define

$$W_i = F_{s(i)-s(i-1)}(W_i - 1) \cap U_{i-1} \cap U_i, \quad i = 1, \ldots, n,$$

which are open sets. Since $s(1) < \alpha(0)$ and $W_1 \subset U_0$, the domain of $F_{-s(1)}$ includes W_1 and thus $V_1 = F_{-s(1)}(W_1) \cap U_0$ is an open neighborhood of m_0. Since $s(2) - s(1) < \alpha(1)$ and $W_2 \subset U_1$, the domain of $F_{-s(2)+s(1)}$ contains W_2 and so $F_{-s(2)+s(1)}(W_2) \subset W_1$ makes sense. Therefore

$$F_{-s(2)}(W_2) = F_{-s(1)}(F_{-s(2)+s(1)}(W_2)) \subset F_{-s(1)}(W_1)$$

exists and is an open neighborhood of m_0. Put $V_2 = F_{-s(2)}(W_2) \cap U_0$. Now proceed inductively to show that $F_{-s(i)}(W_i) \subset F_{-s(i-1)}(W_{i-1})$ makes sense and is an open neighborhood of m_0, $i = 1, 2, \ldots, n$; see Figure 4.1.4. Let $V_i = F_{-s(i)}(W_i) \cap U_0$, $i = 1, \ldots, n$, open neighborhoods containing m_0. Any integral curve $c(t)$ starting in V_n exists thus for time at least $s(n)$ and $F_{s(n)}(V_n) \subset W_n \subset U_n$. Now consider the integral curve starting at $c(s(n))$ whose time of existence is at least $\alpha(n)$. By uniqueness, $c(t)$ can be smoothly extended to an integral curve which exists for time at least $s(n) + \alpha(n) > t_n = T^+(m_0) - \varepsilon$. ■

The same result and proof hold for time dependent vector fields depending on a parameter.

4.1.25 Corollary. *Let X_t be a C^r time-dependent vector field on M, $r \geq 1$, and let m_0 be an **equilibrium** of X_t, that is, $X_t(m_0) = 0$ for all t. Then for any T there exists a neighborhood V of m_0 such that any $m \in V$ has integral curve existing for time $t \in [-T, T]$.*

Proof. Since $T^+(m_0) = +\infty$, $T^-(m_0) = -\infty$, the previous proposition gives the result. ■

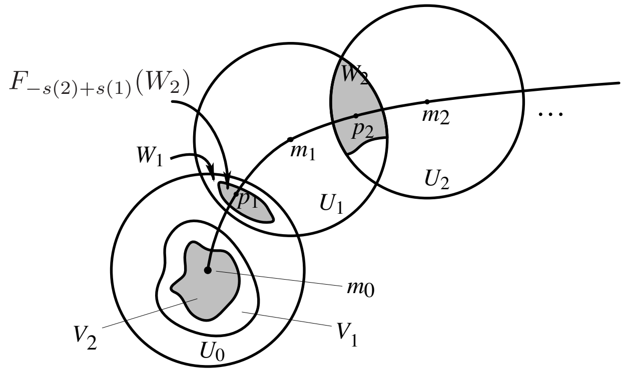


FIGURE 4.1.4. Semicontinuity of lifetimes

SUPPLEMENT 4.1A

Product Formulas

A result of some importance in both theoretical and numerical work concerns writing a flow in terms of iterates of a known mapping. Let $X \in \mathfrak{X}(M)$ with flow F_t (maximally extended). Let $K_\varepsilon(x)$ be a given map defined in some open set of $[0,\infty[\times M$ containing $\{0\} \times M$ and taking values in M, and assume that

(i) $K_0(x) = x$ and

(ii) $K_\varepsilon(x)$ is C^1 in ε with derivative continuous in (ε, x).

We call K the ***algorithm***.

4.1.26 Theorem. *Let X be a C^r vector field, $r \geq 1$. Assume that the algorithm $K_\varepsilon(x)$ is* ***consistent with*** *X in the sense that*

$$X(x) = \frac{\partial}{\partial \varepsilon} K_\varepsilon(x)\bigg|_{\varepsilon=0}$$

Then, if (t,x) is in the domain of $F_t(x)$, $(K_{t/n})^n(x)$ is defined for n sufficiently large and converges to $F_t(x)$ as $n \to \infty$. Conversely, if $(K_{t/n})^n(x)$ is defined and converges for $0 \leq t \leq T$, then (T,x) is in the domain of F and the limit is $F_t(x)$.

In the following proof the notation $O(x)$, $x \in \mathbb{R}$ is used for any continuous function in a neighborhood of the origin such that $O(x)/x$ is bounded. Recall from §2.1 that $o(x)$ denotes a continuous function in a neighborhood of the origin satisfying $\lim_{x\to 0} o(x)/x = 0$.

Proof. First, we prove that convergence holds locally. We begin by showing that for any x_0, the iterates $(K_{t/n})^n(x_0)$ are defined if t is sufficiently small. Indeed, on a neighborhood of x_0, $K_\varepsilon(x) = x + O(\varepsilon)$, so if $(K_{t/j})^j(x)$ is defined for x in a neighborhood of x_0, for $j = 1, \ldots, n-1$, then

$$\begin{aligned}(K_{t/n})^n(x) - x &= ((K_{t/n})^n x - (K_{t/n})^{n-1} x) + ((K_{t/n})^{n-1} - (K_{t/n})^{n-2} x)\\ &\quad + \cdots + (K_{t/n}(x) - x)\\ &= O(t/n) + \cdots + O(t/n) = O(t).\end{aligned}$$

This is small, independent of n for t sufficiently small; so, inductively, $(K_{t/n})^n(x)$ is defined and remains in a neighborhood of x_0 for x near x_0.

Let β be a local Lipschitz constant for X so that $\|F_t(x) - F_t(y)\| \le e^{\beta|t|}\|x - y\|$. Now write

$$\begin{aligned} F_t(x) - (K_{t/n})^n(x) &= (F_{t/n})^n(x) - (K_{t/n})^n(x) \\ &= (F_{t/n})^{n-1}F_{t/n}(x) - (F_{t/n})^{n-1}K_{t/n}(x) \\ &\quad + (F_{t/n})^{n-2}F_{t/n}(y_1) - (F_{t/n})^{n-2}K_{t/n}(y_1) \\ &\quad + \cdots + (F_{t/n})^{n-k}F_{t/n}(y_{k-1}) - (F_{t/n})^{n-k}K_{t/n}(Y_{k-1}) \\ &\quad + \cdots + F_{t/n}(y_{n-1}) - K_{t/n}(y_{n-1}) \end{aligned}$$

where $y_k = (K_{t/n})^k(x)$. Thus

$$\begin{aligned} \|F_t(x) - (K_{t/n})^n(x)\| &\le \sum_{k=1}^{n} e^{\beta|t|(n-k)/n}\|F_{t/n}(t_{k-1}) - K_{t/n}(y_{k-1})\| \\ &\le ne^{\beta|t|}o(t/n) \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

since $F_\varepsilon(y) - K_\varepsilon(y) = o(\varepsilon)$ by the consistency hypothesis.

Now suppose $F_t(x)$ is defined for $0 \le t \le T$. We shall show $(K_{t/n})^n(x)$ converges to $F_t(x)$. By the foregoing proof and compactness, if N is large enough, $F_{t/N} = \lim_{n\to\infty}(K_{t/nN})^n$ uniformly on a neighborhood of the curve $t \mapsto F_t(x)$. Thus, for $0 \le t \le T$,

$$F_t(x) = (F_{t/N})^N(x) = \lim_{n\to\infty}(K_{t/nN})^N(x).$$

By uniformity in t,

$$F_T(x) = \lim_{j\to\infty}(K_{T/j})^j(x).$$

Conversely, let $(K_{t/n})^n(x)$ converge to a curve $c(t)$, $0 \le t \le T$. Let $S = \{\, t \mid F_t(x)$ is defined and $c(t) = F_t(x)\,\}$. From the local result, S is a nonempty open set. Let $t(k) \in S$, $t(k) \to t$. Thus $F_{t(k)}(x)$ converges to $c(t)$, so by local existence theory, $F_t(x)$ is defined, and by continuity, $F_t(x) = c(t)$. Hence $S = [0, T]$ and the proof is complete. ∎

4.1.27 Corollary. *Let $X, Y \in \mathfrak{X}(M)$ with flows F_t and G_t. Let S_t be the flow of $X + Y$. Then for $x \in M$,*

$$S_t(x) = \lim_{n\to\infty}(F_{t/n} \circ G_{t/n})^n(x).$$

The left-hand side is defined iff the right-hand side is. This follows from Theorem 4.1.26 by setting $K_\varepsilon(x) = (F_\varepsilon \circ G_\varepsilon)(x)$. For example, for $n \times n$ matrices A and B, Corollary 4.1.27 yields the classical formula

$$e^{(A+B)} = \lim_{n\to\infty}(e^{A/n}e^{B/n})^n.$$

To see this, define for any $n \times n$ matrix C a vector field $X_C \in \mathfrak{X}(\mathbb{R}^n)$ by $X_C(x) = Cx$. Since X_C is linear in C and has flow $F_t(x) = e^{tC}x$, the formula follows from Corollary 4.1.27 by letting $t = 1$.

The topic of this supplement will continue in Supplement 4.2A. The foregoing proofs were inspired by Nelson [1969] and Chorin, Hughes, McCracken, and Marsden [1978].

Supplement 4.1B

Invariant Sets

If X is a smooth vector field on a manifold M and $N \subset M$ is a submanifold, the flow of X will leave N invariant (as a set) iff X is tangent to N. If N is not a submanifold (e.g., N is an open subset together with a non-smooth boundary) the situation is not so simple; however, for this there is a nice criterion going back to Nagumo [1942]. Our proof follows Brezis [1970].

4.1.28 Theorem. *Let X be a locally Lipschitz vector field on an open set $U \subset \mathbf{E}$, where $\mathbf{E}$ is a Banach space. Let $G \subset U$ be relatively closed and set $d(x, G) = \inf\{\, \|x - y\| \mid y \in G\,\}$. The following are equivalent:*

(i) $\lim_{h \downarrow 0}(d(x + hX(x), G)/h) = 0$ *locally uniformly in $x \in G$ (or pointwise if $\mathbf{E} = \mathbb{R}^n$);*

(ii) *if $x(t)$ is the integral curve of X starting in G, then $x(t) \in G$ for all $t \geq 0$ in the domain of $x(\cdot)$.*

Note that $x(t)$ need not lie in G for $t \leq 0$; so G is only + invariant. (We remark that if X is only continuous the theorem fails.) We give the proof assuming $\mathbf{E} = \mathbb{R}^n$ for simplicity.

Proof. Assume (ii) holds. Setting $x(t) = F_t(x)$, where F_t is the flow of X and $x \in G$, for small h we get

$$d(x + hX(x), G) \leq \|x(h) - x - hX(x)\| = |h| \left\| \frac{x(h) - x}{h} - X(x) \right\|,$$

from which (i) follows.

Now assume (i). It suffices to show $x(t) \in G$ for small t. Near $x = x(0) \in G$, say on a ball of radius r, we have

$$\|X(x_1) - X(x_2)\| \leq K\|x_1 - x_2\|$$

and

$$\|F_t(x_1) - F_t(x_2)\| \leq e^{Kt}\|x_1 - x_2\|.$$

We can assume $\|F_t(x) - x\| < r/2$. Set $\varphi(t) = d(F_t(x), G)$ and note that $\varphi(0) = 0$, so that for small t, $\varphi(t) < r/2$. Since G is relatively closed, and $\mathbf{E} = \mathbb{R}^n$, $d(F_t(x), G) = \|F_t(x) - y_t\|$ for some $y_t \in G$. (In the general Banach space case an approximation argument is needed here.) Thus, $\|y_t - x\| < r$. For small h, $\|F_h(y_t) - x\| < r$, so that

$$\begin{aligned}
\varphi(t+h) &= \inf_{z \in G} \|F_{t+h}(x) - z\| \\
&\leq \inf_{z \in G}\{\|F_{t+h}(x) - F_h(y_t)\| + \|F_h(y_t) - y_t - hX(y_t)\| \\
&\qquad\qquad + \|y_t + hX(y_t) - z\|\} \\
&= \|F_{t+h}(x) - F_h(y_t)\| + \|F_h(y_t) - y_t - hX(y_t)\| \\
&\quad + d(y_t + hX(y_t), G) \\
&\leq e^{Kh}\|y_t - F_t(x)\| + \|F_h(y_t) - y_t - hX(y_t)\| \\
&\quad + d(y_t + hX(y_t), G)
\end{aligned}$$

or

$$\begin{aligned}
\frac{\varphi(t+h) - \varphi(t)}{h} \leq \left(\frac{e^{Kh} - 1}{h}\right)\varphi(t) + \left\|\frac{F_h(y_t) - y_t}{h} - X(y_t)\right\| \\
+ \frac{1}{h} d(y_t + hX(y_t), G).
\end{aligned}$$

Hence

$$\limsup_{h\downarrow 0} \frac{\varphi(t+h)-\varphi(t)}{h} \le K\varphi(t).$$

As in Gronwall's inequality, we may conclude that

$$\varphi(t) \le e^{Kt}\varphi(0),$$

so $\varphi(t) = 0$. ■

4.1.29 Example. Let X be a C^∞ vector field on $\mathbb{R}^n$, let $g : \mathbb{R}^n \to \mathbb{R}$ be smooth, and let $\lambda \in \mathbb{R}$ be a regular value for g, so $g^{-1}(\lambda)$ is a submanifold; see Figure 4.1.5.

Let $G = g^{-1}(]-\infty, \lambda])$ and suppose that on $g^{-1}(\lambda)$,

$$\langle X, \operatorname{grad} g\rangle \le 0.$$

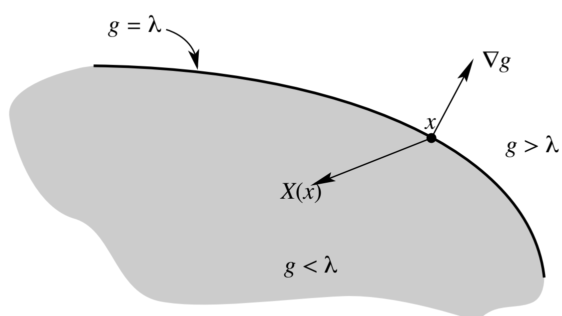


FIGURE 4.1.5. The set $G = g^{-1}(]-\infty, \lambda[)$ is invariant if X does not point strictly outwards at ∂G

Then G is + invariant under F_t as may be seen by using Theorem 4.1.28. This result has been generalized to the case where ∂G might not be smooth by Bony [1969]. See also Redheffer [1972] and Martin [1973]. Related references are Yorke [1967], Hartman [1972], and Crandall [1972]. ♦

SUPPLEMENT 4.1C
A second Proof of the Existence and Uniqueness of Flow Boxes

We now give an alternative "modern" proof of Theorem 4.1.5 and Proposition 4.1.13, namely, if $X \in \mathfrak{X}^k(M)$, $k \ge 1$, then for each $m \in M$ there exists a unique C^k flow box at m. The basic idea is due to Robbin [1968] although similar alternative proofs were simultaneously discovered by Abraham and Pugh [unpublished] and Marsden [1968b, p. 368]. The present exposition follows Robbin [1968] and Ebin and Marsden [1970].

Step 1 *Existence and uniqueness of integral curves for C^1 vector fields.*

Proof. Working in a local chart, we may assume that $X : D_r(0) \to \mathbf{E}$, where $D_r(0)$ is the open disk at the origin of radius r in the Banach space $\mathbf{E}$. Let $U = D_{r/2}(0)$, $I = [-1, 1]$ and define

$$\Phi : \mathbb{R} \times C^1_0(I, U) \to C^0(I, \mathbf{E})$$

by

$$\Phi(s, \gamma)(t) = \frac{d\gamma}{dt}(t) - sX(\gamma(t)),$$

where $C^i(I, \mathbf{E})$ is the Banach space of C^i-maps of I into $\mathbf{E}$, endowed with the $\| \cdot \|_i$-norm (see Supplement 2.4B),

$$C^i_0(I, \mathbf{E}) = \{ f \in C^i(I, \mathbf{E}) \mid f(0) = 0 \}$$

is a closed subspace of $C^i(I, \mathbf{E})$ and

$$C^i_0(I, U) = \{ f \in C^i_0(I, \mathbf{E}) \mid f(I) \subset U \}$$

is open in $C^i_0(I, \mathbf{E})$. We first show that Φ is a C^1-map.

The map $d/dt : C^1_0(I, \mathbf{E}) \to C^0(I, \mathbf{E})$ is clearly linear and is continuous since $\|d/dt\| \leq 1$. Moreover, if $d\gamma/dt = 0$ on I, then γ is constant and since $\gamma(0) = 0$, it follows $\gamma = 0$; that is, d/dt is injective. Given $\delta \in C^0(I, \mathbf{E})$,

$$\gamma(t) = \int_0^t \delta(s)\, ds$$

defines an element of $C^i_0(I, \mathbf{E})$ with $d\gamma/dt = \delta$, that is, d/dt is a Banach space isomorphism from $C^1_0(I, \mathbf{E})$ to $C^0(I, \mathbf{E})$.

From these remarks and the Ω lemma 2.4.18, it follows that Φ is a C^1-map. Moreover, $\mathbf{D}_\gamma \Phi(0, 0) = d/dt$ is an isommorphism of $C^1_0(I, \mathbf{E})$ with $C^0(I, \mathbf{E})$. Since $\Phi(0, 0) = 0$, by the implicit function theorem there is an $\varepsilon > 0$ such that $\Phi(\varepsilon, \gamma) = 0$ has a unique solution $\gamma_\varepsilon(t)$ in $C^1_0(I, U)$. The unique integral curve sought is $\gamma(t) = \gamma_\varepsilon(t/\varepsilon)$, $-\varepsilon \leq t \leq \varepsilon$. ■

The same argument also works in the time-dependent case. It also shows that γ varies continuously with X.

Step 2. *The local flow of a C^k vector field X is C^k.*

Proof. First, suppose $k = 1$. Modify the definition of Φ in Step 1 by setting $\Psi : \mathbb{R} \times U \times C^1_0(I, U) \to C^0(I, \mathbf{E})$,

$$\Psi(s, x, \gamma)(t) = \gamma'(t) - sX(x + \gamma(t)).$$

As in Step 1, Ψ is a C^1-map and $\mathbf{D}_\gamma \Psi(0, 0, 0)$ is an isomorphism, so $\Psi(\varepsilon, x, \gamma) = 0$ can be locally solved for γ giving a map

$$H_\varepsilon : U \to C^1_0(I, U), \quad \varepsilon > 0.$$

The local flow is $F(x, t) = x + H_\varepsilon(x)(t/\varepsilon)$, as in Step 1. By Proposition 2.4.17 (differentiability of the evaluation map), F is C^1. Moreover, if $v \in \mathbf{E}$, we have $\mathbf{D}F_t(x) \cdot v = v + (\mathbf{D}H_\varepsilon(x) \cdot v)(t/\varepsilon)$, so that the mixed partial derivative

$$\frac{d}{dt} \mathbf{D}F_t(x) \cdot v = \frac{1}{\varepsilon} (\mathbf{D}H_\varepsilon(x) \cdot v) \left(\frac{t}{\varepsilon} \right)$$

exists and is jointly continuous in (t, x). By Exercise 2.4-7, $\mathbf{D}(dF_t(x)/dt)$ exists and equals $(d/dt)(\mathbf{D}F_t(x))$.

Next we prove the result for $k \geq 2$. Consider the Banach space $F = C^{k-1}(\mathrm{cl}(U), \mathbf{E})$ and the map $\omega_X : F \to F$; $\eta \mapsto X \circ \eta$. This map is C^1 by the Ω lemma (remarks following Lemma 2.4.18). Regarding ωX as a vector field on F, it has a unique C^1 integral curve η_t with $\eta_0 =$ identity, by Step 1. This integral curve is the local flow of X and is C^{k-1} since it lies in F. Since $k \geq 2$, η_t is at least C^1 and so one sees that $\mathbf{D}\eta_t = u_t$ satisfies $du_t/dt = \mathbf{D}X(\eta_t) \cdot u_t$, so by Step 1 again, u_t lies in C^{k-1}. Hence η_t is C^k.

The following is a useful alternative argument for proving the result for $k = 1$ from that for $k \geq 2$. For $k = 1$, let $X^n \to X$ in C^1, where X^n are C^2. By the above, the flows of X^n are C^2 and by Step 1, converge uniformly that is, in C^0, to the flow of X. From the equations for $\mathbf{D}\eta t_n$, we likewise see that $\mathbf{D}\eta_t n$ converges uniformly to the solution of $du_t/dt = \mathbf{D}X(\eta_t) \cdot u_t$, $u_0 =$ identity. It follows by elementary analysis (see Exercise 2.4-10 or Marsden and Hoffman [1993, p. 109]) that η_t is C^1 and $\mathbf{D}\eta_t = u_t$. ■

This proof works with minor modifications on manifolds with vector fields and flows of Sobolev class H^s or Holder class $C^{k+\alpha}$; see Ebin and Marsden [1970] and Bourguignon and Brezis [1974]. In fact the foregoing proof works in any function spaces for which the Ω lemma can be proved. Abstract axioms guaranteeing this are given in Palais [1968].

Exercises

◇ **4.1-1.** Find an explicit formula for the flow $F_t : \mathbb{R}^2 \to \mathbb{R}^2$ of the harmonic oscillator equation $\ddot{x} + \omega^2 x = 0$, $\omega \in \mathbb{R}$ a constant.

◇ **4.1-2.** Show that if (U_0, a, F) is a flow box for X, then (U_0, a, F_-) is a flow box for $-X$, where $F_-(u, t) = F(u, -t)$ and $(-X)(m) = -(X(m))$.

◇ **4.1-3.** Show that the integral curves of a C^r vector field X on an n-manifold can be defined locally in the neighborhood of a point where X is nonzero by n equations $\psi_i(m, t) = c_i =$ constant, $i = 1, \ldots, n$ in the $n + 1$ unknowns (m, t). Such a system of equations is called a ***local complete system of integrals***.
Hint: Use the straightening-out theorem.

◇ **4.1-4.** Prove the following generalization of Gronwall's inequality. *Supppose* $v(t) \geq 0$ *satisfies*

$$v(t) \leq C + \left| \int_0^t p(s)\, v(s)\, ds \right|,$$

where $C \geq 0$ *and* $p \in L^1$. *Then*

$$v(t) \leq C \exp\left(\int_0^t |p(s)|\, ds \right).$$

Use this to generalize Example 4.1.23C to allow A to be a time-dependent matrix.

◇ **4.1-5.** Let $F_t = e^{tX}$ be the flow of a linear vector field X on $\mathbf{E}$. Show that the solution of the equation

$$\dot{x} = X(x) + f(x)$$

with initial conditions x_0 satisfies the ***variation of constants formula***

$$x(t) = e^{tx} x_0 + \int_0^t e^{(t-s)X} f(x(s))\, ds$$

◇ **4.1-6.** Let $F(m, t)$ be a C^∞ mapping of $M \times \mathbb{R}$ to M such that $F_{t+s} = F_t \circ F_s$ and $F_0 =$ identity (where $F_t(m) = F(m, t)$). Show that there is a unique C^∞ vector field X whose flow is F.

◇ **4.1-7.** Let $\sigma(t)$ be an integral curve of a vector field X and let $g : M \to \mathbb{R}$. Let $\tau(t)$ satisfy $\tau'(t) = g(\sigma(\tau(t)))$. Then show $t \mapsto \sigma(\tau(t))$ is an integral curve of gX. Show by example that even if X is complete, gX need not be.

◇ **4.1-8.**

(i) (**Gradient Flows.**) Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^1 and let $X = (\partial f/\partial x^1, \ldots, \partial f/\partial x^n)$ be the gradient of f. Let F be the flow of X. Show that $f(F_t(x) \geq f(F_s(x))$ if $t \geq s$.

(ii) Use (i) to find a vector field X on $\mathbb{R}^n$ such that $X(0) = 0$, $X'(0) = 0$, yet 0 is globally attracting; that is, every integral curve converges to 0 as $t \to \infty$. This exercise continues in Exercise 4.3-11.

◇ **4.1-9.** Let c be a locally Lipschitz increasing function, $c(t) > 0$ for $t \geq 0$ and assume that the differential equation $r'(t) = c(r(t))$ has the solution with $r(0) = r_0 \geq 0$ existing for time $t \in [0, T]$. Conclude that $r(t) \geq 0$ for $t \in [0, T]$. Prove the following ***comparison lemmas***.

(i) If $h(t)$ is a continuous function on $[0, T]$, $h(t) \geq 0$, satisfying $h'(t) \leq c(h(t))$ on $[0, T]$, $h(0) = r_0$, then show that $h(t) \leq r(t)$.

Hint: Prove that

$$\int_{r_0}^{h(t)} \frac{dx}{c(x)} \leq t = \int_{r_0}^{r(t)} \frac{dx}{c(x)}$$

and use strict positivity of the integrand.

(ii) Generalize (i) to the case $h(0) \leq r_0$.

Hint: The function $h(t) = h(t) + r_0 - h(0) \leq h(t)$ satisfies the hypotheses in (i).

(iii) If $f(t)$ is a continuous function on $[0, T]$, $f(t) \geq 0$, satisfying

$$f(t) \leq r_0 + \int_0^1 c(f(s))\, ds$$

on $[0, T]$, then show that $f(t) \leq r(t)$.

Hint:

$$h(t) = r_0 + \int_0^1 c(f(s))\, ds \geq f(t)$$

satisfies the hypothesis in (i) since $h'(t) = c(f(t)) \leq c(h(t))$.

(iv) If in addition

$$\int_0^\infty \frac{dx}{c(x)} = +\infty$$

show that the solution $f(t) \geq 0$ exists for all $t \geq 0$.

(v) If $h(t)$ is only continuous on $[0, T]$ and

$$h(t) \leq r_0 + \int_0^t c(h(s))\, ds,$$

show that $h(t) \leq r(t)$ on $[0, T]$.

Hint: Approximate $h(t)$ by a C^1-function $g(t)$ and show that $g(t)$ satisfies the same inequality as $h(t)$.

◇ **4.1-10.**

(i) Let $X = y^2\partial/\partial x$ and $Y = x^2\partial/\partial y$. Show that X and Y are complete on $\mathbb{R}^2$ but $X + Y$ is not.
Hint: Note that $x^3 - y^3 =$ constant and consider an integral curve with $x(0) = y(0)$.

(ii) Prove the following theorem:
Let H be a Hilbert space and X and Y be locally Lipschitz vector fields that satisfy the following:

(a) *X and Y are bounded and Lipschitz on bounded sets;*

(b) *there is a constant $\beta \geq 0$ such that*

$$\langle Y(x), x\rangle \leq \beta\|x\|^2 \quad \textit{for all } x \in H;$$

(c) *there is a locally Lipschitz monotone increasing function $c(t) > 0$, $t \geq 0$, such that*

$$\int_0^\infty \frac{dx}{c(x)} = +\infty$$

and if $x(t)$ is an integral curve of X,

$$\frac{d}{dt}\|x(t)\| \leq c(\|x(t)\|).$$

Then X, Y and $X + Y$ are positively complete.

Note: One may assume $\|X(x_0)\| \leq c(\|x_0\|)$ in (c) instead of $(d/dt)\|x(t)\| \leq c(\|x(t)\|)$.
Hint: Find a differential inequality for $(1/2)(d/dt)\|u(t)\|^2$, where $u(t)$ is an integral curve of $X + Y$ and then use Exercise 4.1-9iii.

◇ **4.1-11.** Prove the following result on the ***convergence of flows***:
Let X_α be locally Lipschitz vector fields on M for α in some topological space. Suppose the Lipschitz constants of X_α are locally bounded as $\alpha \to \alpha(0)$ and $X_\alpha \to X_{\alpha(0)}$ locally uniformly. Let $c(t)$ be an integral curve of $X_{\alpha(0)}$, $0 \leq t \leq T$ and $\varepsilon > 0$. Then the integral curves $c_\alpha(t)$ of X_α with $c_{\alpha(0)} = c(0)$ are defined for the interval $t \in [0, T - \varepsilon]$ for α sufficiently close to $\alpha(0)$ and $c_\alpha(t) \to c(t)$ uniformly in $t \in [0, T - \varepsilon]$ as $\alpha \to \alpha(0)$. If the flows are complete $F_t^\alpha \to F_t$ locally uniformly. (The vector fields may be time dependent if the estimates are locally t-uniform.)
Hint: Show that

$$\|c_\alpha(t) - c(t)\| \leq k\int_0^t \|c_\alpha(\tau) - c(\tau)\|\, d\tau + \int_0^t \|X_\alpha(c(\tau)) - X_{\alpha(0)}(c(\tau))\|\, d\tau$$

and conclude from Gronwall's inequality that $c_\alpha(t) \to c(t)$ for $\alpha \to \alpha(0)$ since the second term $\to 0$. This estimate shows that $c_\alpha(t)$ exists as long as $c(t)$ does on any compact subinterval of $[0, T[$.

◇ **4.1-12.** Prove that the C^r flow of a C^{r+1} vector field is a C^1 function of the vector field by utilizing Supplement 4.1C. (*Caution*. It is known that the C^k flow of a C^k vector field cannot be a C^1 function of the vector field; see Ebin and Marsden [1970] for the explanation and further references).

◇ **4.1-13** (Nonunique integral curves on non-Hausdorff manifolds). Let M be the line with two origins (see Exercise 3.5-8) and consider the vector field $X : M \to TM$ which is defined by $X([x, i]) = x$, $i = 1, 2$; here $[x, i]$ denotes a point of the quotient manifold M. Show that through every point other than $[0, 0]$ and $[0, 1]$, there are exactly two integral curves of X. Show that X is complete.
Hint: The two distinct integral curves pass respectively through $[0, 0]$ and $[0, 1]$.

◇ **4.1-14.** Give another proof of Theorem 4.1.5 using Exercise 2.5-10.

◇ **4.1-15.** Give examples of vector fields satisfying the following conditions:

(i) on $\mathbb{R}$ and S^1 with no critical points; generalize to $\mathbb{R}^n$ and $\mathbb{T}^n$;

(ii) on S^1 with exactly k critical points; generalize to $\mathbb{T}^n$;

(iii) on $\mathbb{R}^2$ and $\mathbb{RP}^2$ with exactly one critical point and all other orbits closed;

(iv) on the Möbius band with no critical points and such that the only integral curve intersecting the zero section is the zero section itself;

(v) on S^2 with precisely two critical points and one closed orbit;

(vi) on S^2 with precisely one critical point and no closed orbit;

(vii) on S^2 with no critical points on a great circle and nowhere tangent to it; show that any such vector field has its integral curves intersecting this great circle at most once;

(viii) on $\mathbb{T}^2$ with no critical points, all orbits closed and winding exactly k times around $\mathbb{T}^2$.

◇ **4.1-16.** Let $\pi : M \to N$ be a surjective submersion. A vector field $X \in \mathfrak{X}(M)$ is called ***π-vertical*** if $T\pi \circ X = 0$. If $\mathbb{K}$ is the Klein bottle, show that $\pi : \mathbb{K} \to S^1$ given by $\pi([a, b]) = e^{2\pi i a}$ is a surjective submersion; see Figure 4.1.6. Prove that $\mathbb{K}$ is a non trivial S^1-bundle.

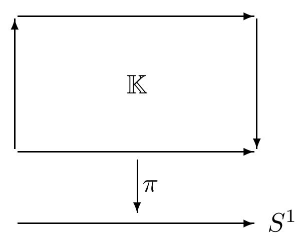


FIGURE 4.1.6. The Klein bottle as an S^1-bundle

HINT: If it were trivial, there would exist a nowhere zero vertical vector field on $\mathbb{K}$. In Figure 4.1.6, this means that arrows go up on the left and down on the right hand side. Follow a path from left to right and argue by the intermediate value theorem that the vector field must vanish somewhere.

4.2 Vector Fields as Differential Operators

In the previous section vector fields were studied from the point of view of dynamics; that is in terms of the flows they generate. Before continuing the development of dynamics, we shall treat some of the algebraic aspects of vector fields. The specific goal of the section is the development of the Lie derivative of functions and vector fields and its relationship with flows. One important feature is the behavior of the constructions under mappings. The operations should be as natural or covariant as possible when subjected to a mapping.

We begin with a discussion of the action of mappings on functions and vector fields. First, recall some notation. Let $C^r(M, \mathbf{F})$ denote the space of C^r maps $f : M \to \mathbf{F}$, where $\mathbf{F}$ is a Banach space, and let $\mathfrak{X}^r(M)$ denote the space of C^r vector fields on M. Both are vector spaces with the obvious operations of addition and scalar multiplication. For brevity we write

$$\mathcal{F}(M) = C^\infty(M, \mathbb{R}), \quad \mathcal{F}^r(M) = C^r(M, \mathbb{R}) \quad \text{and} \quad \mathfrak{X}(M) = \mathfrak{X}^\infty(M).$$

Note that $\mathcal{F}^r(M)$ has an *algebra structure*; that is, for $f, g \in \mathcal{F}^r(M)$ the product fg defined by $(fg)(m) = f(m)g(m)$ obeys the usual algebraic properties of a product such as $fg = gf$ and $f(g+h) = fg + fh$.

4.2.1 Definition.

(i) *Let $\varphi : M \to N$ be a C^r mapping of manifolds and $f \in \mathcal{F}^r(N)$. Define the* ***pull-back*** *of f by φ by*

$$\varphi^* f = f \circ \varphi \in \mathcal{F}^r(M).$$

(ii) *If f is a C^r diffeomorphism and $X \in \mathfrak{X}^r(M)$, the* ***push-forward*** *of X by φ is defined by*

$$\varphi_* X = T\varphi \circ X \circ \varphi^{-1} \in \mathfrak{X}^r(N).$$

Consider local charts (U, χ), $\chi : U \to U' \subset \mathbf{E}$ on M and (V, ψ), $\psi : V \to V' \subset \mathbf{F}$ on N, and let $(T\chi \circ X \circ \chi^{-1})(u) = (u, X(u))$, where $X : U' \to \mathbf{E}$ is the local representative of X. Then from the chain rule and the definition of push–forward, the local representative of $\varphi_* X$ is

$$(T\psi \circ (\varphi_* X) \circ \psi^{-1})(v) = (v, \mathbf{D}(\psi \circ \varphi \circ \chi^{-1})(u) \cdot X(u)),$$

where $v = (\psi \circ \varphi \circ \chi^{-1})(u)$. The different point of evaluation on each side of the equation corresponds to the necessity of having φ^{-1} in the definition. If M and N are finite dimensional, x^i are local coordinates on M and y^j local coordinates on N, the preceding formula gives the components of $\varphi_* X$ by

$$(\varphi_* X)^j(y) = \frac{\partial \varphi^j}{\partial x^i}(x)\, X^i(x)$$

where $y = \varphi(x)$.

We can interchange "pull-back" and "push-forward" by changing φ to φ^{-1}, that is, defining φ_* (resp. φ^*) by $\varphi_* = (\varphi^{-1})^*$ (resp. $\varphi^* = (\varphi^{-1})_*$). Thus the ***push-forward*** of a function f on M is $\varphi_* f = f \circ \varphi^{-1}$ and the ***pull-back*** of a vector field Y on N is $\varphi^* Y = (T\varphi)^{-1} \circ Y \circ \varphi$ (Figure 4.2.1). Notice that φ must be a diffeomorphism in order that the pull-back and push-forward operations make sense, the only exception being pull-back of functions. Thus vector fields can only be pulled back and pushed forward by diffeomorphisms. However, even when φ is not a diffeomorphism we can talk about φ-related vector fields as follows.

4.2.2 Definition. *Let $\varphi : M \to N$ be a C^r mapping of manifolds. The vector fields $X \in \mathfrak{X}^{r-1}(M)$ and $Y \in \mathfrak{X}^{r-1}(N)$ are called φ-**related**, denoted $X \sim_\varphi Y$, if $T\varphi \circ X = Y \circ \varphi$.*

Note that if φ is diffeomorphism and X and Y are φ-related, then $Y = \varphi_* X$. In general however, X can be φ-related to more than one vector field on N. φ-relatedness means that the following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{\quad T\varphi \quad} & TN \\ {\scriptstyle X}\uparrow & & \uparrow{\scriptstyle Y} \\ M & \xrightarrow[\quad \varphi \quad]{} & N \end{array}$$

4.2.3 Proposition.

(i) *Pull-back and push-forward are linear maps, and*

$$\varphi^*(fg) = (\varphi^* f)(\varphi^* g), \quad \varphi_*(fg) = (\varphi_* f)(\varphi_* g).$$

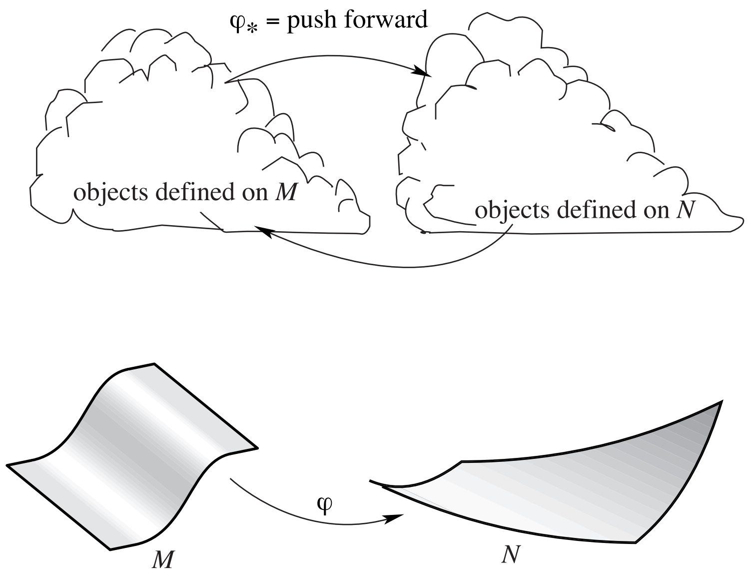


FIGURE 4.2.1. Push–forward and pull–back

(ii) *If* $X_i \sim_\varphi Y_i$, $i = 1, 2$, *and* $a, b \in \mathbb{R}$, *then* $aX_1 + bX_2 \sim_\varphi aY_1 + bY_2$.

(iii) *For* $\varphi : M \to N$ *and* $\psi : N \to P$, *we have*

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^* \quad \textit{and} \quad (\psi \circ \varphi)_* = \psi_* \circ \varphi_*.$$

(iv) *If* $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$, $Z \in \mathfrak{X}(P)$, $X \sim_\varphi Y$, *and* $Y \sim_\psi Z$, *then* $X \sim_{\psi \circ \varphi} Z$.

In this proposition it is understood that all maps are diffeomorphims with the exception of the pull-back of functions and the relatedness of vector fields.

Proof. (i) This consists of straightforward verifications. For example, if $X_i \sim_\varphi Y_i$, $i = 1, 2$, then $T\varphi \circ (aX_1 + bX_2) = aT\varphi \circ X_1 + bT\varphi \circ X_2 = aY_1 \circ \varphi + bY_2 \circ \varphi$, that is, $aX_1 + bX_2 \sim_\varphi aY_1 + bY_2$.

(ii) These relations on functions are simple consequences of the definition, and the ones on $\mathfrak{X}(P)$ and $\mathfrak{X}(M)$ are proved in the following way using the chain rule:

$$T(\psi \circ \varphi) \circ X = T\psi \circ T\varphi \circ X = T\psi \circ Y \circ \varphi = Z \circ \psi \circ \varphi.$$

■

In this development we can replace $\mathcal{F}(M)$ by $C^r(M, \mathbf{F})$ with little change; that is, we can replace real-valued functions by **F**-valued functions.

The behavior of flows under these operations is as follows:

4.2.4 Proposition. *Let* $\varphi : M \to N$ *be a* C^r*-mapping of manifolds,* $X \in \mathfrak{X}^r(M)$ *and* $Y \in \mathfrak{X}^r(N)$. *Let* F_t^X *and* F_t^Y *denote the flows of* X *and* Y *respectively. Then* $X \sim_\varphi Y$ *iff* $\varphi \circ F_t^X = F_t^Y \circ \varphi$. *In particular, if* φ *is a diffeomorphism, then the equality* $Y = \varphi_* X$ *holds iff the flow of* Y *is* $\varphi \circ F_t^X \circ \varphi^{-1}$. *In particular,* $(F_s^X)_* X = X$.

Proof. Taking the time derivative of the relation $(\varphi \circ F_t^X)(m) = (F_t^Y \circ \varphi)(m)$, for $m \in M$, using the chain rule and definition of the flow, we get

$$T\varphi\left(\frac{\partial F_t^X(m)}{\partial t}\right) = \frac{\partial F_y^Y}{\partial t}(\varphi(m)),$$

that is,

$$(T\varphi \circ X \circ F_t^X)(m) = (Y \circ F_t^Y \circ \varphi)(m) = (Y \circ \varphi \circ F_t^X)(m),$$

which is equivalent to $T\varphi \circ X = Y \circ \varphi$. Conversely, if this relation is satisfied, let $c(t) = F_t^X(m)$ denote the integral curve of X through $m \in M$. Then

$$\frac{d(\varphi \circ c)(t)}{dt} = T\varphi\left(\frac{dc(t)}{dt}\right) = T\varphi(X(c(t))) = Y((\varphi \circ c)(t))$$

says that $\varphi \circ c$ is the integral curve of Y through $\varphi(c(0)) = \varphi(m)$. By uniqueness of integral curves, we get $(\varphi \circ F_t^X)(m) = (\varphi \circ c)(t) = F_t^Y(\varphi(m))$. The last statement is obtained by taking $\varphi = F_s^X$ for fixed s. ■

We call $\varphi \circ F_t \circ \varphi^{-1}$ the ***push-forward*** of F_t by φ since it is the natural way to construct a diffeomorphism on N out of one on M. See Figure 4.2.2. Thus, Proposition 4.2.4 says that *the flow of the push-forward of a vector field is the push-forward of its flow*.

Next we define how vector fields operate on functions. This is done by means of the directional derivative. Let $f : M \to \mathbb{R}$, so $Tf : TM \to T\,\mathbb{R} = \mathbb{R} \times \mathbb{R}$. Recall that a tangent vector to $\mathbb{R}$ at a base point $\lambda \in \mathbb{R}$ is a pair (λ, μ), the number μ being the principal part. Thus we can write Tf acting on a vector $v \in T_mM$ in the form

$$Tf \cdot v = (f(m), \mathbf{d}f(m) \cdot v).$$

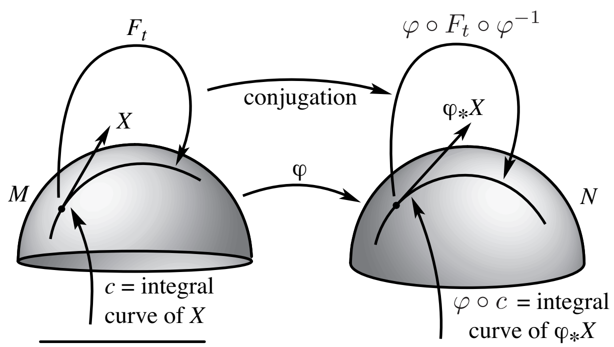


FIGURE 4.2.2. Pushing forward vector fields and integral curves

This defines, for each $m \in M$, the element $\mathbf{d}f(m) \in T_m^*M$. Thus $\mathbf{d}f$ is a section of T^*M, ***a covector field***, or ***one-form***.

4.2.5 Definition. *The covector field $\mathbf{d}f : M \to T^*M$ defined this way is called the* ***differential*** *of f.*

For $\mathbf{F}$-valued functions, $f : M \to \mathbf{F}$, where $\mathbf{F}$ is a Banach space, a similar definition gives $\mathbf{d}f(m) \in L(T_mM, \mathbf{F})$ and we speak of $\mathbf{d}f$ as an **F*-valued one-form***.

Clearly if f is C^r, then $\mathbf{d}f$ is C^{r-1}. Let us now work out $\mathbf{d}f$ in local charts for $f \in \mathcal{F}(M)$. If $\varphi : U \subset M \to V \subset \mathbf{E}$ is a local chart for M, then the local representative of f is the map $f : V \to \mathbb{R}$ defined by $f = f \circ \varphi^{-1}$. The local representative of Tf is the tangent map for local manifolds:

$$Tf(x, v) = (f(x), \mathbf{D}f(x) \cdot v).$$

Thus, the local representative of $\mathbf{d}f$ is the derivative of the local representative of f. In particular, if M is finite dimensional and local coordinates are denoted $(x^1, \ldots, x^n)$, then the local components of $\mathbf{d}f$ are

$$(\mathbf{d}f)_i = \frac{\partial f}{\partial x^i}$$

The introduction of $\mathbf{d}f$ leads to the following.

4.2.6 Definition. *Let $f \in \mathcal{F}^r(M)$ and $X \in \mathfrak{X}^{r-1}(M)$, $r \geq 1$. Define the **directional** or **Lie derivative** of f along X by*

$$\pounds_X f(m) \equiv X[f](m) = \mathbf{d}f(m) \cdot X(m),$$

for any $m \in M$. Denote by $X[f] = \mathbf{d}f(X)$ the map $m \in M \mapsto X[f](m) \in \mathbb{R}$. If f is $\mathbf{F}$-valued, the same definition is used, but now $X[f]$ is $\mathbf{F}$-valued.

The local representative of $X[f]$ in a chart is given by the function $x \mapsto \mathbf{D}f(x) \cdot X(x)$, where f and X are the local representatives of f and X. In particular, if M is finite dimensional then we have

$$X[f] \equiv \pounds_X f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} X^i.$$

Evidently if f is C^r and X is C^{r-1} then $X[f]$ is C^{r-1}.

From the chain rule, $\mathbf{d}(f \circ \varphi) = \mathbf{d}f \circ T\varphi$, where $\varphi : N \to M$ is a C^r map of manifolds, $r \geq 1$. For real–valued functions, Leibniz' rule gives

$$\mathbf{d}(fg) = f\mathbf{d}g + g\mathbf{d}f.$$

(If f is $\mathbf{F}$-valued, g is G-valued and $B : \mathbf{F} \times \mathbf{G} \to \mathbf{H}$ is a continuous bilinear map of Banach spaces, this generalizes to $\mathbf{d}(B(f, g)) = B(\mathbf{d}f, g) + B(f, \mathbf{d}g)$.)

4.2.7 Proposition.

(i) *Suppose $\varphi : M \to N$ is a diffeomorphism. Then $\pounds_X$ is natural with respect to **push-forward** by φ. That is, for each $f \in \mathcal{F}(M)$,*

$$\pounds_{\varphi_* X}(\varphi_* f) = \varphi_* \pounds_X f;$$

in other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\ \varphi_*\ } & \mathcal{F}(N) \\ \Big\downarrow{\scriptstyle \pounds_X} & & \Big\downarrow{\scriptstyle \pounds_{\varphi_* X}} \\ \mathcal{F}(M) & \xrightarrow[\ \varphi_*\ ]{} & \mathcal{F}(N) \end{array}$$

(ii) *$\pounds_X$ is **natural with respect to restrictions**. That is, for U open in M and $f \in \mathcal{F}(M)$, $\pounds_{X|U}(f|U) = (\pounds_X f)|U$; or if $|U : \mathcal{F}(M) \to \mathcal{F}(U)$ denotes restriction to U, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\;|U\;} & \mathcal{F}(U) \\ \Big\downarrow \pounds_X & & \Big\downarrow \pounds_{X|U} \\ \mathcal{F}(M) & \xrightarrow[\;|U\;]{} & \mathcal{F}(U) \end{array}$$

Proof. For (i), if $n \in N$ then

$$\begin{aligned} \pounds_{\varphi_* X}(\varphi_* f)(n) &= \mathbf{d}(f \circ \varphi^{-1}) \cdot (\varphi_* X)(n) \\ &= \mathbf{d}(f \circ \varphi^{-1})(n) \cdot (T\varphi \circ X \circ \varphi^{-1})(n) \\ &= \mathbf{d}f(\varphi^{-1}(n)) \cdot (X \circ \varphi^{-1})(n) = \varphi_*(\pounds_X f)(n). \end{aligned}$$

(ii) follows from $\mathbf{d}(f|U) = (\mathbf{d}f)|U$, which itself is clear from the definition of $\mathbf{d}$. ■

This proposition is readily generalized to $\mathbf{F}$-valued C^r functions.

Since $\varphi^* = (\varphi^{-1})_*$, the Lie derivative is also natural with respect to pull-back by φ. This has a generalization to φ-related vector fields as follows.

4.2.8 Proposition. *Let $\varphi : M \to N$ be a C^r map, $X \in \mathfrak{X}^{r-1}(M)$ and $Y \in \mathfrak{X}^{r-1}(N)$. If $X \sim_\varphi Y$, then*

$$\pounds_X(\varphi^* f) = \varphi^* \pounds_Y f$$

for all $f \in C^r(N, \mathbf{F})$; that is, the following diagram commutes:

$$\begin{array}{ccc} C^r(N, \mathbf{F}) & \xrightarrow{\;\varphi^*\;} & C^r(M, \mathbf{F}) \\ \Big\downarrow \mathcal{C}_Y & & \Big\downarrow \mathcal{C}_X \\ C^{r-1}(N, \mathbf{F}) & \xrightarrow[\;\varphi^*\;]{} & C^{r-1}(M, \mathbf{F}) \end{array}$$

Proof. For $m \in M$,

$$\begin{aligned} \pounds_X(\varphi^* f)(m) &= \mathbf{d}(f \circ \varphi)(m) \cdot X(m) = \mathbf{d}f(\varphi(m)) \cdot (T_m\varphi(X(m))) \\ &= \mathbf{d}f(\varphi(m)) \cdot Y(\varphi(m)) = \mathbf{d}f(Y)(\varphi(m)) = (\varphi^* \pounds_Y f)(m). \end{aligned}$$ ■

Next we show that $\pounds_X$ satisfies the Leibniz rule.

4.2.9 Proposition.

(i) *The mapping $\pounds_X : C^r(M, \mathbf{F}) \to C^{r-1}(M, \mathbf{F})$ is a **derivation**. That is $\pounds_X$ is $\mathbb{R}$-linear and for $f \in C^r(M, \mathbf{F})$, $g \in C^r(M, \mathbf{G})$ and $B : \mathbf{F} \times \mathbf{G} \to \mathbf{H}$ a bilinear map*

$$\pounds_X(B(f, g)) = B(\pounds_X f, g) + B(f, \pounds_X g).$$

In particular, for real-valued functions, $\pounds_X(fg) = g\pounds_X f + f\pounds_X g$.

(ii) *If c is a constant function, $\pounds_X c = 0$.*

Proof. Part (i) follows from the Leibniz rule for $\mathbf{d}$ and the definition $\pounds_X f$. Part (ii) results from the definition. ■

The connection between $\pounds_X f$ and the flow of X is as follows.

4.2.10 Theorem (Lie Derivative Formula for Functions). *Suppose $f \in C^r(M, \mathbf{F})$, $X \in \mathfrak{X}^{r-1}(M)$, and X has a flow F_t. Then*

$$\frac{d}{dt} F_t^* f = F_t^* \pounds_X f$$

Proof. By the chain rule, the definition of the differential of a function and the flow of a vector field,

$$\begin{aligned}\frac{d}{dt}(F_t^* f)(m) &= \frac{d}{dt}(f \circ F_t)(m) = \mathbf{d}f(F_t(m)) \cdot \frac{dF_t(m)}{dt} \\ &= \mathbf{d}f(F_t(m)) \cdot X(F_t(m)) = \mathbf{d}f(X)(F_t(m)) \\ &= (\pounds_X f)(F_t(m)) = (F_t^* \pounds_X f)(m).\end{aligned}$$ ■

As an application of the Lie derivative formula, we consider the problem of solving a partial differential equation on $\mathbb{R}^{n+1}$ of the form

$$\frac{\partial f}{\partial t}(x,t) = \sum_{i=1}^{n} X^i(x) \frac{\partial f}{\partial x^i}(x,t) \tag{P}$$

with initial condition $f(x, 0) = g(x)$ for given smooth functions $X^i(x)$, $i = 1, \ldots, n$, $g(x)$ and a scalar unknown $f(x, t)$.

4.2.11 Proposition. *Suppose $X = (X^1, \ldots, X^n)$ has a complete flow F_t. Then $f(x,t) = g(F_t(x))$ is a solution of the foregoing problem (P). (See Exercise 4.2-3 for uniqueness.)*

Proof.

$$\frac{\partial f}{\partial t} = \frac{d}{dt} F_t^* g = F_t^* \pounds_X g = \pounds_X (F_t^* g) = X[f].$$ ■

Thus, one can solve this *scalar* equation by computing the orbits of X and pushing (or "dragging along") the graph of g by the flow of X; see Figure 4.2.3. These trajectories of X are called *characteristics* of (P). (As we shall see below, the vector field X in (P) can be time dependent.)

4.2.12 Example. Solve the partial differential equation

$$\frac{\partial f}{\partial t} = (x+y)\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right),$$

with initial condition $f(x, y, 0) = x^2 + y^2$. ♦

Solution. The vector field $X(x,y) = (x+y, -x-y)$ has a complete flow $F_t(x,y) = ((x+y)t + x, -(x+y)t + y)$, so that the solution of the previous partial differential equation is given by

$$f(x,y,t) = 2(x+y)^2 t^2 + x^2 + y^2 + 2(x^2 - y^2)t.$$ ♦

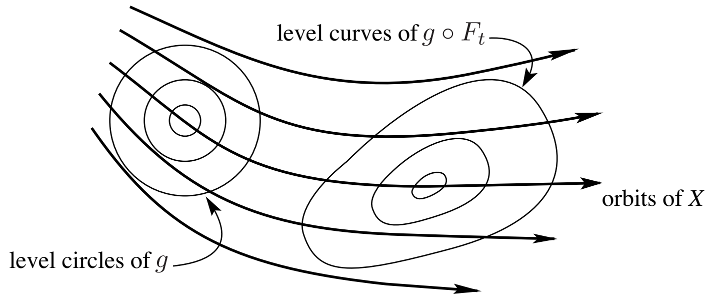


FIGURE 4.2.3. Solving a PDE using characteristics.

Now we turn to the question of using the Lie derivative to characterize vector fields. We will prove that any derivation on functions uniquely defines a vector field. Because of this, derivations can be (and often are) used to *define* vector fields. (See the introduction to §3.3.) In the proof we shall need to localize things in a smooth way, hence the following lemma of general utility is proved first.

4.2.13 Lemma. *Let $\mathbf{E}$ be a C^r Banach space, that is, one whose norm is C^r on $\mathbf{E}\backslash\{0\}$, $r \geq 1$. Let U_1 be an open ball of radius r_1 about x_0 and U_2 an open ball of radius r_2, $r_1 < r_2$. Then there is a C^r function $h : \mathbf{E} \to \mathbb{R}$ such that h is one on U_1 and zero outside U_2.*

We call h a ***bump function***. Later we will prove more generally that on a manifold M, if U_1 and U_2 are two open sets with $\mathrm{cl}(U_1) \subset U_2$, there is an $h \in \mathcal{F}^r(M)$ such that h is one on U_1 and is zero outside U_2.

Proof. By a scaling and translation, we can assume that U_1 and U_2 are balls of radii 1 and 3 and centered at the origin. Let $\theta : \mathbb{R} \to \mathbb{R}$ be given by

$$\theta(x) = \exp\left(\frac{-1}{1-|x|^2}\right) \quad \text{if } |x| < 1$$

and set

$$\theta(x) = 0, \quad \text{if } |x| \geq 1.$$

(See the remarks following Theorem 2.4.15.) Now set

$$\theta_1(s) = \frac{\displaystyle\int_{-\infty}^{s} \theta(t)\,dt}{\displaystyle\int_{-\infty}^{\infty} \theta(t)\,dt}$$

so θ_1 is a C^∞ function and is 0 if $s < -1$, and 1 if $s > 1$. Let $\theta_2(s) = \theta_1(2-s)$, so θ_2 is a C^∞ function that is 1 if $s < 1$ and 0 if $s > 3$. Finally, let $h(x) = \theta_2(\|x\|)$. ■

The norm on a real Hilbert space is C^∞ away from the origin. The order of differentiability of the norms of some concrete Banach spaces is also known; see Bonic and Frampton [1966], Yamamuro [1974], and Supplement 5.5B.

4.2.14 Corollary. *Let M be a C^r manifold modeled on a C^r Banach space. If $\alpha_m \in T_m^*M$, then there is an $f \in \mathcal{F}^r(M)$ such that $\mathbf{d}f(m) = \alpha_m$.*

Proof. If $M = \mathbf{E}$, so $T_m\mathbf{E} \cong \mathbf{E}$, let $f(x) = \alpha_m(x)$, a linear function on $\mathbf{E}$. Then $\mathbf{d}f$ is constant and equals α_m.

The general case can be reduced to $\mathbf{E}$ using a local chart and a bump function as follows. Let $\varphi : U \to U' \subset \mathbf{E}$ be a local chart at m with $\varphi(m) = 0$ and such that U' contains the ball of radius 3. Let $\tilde{\alpha}_m$ be the local representative of α_m and let h be a bump function, 1 on the ball of radius 1 and zero outside the ball of radius 2. let $f(x) = \tilde{\alpha}_m(x)$ and let

$$f = \begin{cases} (hf) \circ \varphi, & \text{on } U \\ 0, & \text{on } M \backslash U. \end{cases}$$

It is easily verified that f is C^r and $\mathbf{d}f(m) = \alpha_m$. ■

4.2.15 Proposition.

(i) *Let M be a C^r manifold modeled on a C^r Banach space. The collection of operators $\pounds_X$ for $X \in \mathfrak{X}^r(M)$, defined on $C^r(M, F)$ and taking values in $C^{r-1}(M, F)$ forms a real vector space and an $\mathcal{F}^r(M)$-module with $(f\pounds_X)(g) = f(\pounds_X g)$, and is isomorphic to $\mathfrak{X}^r(M)$ as a real vector space and as an $\mathcal{F}^r(M)$-module. In particular, $\pounds_X = 0$ iff $X = 0$; and $\pounds_{fX} = f\pounds_X$.*

(ii) *Let M be any C^r manifold. If $\pounds_X f = 0$ for all $f \in C^r(U, F)$, for all open subsets U of M, then $X = 0$.*

Proof. (i) Consider the map $\sigma : X \mapsto \pounds_X$. It is obviously $\mathbb{R}$ and $\mathcal{F}^r(M)$ linear; that is,

$$\pounds_{X_1 + fX_2} = \pounds_{X_1} + f\pounds_{X_2}.$$

To show that it is one–to–one, we must show that $\pounds_X = 0$ implies $X = 0$. But if $\pounds_X f(m) = 0$, then $\mathbf{d}f(m) \cdot X(m) = 0$ for all f. Hence, $\alpha_m(X(m)) = 0$ for all $\alpha_m \in T_m^*M$ by Corollary 4.2.14. Thus $X(m) = 0$ by the Hahn–Banach theorem.

(ii) This has an identical proof with the only exception that one works in a local chart, so it is not necessary to extend a linear functional to the entire manifold M as in Corollary 4.2.14. Thus the condition on the differentiability of the norm of the model space of M can be dropped. ■

4.2.16 Theorem (Derivation Theorem)**.** (i) *If M is finite dimensional and C^∞, the collection of all derivations on $\mathcal{F}(M)$ is a real vector space isomorphic to $\mathfrak{X}(M)$. In particular, for each derivation θ there is a unique $X \in \mathfrak{X}(M)$ such that $\theta = \pounds_X$.*

(ii) *Let M be a C^∞ manifold modeled on a C^∞ Banach space $\mathbf{E}$, that is, $\mathbf{E}$ has a C^∞ norm away from the origin. The collection of all ($\mathbb{R}$-linear) derivations on $C^\infty(M, \mathbf{F})$ (for all Banach spaces $\mathbf{F}$) forms a real vector space isomorphic to $\mathfrak{X}(M)$.*

Proof. We prove (ii) first. Let θ be a derivation. We wish to construct X such that $\theta = \pounds_X$. First of all, note that θ is a ***local operator***; that is, if $h \in C^\infty(M, \mathbf{F})$ vanishes on a neighborhood V of m, then $\theta(h)(m) = 0$. Indeed, let g be a bump function equal to one on a neighborhood of m and zero outside V. Thus $h = (1 - g)h$ and so

$$\theta(h)(m) = \theta(1 - g)(m) \cdot h(m) + \theta(h)(m)(1 - g(m)) = 0. \tag{4.2.1}$$

If U is an open set in M, and $f \in C^\infty(U, \mathbf{F})$ define $(\theta|U)(f)(m) = \theta(gf)(m)$, where g is a bump function equal to one on a neighborhood of m and zero outside U. By the previous remark, $(\theta|U)(f)(m)$ is independent of g, so $\theta|U$ is well defined. For convenience we write $\theta = \theta|U$.

Let (U, φ) be a chart on M, $m \in U$, and $f \in C^\infty(M, \mathbf{F})$ where $\varphi : U \to U' \subset \mathbf{E}$; we can write, for $x \in U'$ and $a = \varphi(m)$,

$$\begin{aligned}(\varphi_* f)(x) &= (\varphi_* f)(a) + \int_0^1 \frac{\partial}{\partial t}(\varphi_* f)[a + t(x - a)]\, dt \\ &= (\varphi_* f)(a) + \int_0^1 \mathbf{D}(\varphi_* f)[a + t(x - a)] \cdot (x - a)\, dt\end{aligned}$$

This formula holds in some neighborhood $\varphi(V)$ of a. Hence for $u \in V$ we have

$$f(u) = f(m) + g(u) \cdot (\varphi(u) - a), \tag{4.2.2}$$

where $g \in C^\infty(V, L(\mathbf{E}, \mathbf{F}))$ is given by

$$g(u) = \int_0^1 \mathbf{D}(\varphi_* f)[a + t(\varphi(u) - a)]dt.$$

Applying θ to equation (4.2.2) at $u = m$ gives

$$\theta f(m) = g(m) \cdot (\theta\varphi)(m) = \mathbf{D}(\varphi_* f)(a) \cdot (\theta\varphi)(m). \tag{4.2.3}$$

Since θ was given globally, the right hand side of equation (4.2.3) is independent of the chart. Now define X on U by its local representative

$$X_\varphi(x) = (x, \theta(\varphi)(u)),$$

where $x = \varphi(u) \in U'$. It follows that $X|U$ is independent of the chart φ and hence $X \in \mathfrak{X}(M)$. Then, for $f \in C^\infty(M, \mathbf{F})$, the local representative of $\pounds_X f$ is

$$\mathbf{D}(f \circ \varphi^{-1})(x) \cdot X_\varphi(x) = \mathbf{D}(f \circ \varphi^{-1})(x) \cdot (\theta\varphi)(u) = \theta f(u).$$

Hence $\pounds_X = \theta$. Finally, uniqueness follows from Proposition 4.2.15.

The vector derivative property was used only in establishing equations (4.2.1) and (4.2.3). Thus, if M is finite dimensional and θ is a derivation on $\mathcal{F}(M)$, we have as before

$$f(u) = f(m) + g(u) \cdot (\varphi(u) - a) = f(m) + \sum_{i=1}^n (\varphi^i(u) - a^i) g_i(u),$$

where $g_i \in \mathcal{F}(V)$ and

$$g_i(m) = \left.\frac{\partial(\varphi_* f)(u)}{\partial X^i}\right|_{u=a}, \quad a = (a^1, \ldots, a^n).$$

Hence equation (4.2.3) becomes

$$\theta f(m) = \sum_{i=1}^n g_i(m)\, \theta(\varphi^i)(m) = \sum_{i=1}^n \frac{\partial}{\partial x^i}(\varphi_* f)(a)\, \theta(\varphi^i)(m)$$

and this is again independent of the chart. Now define X on U by its local representative

$$(x, \theta(\varphi^1)(u), \ldots, \theta(\varphi^n)(u))$$

and proceed as before. ■

Remark. There is a difficulty with this proof for C^r manifolds and derivations mapping C^r to C^{r-1}. Indeed in equation (4.2.2), g is only C^{r-1} if f is C^r, so θ need not be defined on g. Thus, one has to regard θ as defined on C^{r-1}-functions and taking values in C^{r-2}-functions. Therefore $\theta(\varphi)$ is only C^{r-1} and so the vector field it defines is also only C^{r-2}. Then the above proof shows that $\pounds_X$ and θ coincide on C^r-functions on M. In Supplement 4.2D we will prove that $\pounds_X$ and θ are in fact equal on C^{r-1}-functions, but the proof requires a different argument. ♦

For finite-dimensional manifolds, the preceding theorem provides a local basis for vector fields. If (U, φ), $\varphi : U \to V \subset \mathbb{R}^n$ is a chart on M defining the coordinate functions $x^i : U \to \mathbb{R}$, define n derivations $\partial/\partial x^i$ on $\mathcal{F}(U)$ by

$$\frac{\partial f}{\partial x^i} = \frac{\partial (f \circ \varphi^{-1})}{\partial x^i} \circ \varphi.$$

These derivations are linearly independent with coefficients in $\mathcal{F}(U)$, for if

$$\sum_{i=1}^n f^i \frac{\partial}{\partial x^i} = 0, \text{ then } \left(\sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \right) (x^j) = f^j = 0 \text{ for all } j = 1, \dots, n,$$

since $(\partial/\partial x^i)x^j = \delta_i^j$. By Theorem 4.2.16, $(\partial/\partial x^i)$ can be identified with vector fields on U. Moreover, if $X \in \mathfrak{X}(M)$ has components $X^1, \dots, X^n$ in the chart φ, then

$$\pounds_X f = X[f] = \sum_{i=1}^n X^i \frac{\partial f}{\partial x^i} = \left(\sum_{i=1}^n X^i \frac{\partial}{\partial x^i} \right) f, \quad \text{i.e.,} \quad X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}.$$

Thus the vector fields $(\partial/\partial x^i)$, $i = 1, \dots, n$ form a ***local basis*** for the vector fields on M. It should be mentioned however that a ***global basis*** of $\mathfrak{X}(M)$, that is, n vector fields, $X_1, \dots, X_n \in \mathfrak{X}(M)$ that are linearly independent over $\mathcal{F}(M)$ and span $\mathfrak{X}(M)$, does not exist in general. Manifolds that do admit such a global basis for $\mathfrak{X}(M)$ are called ***parallelizable***. It is straightforward to show that a finite-dimensional manifold is parallelizable iff its tangent bundle is trivial. For example, it is shown in differential topology that S^3 is parallelizable but S^2 is not (see Supplement 7.5A).

This completes the discussion of the Lie derivative on functions. Turning to the Lie derivative on vector fields, let us begin with the following.

4.2.17 Proposition. *If X and Y are C^r vector fields on M, then*

$$[\pounds_X, \pounds_Y] = \pounds_X \circ \pounds_Y - \pounds_Y \circ \pounds_X$$

is a derivation mapping $C^{r+1}(M, \mathbf{F})$ to $C^{r-1}(M, \mathbf{F})$.

Proof. More generally, let θ_1 and θ_2 be two derivations mapping C^{r+1} to C^r and C^r to C^{r-1}. Clearly $[\theta_1, \theta_2] = \theta_1 \circ \theta_2 - \theta_2 \circ \theta_1$ is linear and maps C^{r+1} to C^{r-1}. Also, if $f \in C^{r+1}(M, \mathbf{F})$, $g \in C^{r+1}(M, \mathbf{G})$, and $B \in L(\mathbf{F}, \mathbf{G}; H)$, then

$$\begin{aligned}
[\theta_1, \theta_2] (B(f, g)) &= (\theta_1 \circ \theta_2)(B(f, g)) - (\theta_2 \circ \theta_1)(B(f, g)) \\
&= \theta_1\{B(\theta_2(f), g) + B(f, \theta_2(g))\} - \theta_2\{B(\theta_1(f), g) \\
&\quad + B(f, \theta_1(g))\} \\
&= B(\theta_1(\theta_2(f)), g) + B(\theta_2(f), \theta_1(g)) + B(\theta_1(f), \theta_2(g)) \\
&\quad + B(f, \theta_1(\theta_2(g))) - B(\theta_2(\theta_1(f)), g) - B(\theta_1(f), \theta_2(g)) \\
&\quad - B(\theta_2(f), \theta_1(g)) - B(f, \theta_2(\theta_1(g))) \\
&= B([\theta_1, \theta_2] (f), g) + B(f, [\theta_1, \theta_2] (g)).
\end{aligned}$$

■

Because of Theorem 4.2.16 the following definition can be given.

4.2.18 Definition. *Let M be a manifold modeled on a C^∞ Banach space and $X, Y \in \mathfrak{X}^\infty(M)$. Then $[X, Y]$ is the unique vector field such that $\pounds_{[X,Y]} = [\pounds_X, \pounds_Y]$. This vector field is also denoted $\pounds_X Y$ and is called the **Lie derivative of** Y **with respect to** X, **or the Jacobi–Lie bracket of** X **and** Y.*

Even though this definition is useful for Hilbert manifolds (in particular for finite-dimensional manifolds), it excludes consideration of C^r vector fields on Banach manifolds modeled on nonsmooth Banach spaces, such as L^p function spaces for p not even. We shall, however, establish an equivalent definition, which makes sense on any Banach manifold and works for C^r vector fields. This alternative definition is based on the following result.

4.2.19 Theorem (Lie Derivative Formula for Vector Fields). *Let M be as in Definition 4.2.18, $X, Y \in \mathfrak{X}(M)$, and let X have (local) flow F_t. Then*

$$\frac{d}{dt}(F_t^* Y) = F_t^*(\pounds_X Y)$$

(at those points where F_t is defined).

Proof. If $t = 0$ this formula becomes

$$\left.\frac{d}{dt}\right|_{t=0} F_t^* Y = \pounds_X Y. \tag{4.2.4}$$

Assuming equation (4.2.4) for the moment,

$$\frac{d}{dt}(F_t^* Y) = \left.\frac{d}{ds}\right|_{s=0} F_{t+s}^* Y = F_t^* \left.\frac{d}{ds}\right|_{s=0} F_s^* Y = F_t^* \pounds_X Y.$$

Thus the formula in the theorem is equivalent to equation (4.2.4), which is proved in the following way. Both sides of equation (4.2.4) are clearly vector derivations. In view of Theorem 4.2.16, it suffices then to prove that both sides are equal when acting on an arbitrary function $f \in C^\infty(M, \mathbf{F})$. Now

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} (F_t^* Y)[f](m) &= \left.\frac{d}{dt}\right|_{t=0} \{\pounds_{F_t^* Y}[F_t^*(F_{-t}^* f)]\}(m) \\ &= \left.\frac{d}{dt}\right|_{t=0} F_t^*(Y[F_{-t}^* f])(m),\end{aligned}$$

by Proposition 4.2.7(i). Using Theorem 4.2.10 and Leibniz' rule, this becomes

$$X[Y[f]](m) - Y[X[f]](m) = [X, Y][f](m). \qquad \blacksquare$$

Since the formula for $\pounds_X Y$ in equation (4.2.4) does not use the fact that the norm of $\mathbf{E}$ is C^∞ away from the origin, we can state the following definition of the Lie derivative on any Banach manifold M.

4.2.20 Definition (Dynamic Definition of Jacobi–Lie bracket). *If $X, Y \in \mathfrak{X}^r(M)$, $r \geq 1$ and X has flow F_t, the C^{r-1} vector field $\pounds_X Y = [X, Y]$ on M defined by*

$$[X, Y] = \left.\frac{d}{dt}\right|_{t=0} (F_t^* Y)$$

*is called the **Lie derivative** of Y with respect to X, or the **Lie bracket** of X and Y.*

Theorem 4.2.19 then shows that this definition agrees with the earlier one, Definition 4.2.18:

4.2.21 Proposition. *Let $X, Y \in \mathfrak{X}^r(M)$, $r \geq 1$. Then $[X, Y] = \pounds_X Y$ is the unique C^{r-1} vector field on M satisfying*

$$[X, Y][f] = X[Y[f]] - Y[X[f]]$$

for all $f \in C^{r+1}(U, \mathbf{F})$, where U is open in M.

The derivation approach suggests that if $X, Y \in \mathfrak{X}^r(M)$ then $[X, Y]$ might only be C^{r-2}, since $[X, Y]$ maps C^{r+1} functions to C^{r-1} functions, and differentiates them twice. However Definition 4.2.20 (and the local expression equation (4.2.6) below) show that $[X, Y]$ is in fact C^{r-1}.

4.2.22 Proposition. *The bracket $[X, Y]$ on $\mathfrak{X}(M)$, together with the real vector space structure $\mathfrak{X}(M)$, form a* ***Lie algebra****. That is,*

(i) $[\,,\,]$ *is* $\mathbb{R}$ *bilinear;*

(ii) $[X, X] = 0$ *for all* $X \in \mathfrak{X}(M)$*;*

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *for all* $X, Y, Z \in \mathfrak{X}(M)$ *(****Jacobi identity****).*

The proof is straightforward, applying the brackets in question to an arbitrary function. Unlike $\mathfrak{X}(M)$, the space $\mathfrak{X}^r(M)$ is not a Lie algebra since $[X, Y] \in \mathfrak{X}^{r-1}(M)$ for $X, Y \in \mathfrak{X}^r(M)$. (i) and (ii) imply that $[X, Y] = -[Y, X]$, since $[X + Y, X + Y] = 0 = [X, X] + [X, Y] + [Y, X] + [Y, Y]$. We can describe (iii) by writing $\pounds_X$ as a ***Lie bracket derivation***:

$$\pounds_X [Y, Z] = [\pounds_X Y, Z] + [Y, \pounds_X Z].$$

Strictly speaking we should be careful using the same symbol $\pounds_X$ for both definitions of $\pounds_X f$ and $\pounds_X Y$. However, the meaning is generally clear from the context. The analog of Proposition 4.2.7 on the vector field level is the following.

4.2.23 Proposition. (i) *Let $\varphi : M \to N$ be a diffeomorphism and $X \in \mathfrak{X}(M)$. Then $\pounds_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is* ***natural with respect to push-forward*** *by φ. That is,*

$$\pounds_{\varphi_* X} \varphi_* Y = \varphi_* \pounds_X Y,$$

i.e., $[\varphi_* X, \varphi_* Y] = \varphi_* [X, Y]$, i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{X}(M) & \xrightarrow{\varphi_*} & \mathfrak{X}(N) \\ \downarrow{\scriptstyle \pounds_X} & & \downarrow{\scriptstyle \pounds_{\varphi_* X}} \\ \mathfrak{X}(M) & \xrightarrow[\varphi_*]{} & \mathfrak{X}(N) \end{array}$$

(ii) *$\pounds_X$ is* ***natural with respect to restrictions****. That is, for $U \subset M$ open, $[X|U, Y|U] = [X, Y]|U$; or the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{X}(M) & \xrightarrow{|U} & \mathfrak{X}(U) \\ \downarrow{\scriptstyle \pounds_X} & & \downarrow{\scriptstyle \pounds_{X|U}} \\ \mathfrak{X}(M) & \xrightarrow[|U]{} & \mathfrak{X}(U) \end{array}$$

Proof. For (i), let $f \in \mathcal{F}(V)$, V be open in N, and $\varphi(m) = n \in V$. By Proposition 4.2.7(i), for any $Z \in \mathfrak{X}(M)$

$$((\varphi_* Z)[f])(n) = Z[f \circ \varphi](m),$$

so we get from Proposition 4.2.21

$$\begin{aligned}(\varphi_*[X,Y])[f](n) &= [X,Y][f\circ\varphi](m)\\ &= X[Y[f\circ\varphi]](m) - Y[X[f\circ\varphi]](m)\\ &= X[(\varphi_*Y)[f]\circ\varphi](m) - Y[(\varphi_*X)[f]\circ\varphi](m)\\ &= (\varphi_*X)[(\varphi_*Y)[f]](n) - (\varphi_*Y)[(\varphi_*X)[f]](n)\\ &= [\varphi_*X, \varphi_*Y][f](n).\end{aligned}$$

Thus, $\varphi_*[X,Y] = [\varphi_*X, \varphi_*Y]$ by Proposition 4.2.15(ii). (ii) follows from the fact that $\mathbf{d}(f|U) = \mathbf{d}f|U$. ■

Let us compute the local expression for $[X,Y]$. Let $\varphi : U \to V \subset \mathbf{E}$ be a chart on M and let the local representatives of X and Y be X and Y respectively, so $X, Y : V \to \mathbf{E}$. By Proposition 4.2.23, the local representative of $[X,Y]$ is $[X,Y]$. Thus,

$$\begin{aligned}[X,Y][f](x) &= X[Y[f]](x) - Y[X[f]](x)\\ &= \mathbf{D}(Y[f])(x)\cdot X(x) - \mathbf{D}(X[f])(x)\cdot Y(x).\end{aligned}$$

Now $Y[f](x) = \mathbf{D}f(x)\cdot Y(x)$ and its derivative may be computed by the product rule. The terms involving the second derivative of f cancel by symmetry of $\mathbf{D}^2 f(x)$ and so we are left with

$$\mathbf{D}f(x)\cdot\{\mathbf{D}Y(x)\cdot X(x) - \mathbf{D}X(x)\cdot Y(x)\}.$$

Thus the local representative of $[X,Y]$ is

$$[X,Y] = \mathbf{D}Y\cdot X - \mathbf{D}X\cdot Y. \tag{4.2.5}$$

If M is n-dimensional and the chart φ gives local coordinates $(x^1,\ldots,x^n)$ then this calculation gives the components of $[X,Y]$ as

$$[X,Y]^j = \sum_{i=1}^n X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i} \tag{4.2.6}$$

that is, $[X,Y] = (X\cdot\nabla)Y - (Y\cdot\nabla)X$.

Part (i) of Proposition 4.2.23 has an important generalization to φ-related vector fields. For this, however, we need first the following preparatory proposition.

4.2.24 Proposition. *Let $\varphi : M \to N$ be a C^r map of C^r manifolds, $X \in \mathfrak{X}^{r-1}(M)$, and $X' \in \mathfrak{X}^{r-1}(N)$. Then $X \sim_\varphi X'$ iff $(X'[f])\circ\varphi = X[f\circ\varphi]$ for all $f \in \mathcal{F}^1(V)$, where V is open in N.*

Proof. By definition, $((X'[f])\circ\varphi)(m) = \mathbf{d}f(\varphi(m))\cdot X'(\varphi(m))$. By the chain rule,

$$X[f\circ\varphi](m) = \mathbf{d}(f\circ\varphi)(m)\cdot X(m) = \mathbf{d}f(\varphi(m))\cdot T_m\varphi(X(m)).$$

If $X \sim_\varphi X'$, then $T\varphi\circ X = X'\circ\varphi$ and we have the desired equality. Conversely, if $X[f\circ\varphi] = (X'[f])\circ\varphi$ for all $f \in \mathcal{F}^1(V)$, and all V open in N, choosing V to be a chart domain and f the pull-back to V of linear functionals on the model space of N, we conclude that $\alpha_n\cdot(X'\circ\varphi)(m) = \alpha_n\cdot(T\varphi\circ X)(m)$, where $n = \varphi(m)$, for all $\alpha_n \in T_n^*N$. Using the Hahn–Banach theorem, we deduce that $(X'\circ\varphi)(m) = (T\varphi\circ X)(m)$, for all $m \in M$. ■

It is to be noted that under differentiability assumptions on the norm on the model space of N (as in Theorem 4.2.16), the condition "for all $f \in \mathcal{F}^1(V)$ and all $V \subset N$" can be replaced by "for all $f \in \mathcal{F}^1(N)$" by using bump functions. This holds in particular for Hilbert (and hence for finite-dimensional) manifolds.

4.2.25 Proposition. *Let $\varphi : M \to N$ be a C^r map of manifolds, $X, Y \in \mathfrak{X}^{r-1}(M)$, and $X', Y' \in \mathfrak{X}^{r-1}(N)$. If $X \sim_\varphi X'$ and $Y \sim_\varphi Y'$, then $[X, Y] \sim_\varphi [X', Y']$.*

Proof. By Proposition 4.2.24 it suffices to show that $([X', Y'][f]) \circ \varphi = [X, Y][f \circ \varphi]$ for all $f \in \mathcal{F}^1(V)$, where V is open in N. We have

$$\begin{aligned}([X', Y'][f]) \circ \varphi &= X'[Y'[f]] \circ \varphi - Y'[X'[f]] \circ \varphi \\ &= X[(Y'[f]) \circ \varphi] - Y[(X'[f]) \circ \varphi] \\ &= X[Y[f \circ \varphi]] - Y[X[f \circ \varphi]] \\ &= [X, Y][f \circ \varphi].\end{aligned}$$

■

The analog of Proposition 4.2.9 is the following.

4.2.26 Proposition. *For every $X \in \mathfrak{X}(M)$, the operator $\pounds_X$ is a derivation on $(\mathcal{F}(M), \mathfrak{X}(M))$. That is, $\pounds_X$ is $\mathbb{R}$-linear and $\pounds_X(fY) = (\pounds_X f)Y + f(\pounds_X Y)$.*

Proof. For $g \in C^\infty(U, \mathbf{E})$, where U is open in M, we have

$$\begin{aligned}[X, fY][g] &= \pounds_X(\pounds_{fY} g) - \pounds_{fY} \pounds_X g \\ &= \pounds_X(f \pounds_Y g) - f \pounds_Y \pounds_X g \\ &= (\pounds_X f)\pounds_Y g + f \pounds_X \pounds_Y g - f \pounds_Y \pounds_X g,\end{aligned}$$

so

$$[X, fY] = (\pounds_X f)\, Y + f[X, Y] \text{ by Proposition 4.2.15(ii).}$$

■

Commutation of vector fields is characterized by their flows in the following way.

4.2.27 Proposition. *Let $X, Y \in \mathfrak{X}^r(M)$, $r \geq 1$, and let F_t, G_t denote their flows. The following are equivalent.*

(i) $[X, Y] = 0$;

(ii) $F_t^* Y = Y$;

(iii) $G_t^* X = X$;

(iv) $F_t \circ G_s = G_s \circ F_t$.

(*In* (ii)–(iv)*, equality is understood, as usual, where the expressions are defined.*)

Proof. $F_t \circ G_s = G_s \circ F_t$ iff $G_s = F_t \circ G_s \circ F_t^{-1}$, which by Proposition 4.2.4 is equivalent to $Y = F_t^* Y$; that is, (iv) is equivalent to (ii). Similarly (iv) is equivalent to (iii). If $F_t^* Y = Y$, then

$$[X, Y] = \left.\frac{d}{dt}\right|_{t=0} F_t^* Y = 0.$$

Conversely, if $[X, Y] = \pounds_X Y = 0$, then

$$\frac{d}{dt} F_t^* Y = \left.\frac{d}{ds}\right|_{s=0} F_{t+s}^* Y = F_t^* [X, Y] = 0$$

so that $F_t^* Y$ is constant in t. For $t = 0$, however, its value is Y, so that $F_t^* Y = Y$ and we have thus showed that (i) and (ii) are equivalent. Similarly (i) and (iii) are equivalent. ■

Just as in Theorem 4.2.10, the formula for the Lie derivative involving the flow can be used to solve special types of first-order linear $n \times n$ systems of partial differential equations. Consider the first-order system:

$$\frac{\partial Y^i}{\partial t}(x,t) = \sum_{j=1}^{n} \left(X^j(x) \frac{\partial Y^i}{\partial x^j}(x,t) - Y^j(x,t) \frac{\partial X^i(x)}{\partial x^j} \right) \qquad (P_n)$$

with initial conditions $Y^i(x,0) = g^i(x)$ for given functions $X^i(x)$, $g^i(x)$ and scalar unknowns $Y^i(x,t)$, $i = 1, \ldots, n$, where $x = (x^1, \ldots, x^n)$.

4.2.28 Proposition. *Suppose $X = (X^1, \ldots, X^n)$ has a complete flow F_t. Then letting $Y = (Y^1, \ldots, Y^n)$ and $G = (g^1, \ldots, g^n)$, $Y = F_t^* G$ is a solution of the foregoing problem (P_n). (See Exercise 4.2-3 for uniqueness.)*

Proof.

$$\frac{\partial Y}{\partial t} = \frac{d}{dt} F_t^* G = F_t^*[X, G] = [F_t^* X, F_t^* G] = [X, Y]$$

since $F_t^* X = X$ and $Y = F_t^* G$. The expression in the problem (P_n) is by equation (4.2.6) the ith component of $[X, Y]$. ■

4.2.29 Example. Solve the system of partial differential equations:

$$\frac{\partial Y^1}{\partial t} = (x+y)\frac{\partial Y^1}{\partial x} - (x+y)\frac{\partial Y^1}{\partial y} - Y^1 - Y^2,$$
$$\frac{\partial Y^2}{\partial t} = (x+y)\frac{\partial Y^2}{\partial x} - (x+y)\frac{\partial Y^2}{\partial y} + Y^1 + Y^2$$

with initial conditions $Y^1(x,y,0) = x$, $Y^2(x,y,0) = y^2$. The vector field $X(x,y) = (x+y, -x-y)$ has the complete flow $F_t(x,y) = ((x+y)t + x, -(x+y)t + y)$, so that the solution is given by $Y(x,y,t) = F_t^*(x, y^2)$; that is,

$$Y^1(x,y,t) = ((x+y)t + x)(1-t) - t[y - (x+y)t]^2$$
$$Y^2(x,y,t) = t((x+y)t + x) + (t+1)[y - (x+y)t]^2. \qquad \blacklozenge$$

In later chapters we will need a flow type formula for the Lie derivative of a time-dependent vector field, In §4.1 we discussed the existence and uniqueness of solutions of a time-dependent vector field. Let us formalize and recall the basic facts.

4.2.30 Definition. *A C^r **time-dependent vector field** is a C^r map $X : \mathbb{R} \times M \to TM$ such that $X(t,m) \in T_m M$ for all $(t,m) \in \mathbb{R} \times M$; that is, $X_t \in \mathfrak{X}^r(M)$, where $X_t(m) = X(t,m)$. The **time-dependent flow** or **evolution operator** $F_{t,s}$ of X is defined by the requirement that $t \mapsto F_{t,s}(m)$ be the integral curve of X starting at m at time $t = s$; that is,*

$$\frac{d}{dt} F_{t,s}(m) = X(t, F_{t,s}(m)) \quad \textit{and} \quad F_{s,s}(m) = m.$$

By uniqueness of integral curves we have $F_{t,s} \circ F_{s,r} = F_{t,r}$ (replacing the flow property $F_{t+s} = F_t \circ F_s$), and $F_{t,t} =$ identity. It is customary to write $F_t = F_{t,0}$. If X is time independent, $F_{t,s} = F_{t-s}$. In general $F_t^* X_t \neq X_t$. However, the basic Lie derivative formulae still hold.

4.2.31 Theorem. *Let $X_t \in \mathfrak{X}^r(M)$, $r \geq 1$ for each t and suppose $X(t,m)$ is continuous in (t,m). Then $F_{t,s}$ is of class C^r and for $f \in C^{r+1}(M, F)$, and $Y \in \mathfrak{X}^r(M)$, we have*

(i) $\dfrac{d}{dt} F_{t,s}^* f = F_{t,s}^*(\pounds_{X_t} f)$, *and*

(ii) $\frac{d}{dt} F_{t,s}^* Y = F_{t,s}^*([X_t, Y]) = F_{t,s}^*(\pounds_{X_t} Y)$.

Proof. That $F_{t,s}$ is C^r was proved in §4.1. The proof of (i) is a repeat of Theorem 4.2.10:

$$\begin{aligned}\frac{d}{dt}(F_{t,s}^* f)(m) &= \frac{d}{dt}(f \circ F_{t,s})(m) \\ &= \mathbf{d}f(F_{t,s}(m)) \frac{dF_{t,s}(m)}{dt} \\ &= \mathbf{d}f(F_{t,s}(m)) \cdot X_t(F_{t,s}(m)) \\ &= (\pounds_{X_t} f)(F_{t,s}(m)) \\ &= F_{t,s}^*(\pounds_{X_t} f)(m).\end{aligned}$$

For vector fields, note that by Proposition 4.2.7(i),

$$(F_{t,s}^* Y)[f] = F_{t,s}^*(Y[F_{t,s}^* f]) \tag{4.2.7}$$

since $F_{s,t} = F_{t,s}^{-1}$. The result (ii) will be shown to follow from (i), equation (4.2.7), and the next lemma.

4.2.32 Lemma. *The following identity holds*:

$$\frac{d}{dt} F_{s,t}^* f = -X_t[F_{s,t}^* f].$$

Proof. Differentiating $F_{s,t} \circ F_{t,s} =$ identity in t, we get the ***backward differential equation***:

$$\frac{d}{dt} F_{s,t} = -TF_{s,t} \circ X$$

Thus

$$\begin{aligned}\frac{d}{dt} F_{s,t}^* f(m) &= -\mathbf{d}f(F_{s,t}(m)) \cdot TF_{s,t}(X_t(m)) \\ &= -\mathbf{d}F(f \circ F_{s,t}) \cdot X_t(m) = -X_t[f \circ F_{s,t}](m).\end{aligned}$$ ▼

Thus from equation (4.2.7) and (i),

$$\frac{d}{dt}(F_{t,s}^* Y)[f] = F_{t,s}^*(X_t[Y[F_{t,s}^* f]]) - F_{t,s}^*(Y[X_t[F_{t,s}^* f]]),$$

By Proposition 4.2.21 and equation (4.2.7), this equals $(F_{t,s}^*[X_t, Y])[f]$. ■

If f and Y are time dependent, then (i) and (ii) read

$$\frac{d}{dt} F_{t,s}^* f = F_{t,s}^* \left(\frac{\partial f}{\partial t} + \pounds_{X_t} f \right) \tag{4.2.8}$$

and

$$\frac{d}{dt} F_{t,s}^* Y = F_{t,s}^* \left(\frac{\partial Y}{\partial t} + [X_t, Y_t] \right). \tag{4.2.9}$$

Unlike the corresponding formula for time-independent vector fields, we generally have

$$F_{t,s}^*(\pounds_{X_t} f) \neq \pounds_{X_t}(F_{t,s}^* f) \quad \text{and} \quad F_{t,s}^*(\pounds_{X_t} Y) \neq \pounds_{F_{t,s}^* X_t} X_t(F_{t,s}^* Y).$$

Time-dependent vector fields on M can be made into time-independent ones on a bigger manifold. Let $t \in \mathfrak{X}(\mathbb{R}\times M)$ denote the vector field which is defined by $t(s,m) = ((s,1), 0_m) \in T_{(s,m)}(\mathbb{R}\times M) \cong T_s\mathbb{R}\times T_m M$. Let the ***suspension*** of X be the vector field $X' \in \mathfrak{X}(\mathbb{R}\times M)$ where $X'(t,m) = ((t,1), X(t,m))$ and observe that $X' = t + X$. Since $b : I \to M$ is an integral curve of X at m iff $b'(t) = X(t, b(t))$ and $b(0) = m$, a curve $c : I \to \mathbb{R}\times M$ is an integral curve of X' at $(0,m)$ iff $c(t) = (t, b(t))$. Indeed, if $c(t) = (a(t), b(t))$ then $c(t)$ is an integral curve of X' iff $c'(t) = (a'(t), b'(t)) = X'(c(t))$; that is $a'(t) = 1$ and $b'(t) = X(a(t), b(t))$. Since $a(0) = 0$, we get $a(t) = t$. These observations are summarized in the following (see Figure 4.2.4).

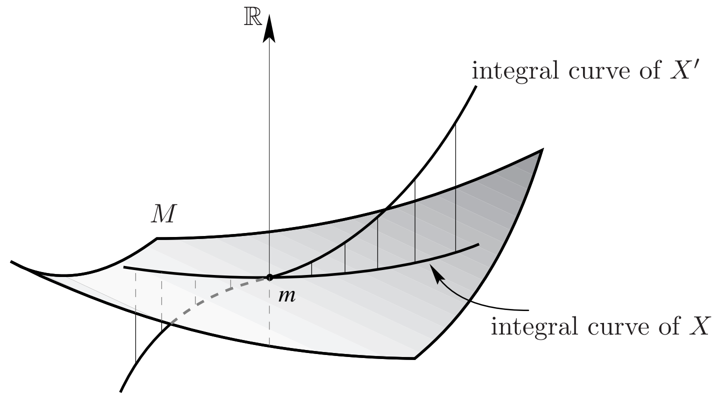


FIGURE 4.2.4. Suspension of a vector field

4.2.33 Proposition. *Let X be a C^r-time-dependent vector field on M with evolution operator $F_{t,s}$. The flow F_t of the suspension $X' \in \mathfrak{X}^r(\mathbb{R}\times M)$ is given by $F_t(s,m) = (t+s, F_{t+s,s}(m))$.*

Proof. In the preceding notations, $b(t) = F_{t,0}(m)$, $c(t) = F_t(0,m) = (t, F_{t,0}(m))$, and so the statement is proved for $s = 0$. In general, note that $F_t(s,m) = F_{t+s}(0, F_{0,s}(m))$ since $t \mapsto F_{t+s}(0, F_{0,s}(m))$ is the integral curve of X', which at $t = 0$ passes through

$$F_s(0, F_{0,s}(m)) = (s, (F_{s,0} \circ F_{0,s})(m)) = (s,m).$$

Thus

$$\begin{aligned} F_t(s,m) &= F_{t+s}(0, F_{0,s}(m)) \\ &= (t+s, (F_{t+s,0} \circ F_{0,s})(m)) \\ &= (t+s, F_{t+s,s}(m)). \end{aligned}$$

■

SUPPLEMENT 4.2A

Product formulas for the Lie bracket

This box is a continuation of Supplement 4.1A and gives the flow of the Lie bracket $[X,Y]$ in terms of the flows of the vector fields $X, Y \in \mathfrak{X}(M)$.

4.2.34 Proposition. *Let $X, Y \in \mathfrak{X}(M)$ have flows F_t and G_t. If B_t denotes the flow of $[X, Y]$, then for $x \in M$,*

$$B_t(x) = \lim_{n\to\infty} \left(G_{-\sqrt{t/n}} \circ F_{-\sqrt{t/n}} \circ G_{\sqrt{t/n}} \circ F_{\sqrt{t/n}} \right)^n (x), \quad t \geq 0.$$

Proof. Let

$$K_\varepsilon(x) = \left(G_{-\sqrt{\varepsilon}} \circ F_{-\sqrt{\varepsilon}} \circ G_{\sqrt{\varepsilon}} \circ F_{\sqrt{\varepsilon}} \right)(x), \quad \varepsilon \geq 0.$$

The claimed formula follows from Proposition 4.1.24 if we show that

$$\left. \frac{\partial}{\partial \varepsilon} K_\varepsilon(x) \right|_{\varepsilon=0} = [X, Y](x)$$

for all $x \in M$. This in turn is equivalent to

$$\left. \frac{\partial}{\partial \varepsilon} K_\varepsilon^* f \right|_{\varepsilon=0} = [X, Y](x)$$

for any $f \in \mathcal{F}(M)$. By the Lie derivative formula,

$$\frac{\partial}{\partial \varepsilon} K_\varepsilon^* f = \frac{1}{2\sqrt{\varepsilon}} \left\{ F^*_{\sqrt{\varepsilon}} \pounds_X \left(G^*_{\sqrt{\varepsilon}} F^*_{-\sqrt{\varepsilon}} G^*_{-\sqrt{\varepsilon}} f \right) + F^*_{\sqrt{\varepsilon}} G^*_{\sqrt{\varepsilon}} \pounds_Y \left(F^*_{-\sqrt{\varepsilon}} G^*_{-\sqrt{\varepsilon}} f \right) \right.$$
$$\left. - F^*_{\sqrt{\varepsilon}} G^*_{\sqrt{\varepsilon}} F^*_{-\sqrt{\varepsilon}} \pounds_X \left(G^*_{-\sqrt{\varepsilon}} f \right) - F^*_{\sqrt{\varepsilon}} G^*_{\sqrt{\varepsilon}} F^*_{-\sqrt{\varepsilon}} G^*_{-\sqrt{\varepsilon}} \pounds_Y (f) \right\}.$$

By the chain rule, the limit of this as $\varepsilon \downarrow 0$ is half the $\partial/\partial\sqrt{\varepsilon}$-derivative of the parenthesis at $\varepsilon = 0$. Again by the Lie derivative formula, this equals a sum of 16 terms, which reduces to the expression $[X, Y][f]$. ■

For example, for $n \times n$ matricess A and B, Proposition 4.2.34 yields the classical formula

$$e^{[A,B]} = \lim_{n\to\infty} \left(e^{-A/\sqrt{n}} e^{-B/\sqrt{n}} e^{A/\sqrt{n}} e^{B/\sqrt{n}} \right)^n,$$

where the commutator is given by $[A, B] = AB - BA$. To see this, define for any $n \times n$ matrix C a vector field $X_C \in \mathfrak{X}(\mathbb{R}^n)$ by $X_C(x) = Cx$. Thus X_C has flow $F_t(x) = e^{tC}x$. Note that X_C is linear in C and satisfies $[X_A, X_B] = -X_{[A,B]}$ as is easily verified. Thus the flow of $[X_B, X_A]$ is $e^{t[A,B]}$ and the formula now follows from Propositon 4.2.34.

The results of Corollary 4.1.27 and Proposition 4.2.34 had their historical origins in Lie group theory, where they are known by the name of ***exponential formulas***. The converse of Corollary 4.1.25, namely expressing $e^{tA}e^{tB}$ as an exponential of some matrix for sufficiently small t is the famous Baker–Campbell–Hausdorff formula (see e.g., Varadarajan [1974, Section 2.15]). The formulas in Corollary 4.1.25 and Proposition 4.2.34 have certain generalizations to unbounded operators and are called ***Trotter product formulas*** after Trotter [1958]. See Chorin et. al. The results of Corollary 4.1.27 and Proposition 4.2.34 had their historical origins in Lie group theory, where they are known by the name of ***exponential formulas***. The converse of Corollary 4.1.25, namely expressing $e^{tA}e^{tB}$ as an exponential of some matrix for sufficiently small t is the famous Baker–Campbell–Hausdorff formula (see e.g., Varadarajan [1974, Section 2.15]). The formulas in Corollary 4.1.25 and Proposition 4.2.34 have certain generalizations to unbounded operators and are called ***Trotter product formulas*** after Trotter [1958]. See Chorin, Hughes, McCracken, and Marsden [1978] for further information.

Supplement 4.2B
The Method of Characteristics

The method used to solve problem (P) also enables one to solve first-order quasi-linear partial differential equations in $\mathbb{R}^n$. Unlike Proposition 4.2.11, the solution will be implicit, not explicit. The equation under consideration in $\mathbb{R}^n$ is

$$\sum_{i=1}^{n} X^i(x^1,\ldots,x^n,f)\frac{\partial f}{\partial x^i} = Y(x^1,\ldots,x^n,f), \tag{Q}$$

where $f = f(x^1,\ldots,x^n)$ is the unknown function and X^i, Y, $i = 1,\ldots,n$ are C^r real-valued functions on $\mathbb{R}^{n+1}$, $r \geq 1$. As initial condition one takes an $(n-1)$-dimensional submanifold Γ in $\mathbb{R}^{n+1}$ that is nowhere tangent to the vector field

$$\sum_{i=1}^{n} X^i\frac{\partial}{\partial x^i} + Y\frac{\partial}{\partial f}$$

called the ***characteristic vector field*** of (Q). Thus, if Γ is given parametrically by

$$x^i = x^i(t_1,\ldots,t_{n-1}),\ i = 1,\ldots,n \quad \text{and} \quad f = f(t_1,\ldots,t_{n-1}),$$

this requirement means that the matrix

$$\begin{bmatrix} X^1 & \cdots & X^n & Y \\ \dfrac{\partial x^1}{\partial t_1} & \cdots & \dfrac{\partial x^n}{\partial t_1} & \dfrac{\partial f}{\partial t_1} \\ \vdots & & \vdots & \vdots \\ \dfrac{\partial x^1}{\partial t_{n-1}} & \cdots & \dfrac{\partial x^n}{\partial t_{n-1}} & \dfrac{\partial f}{\partial t_{n-1}} \end{bmatrix}$$

has rank n. It is customary to require that the determinant obtained by deleting the last column be $\neq 0$, for then, as we shall see, the implicit function theorem gives the solution. The function f is found as follows. Consider F_t, the flow of the vector field $\sum_{i=1,\ldots,n} X^i\partial/\partial x^i + Y\partial/\partial f$ in $\mathbb{R}^{n+1}$ and let S be the manifold obtained by sweeping out Γ by F_t. That is, $S = \bigcup\{\, F_t(\Gamma) \mid t \in \mathbb{R}\,\}$. The condition that $\sum_{i=1,\ldots,n} X^i\partial/\partial x^i + Y\partial/\partial f$ never be tangent to Γ insures that the manifold Γ is "dragged along" by the flow F_t to produce a manifold of dimension n. If S is described by $f = f(x^1,\ldots,x^n)$ then f is the solution of the partial differential equation. Indeed, the tangent space to S contains the vector $\sum_{i=1,\ldots,n} X^i\partial/\partial x^i + Y\partial/\partial f$; that is, this vector is perpendicular to the normal $(\partial f/\partial x^1, \ldots \partial f/\partial x^n, -1)$ to the surface $f = f(x^1,\ldots,x^n)$ and thus (Q) is satisfied.

We work parametrically and write the components of F_t as

$$x^i = x^i(t_1,\ldots,t_{n-1},t),\ i = 1,\ldots,n \quad \text{and} \quad f = f(t_1,\ldots,t_{n-1},t).$$

Assuming that

$$0 \neq \begin{vmatrix} X^1 & \cdots & X^n \\ \dfrac{\partial x^1}{\partial t_1} & \cdots & \dfrac{\partial x^n}{\partial t_1} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial x^1}{\partial t_{n-1}} & & \dfrac{\partial x^n}{\partial t_{n-1}} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x^1}{\partial t} & \cdots & \dfrac{\partial x^n}{\partial t} \\ \dfrac{\partial x^1}{\partial t_1} & \cdots & \dfrac{\partial x^n}{\partial t_1} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial x^1}{\partial t_{n-1}} & & \dfrac{\partial x^n}{\partial t_{n-1}} \end{vmatrix}$$

one can locally invert to give $t = (x^1, \ldots, x^n)$, $t_i = t_i(x^1, \ldots, x^n)$, $i = 1, \ldots, n-1$. Substitution into f yields $f = f(x^1, \ldots, x^n)$.

The fundamental assumption in this construction is that the vector field $\sum_{i=1,\ldots,n} X^i \partial/\partial x^i + Y\partial/\partial f$ is never tangent to the $(n-1)$-manifold Γ. The method breaks down if one uses manifolds Γ, which are tangent to this vector field at some point. The reason is that at such a point, no complete information about the derivative of f in a complementary $(n-1)$-dimensional subspace to the characteristic is known.

4.2.35 Example. Consider the equation in $\mathbb{R}^2$ given by

$$\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y} = 3$$

with initial condition $\Gamma = \{\, (x, y, f) \mid x = s,\ y = (1/2)s^2 - s,\ f = s\,\}$. On this one-manifold

$$\begin{vmatrix} 1 & f \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & s-1 \end{vmatrix} = 1 \neq 0$$

so that the vector field $\partial/\partial x + f\partial/\partial y + 3\partial/\partial f$ is never tangent to Γ. Its flow is $F_t(x, y, f) = (t+x, (3/2)t^2 + ft + y, 3t + f)$ so that the manifold swept out by Γ along F_t is given by $x(t, s) = t + s$, $y(t, s) = (3/2)t^2 + st + (1/2)s^2 - s$, $f(t, s) = 3t + s$. Eliminating t, s we get

$$f(x, y) = x - 1 \pm \sqrt{1 - 2x^2 + 4x + 4y}.$$

The solution is defined only for $1 - 2x^2 + 4x + 4y \geq 0$. ♦

Another interesting phenomenon occurs when S can no longer be described in terms of the graph of f; for example, S "folds over." This corresponds to the formation of shock waves. Further information can be found in Chorin and Marsden [1993], Lax [1973], Guillemin and Sternberg [1977], John [1975], and Smoller [1983].

Supplement 4.2C

Automorphisms of Function Algebras

The property of flows corresponding to the derivation property of vector fields is that they are algebra preserving

$$F_t^*(fg) = (F_t^* f)(F_t^* g).$$

In fact it is obvious that every diffeomorphism induces an algebra automorphism of $\mathcal{F}(M)$. The following theorem shows the converse. (We note that there is an analogous result of Mackey [1962] for measurable functions and measurable automorphisms.)

4.2.36 Theorem. *Let M be a paracompact second–countable finite dimensinal manifold. Let $\alpha : \mathcal{F}(M) \to \mathcal{F}(M)$ be an invertible linear mapping that satisfies $\alpha(fg) = \alpha(f)\alpha(g)$ for all $f, g \in \mathcal{F}(M)$. Then there is a unique C^∞ diffeomorphism $\varphi : M \to M$ such that $\alpha(f) = f \circ \varphi$.*

Remarks.

A. There is a similar theorem for paracompact second–countable Banach manifolds; here we assume that there are invertible linear maps $\alpha_F : C^\infty(M,F) \to C^\infty(M,F)$ for each Banach space F such that for any bilinear continuous map $B : F \times G \to H$ we have $\alpha_H(B(f,g)) = B(\alpha_F(f), \alpha_G(g))$ for $f \in C^\infty(M,F)$ and $g \in C^\infty(M,G)$. The conclusion is the same: there is a C^∞ diffeomorphism $\varphi : M \to M$ such that $\alpha_F(f) = f \circ \varphi$ for all F and all $f \in C^\infty(M,G)$. Alternative to assuming this for all F, one can take $F = \mathbb{R}$ and assume that M is modelled on a Banach space that has a norm that is C^∞ away from the origin. We shall make some additional remarks on the infinite-dimensional case in the course of the proof.

B. Some of the ideas about partitions of unity are needed in the proof. Although the present proof is self–contained, the reader may wish to consult §5.6 simultaneously.

C. In Chapter 5 we shall see that finite-dimensional paracompact manifolds are metrizable, so by Theorem 1.6.14 they are automatically second countable. ♦

Proof of uniqueness. We shall first construct a C^∞ function $\chi : M \to \mathbb{R}$ which takes on the values 1 and 0 at two given points $m_1, m_2 \in M$, $m_1 \neq m_2$. Choose a chart (U, φ) at m_1, such that $m_2 \notin U$ and such that $\varphi(U)$ is a ball of radius r_1 about the origin in $\mathbf{E}$, $\varphi(m_1) = 0$. Let $V \subset U$ by the inverse image by φ of the ball of radius $r_2 < r_1$ and let $\theta : \mathbf{E} \to \mathbb{R}$ be a C^∞-bump function as in Lemma 4.2.13. Then the function $\chi : M \to \mathbb{R}$ given by

$$\chi = \begin{cases} \theta \circ \varphi, & \text{on } U; \\ 0, & \text{on } M \backslash U. \end{cases}$$

is clearly C^∞ and $\chi(m_1) = 1$, $\chi(m_2) = 0$.

Now assume that $\varphi^* f = \psi^* f$ for all $f \in \mathcal{F}(M)$ for two different diffeomorphisms φ, ψ of M. Then there is a point $m \in M$ such that $\varphi(m) \neq \psi(m)$ and thus we can find $\chi \in \mathcal{F}(M)$ such that $(\chi \circ \varphi)(m) = 1$, $(\chi \circ \psi)(m) = 0$ contradicting $\varphi^*\chi = \psi^*\chi$. Hence $\varphi = \psi$. ■

The proof of existence is based on the following key lemma.

4.2.37 Lemma. *Let M be a (finite-dimensional) paracompact second countable manifold and $\beta : \mathcal{F}(M) \to \mathbb{R}$ be a nonzero algebra homomorphism. Then there is a unique point $m \in M$ such that $\beta(f) = f(m)$.*

Proof. (Following suggestions of H. Bercovici.) Uniqueness is clear, as before, since for $m_1 \neq m_2$ there exists a bump function $f \in \mathcal{F}(M)$ satisfying $f(m_1) = 0$, $f(m_2) = 1$.

To show existence, note first that $\beta(1) = 1$. Indeed $\beta(1) = \beta(1^2) = \beta(1)\beta(1)$ so that either $\beta(1) = 0$ or $\beta(1) = 1$. But $\beta(1) = 0$ would imply β is identically zero since $\beta(f) = \beta(1 \cdot f) = \beta(1) \cdot \beta(f)$, contrary to our hypotheses. Therefore we must have $\beta(1) = 1$ and thus $\beta(c) = c$ for $c \in \mathbb{R}$. For $m \in M$, let

$$\operatorname{Ann}(m) = \{ f \in \mathcal{F}(M) \mid f(m) = 0 \}.$$

Second, we claim that it is enough to show that there is an $m \in M$ such that $\ker \beta = \{ f \in \mathcal{F}(M) \mid \beta(f) = 0 \} = \operatorname{Ann}(m)$. Clearly, if $\beta(f) = f(m)$ for some m, then $\ker \beta = \operatorname{Ann}(m)$. Conversely, if this holds for some $m \in M$ and $f \notin \ker \beta$, let $c = \beta(f)$ and note that $f - c \in \ker \beta = \operatorname{Ann}(m)$, so $f(m) = c$ and thus $\beta(f) = f(m)$ for all $f \in \mathcal{F}(M)$.

To prove that $\ker \beta = \operatorname{Ann}(m)$ for some $m \in M$, note that both are ideals in $\mathcal{F}(M)$; that is, if $f \in \ker \beta$, (resp., $\operatorname{Ann}(m)$), and $g \in \mathcal{F}(M)$, then $fg \in \ker \beta$ (resp., $\operatorname{Ann}(m)$). Moreover, both of them are maximal ideals; that is, if I is another ideal of $\mathcal{F}(M)$, with $I \neq \mathcal{F}(M)$, and $\ker \beta \subset I$, (resp., $\operatorname{Ann}(m) \subset I$) then necessarily $\ker \beta = I$ (resp. $\operatorname{Ann}(m) = I$). For $\ker \beta$ this is seen in the following way: since $\mathbb{R}$ is a field, it has no ideals except 0 and itself; but $\beta(I)$ is an ideal in $\mathbb{R}$, so $\beta(I) = 0$, that is, $I = \ker \beta$, or $\beta(I) = \mathbb{R} = \beta(\mathcal{F}(M))$.

If $\beta(I) = \mathbb{R}$, then for every $f \in \mathcal{F}(M)$ there exists $g \in I$ such that $f - g \in \ker \beta \subset I$ and hence $f \in g + I \subset I$; that is, $I = \mathcal{F}(M)$. Similarly, the ideal $\mathrm{Ann}(m)$ is maximal since the quotient $\mathcal{F}(M)/\mathrm{Ann}(m)$ is isomorphic to $\mathbb{R}$.

Assume that $\ker \beta \neq \mathrm{Ann}(m)$ for every $m \in M$. By maximality, neither set can be included in the other, and hence for every $m \in M$ there is a relatively compact open neighborhood U_m of m and $f_m \in \ker \beta$ such that $f_m | U_m > 0$. Let V_m be an open neighborhood of the closure, $\mathrm{cl}(U_m)$. Since M is paracompact, we can assume that $\{ V_m \mid m \in M \}$ is locally finite. Since M is second countable, M can be covered by $\{ V_{m(j)} \mid j \in \mathbb{N} \}$. Let $f_j = f_{m(j)}$ and let χ_j be bump functions which are equal to 1 on $\mathrm{cl}(U_{m(j)})$ and vanishing in $M \backslash V_{m(j)}$. If we have the inequality

$$a_n < \frac{1}{n^2 \sup\{ \chi_n(m) f_n^2(m) \mid m \in M \}},$$

then the function

$$f = \sum_{n \geq 1} a_n \chi_n f_n^2$$

is C^∞ (since the sum is finite in a neighborhood of every point), $f > 0$ on M, and the series defining f is uniformly convergent, being majorized by $\sum_{n \geq 1} n^{-2}$. If we can show that β can be taken inside the sum, then $\beta(f) = 0$. This construction then produces $f \in \ker \beta$, $f > 0$ and hence $1 = (1/f) f \in \ker \beta$; that is, $\ker \beta = \mathcal{F}(M)$, contradicting the hypothesis $\beta \neq 0$.

To show that β can be taken inside the series, it suffices to prove the following "g-estimate": for any $g \in \mathcal{F}(M)$,

$$|\beta(g)| \leq \sup\{ |g(m)| \mid m \in M \}.$$

Once this is done, then

$$\begin{aligned} \left| \sum_{m=1}^{N} \beta(a_n \chi_n f_n^2) - \beta(f) \right| &= \left| \beta \left(\sum_{m=1}^{N} a_n \chi_n f_n^2 - f \right) \right| \\ &\leq \sup \left| \sum_{m=1}^{N} a_n \chi_n f_n^2 - f \right| \to 0 \end{aligned}$$

as $n \to \infty$ by uniform convergence and boundedness of all functions involved. Thus $\beta(f) = \sum_{n \geq 1} \beta(a_n \chi_n f_n^2)$. To prove the g-estimate, let $\lambda > \sup\{ |g(m)| \mid m \in M \}$ so that $\lambda \pm g \neq 0$ on M; that is, $\lambda \pm g$ are both invertible functions on M. Since β is an algebra homomorphism, $0 \neq \beta(\lambda \pm g) = \lambda \pm \beta(g)$. Thus $\pm \beta(g) \neq \lambda$ for all $\lambda > \sup\{ |g(m)| \mid m \in M \}$. Hence we get the estimate

$$|\beta(g)| \leq \sup\{ |g(m)| \mid m \in M \}. \qquad \blacksquare$$

Remark. For infinite-dimensional manifolds, the proof of the lemma is almost identical, with the following changes: we work with $\beta : C^\infty(M, F) \to F$, absolute values are replaced by norms, second countability is in the hypothesis, and the neighborhoods V_m are chosen in such a way that $f_m | V_m$ is a bounded function (which is possible by continuity of f_m). ♦

Proof of existence in Theorem 4.2.36. For each $m \in M$, define the algebra homomorphism $\beta_m : \mathcal{F}(M) \to \mathbb{R}$ by $\beta_m(f) = \alpha(f)(m)$. Since α is invertible, $\alpha(1) \neq 0$ and since $\alpha(1) = \alpha(1^2) = \alpha(1)\alpha(1)$, we have $\alpha(1) = 1$. Thus $\beta_m \neq 0$ for all $m \in M$. By Lemma 4.2.37 there exists a unique point, which we call $\varphi(m) \in M$, such that $\beta_m(f) = f(\varphi(m)) = (\varphi^* f)(m)$. This defines a map $\varphi : M \to M$ such that $\alpha(f) = \varphi^* f$ for all $f \in \mathcal{F}(M)$. Since α is an automorphism, φ is bijective and since $\alpha(f) = \varphi^* f \in \mathcal{F}(M)$, $\alpha^{-1}(f) = \varphi_* f \in \mathcal{F}(M)$ for all $f \in \mathcal{F}(M)$, both φ, φ^{-1} are C^∞ (take for f any coordinate function multiplied by a bump function to show that in every chart the local representatives of φ, φ^{-1} are C^∞). ■

The proof of existence in the infinite-dimensional case proceeds in a similar way.

Supplement 4.2D
Derivations on C^r Functions

This supplement investigates to what extent vector fields and tangent vectors are characterized by their derivation properties on functions, if the underlying manifold is finite dimensional and of a finite differentiability class. We start by studying vector fields. Recall from Proposition 4.2.9 that a *derivation* θ is an $\mathbb{R}$-linear map from $\mathcal{F}^{k+1}(M)$ to $\mathcal{F}^k(M)$ satisfying the Leibniz rule, that is, $\theta(fg) = f\theta(g) + g\theta(f)$ for $f, g \in \mathcal{F}^{k+1}(M)$, if the differentiability class of M is at least $k+1$.

4.2.38 Theorem (A. Blass). *Let M be a C^{k+2} finite-dimensional manifold, where $k \geq 0$. The collection of all derivations θ from $\mathcal{F}^{k+1}(M)$ to $\mathcal{F}^k(M)$ is isomorphic to $\mathfrak{X}^k(M)$ as a real vector space.*

Proof. By the remark following Theorem 4.2.16, there is a unique C^k vector field X with the property that $\theta|\mathcal{F}^{k+2}(M) = \pounds_X|\mathcal{F}^{k+2}(M)$. Thus, all we have to do is show that θ and $\pounds_X$ agree on C^{k+1} functions. Replacing θ with $\theta - \pounds_X$, we can assume that θ annihilates all C^{k+2} functions and we want to show that it also annihilates all C^{k+1} functions. As in the proof of Theorem 4.2.16, it suffices to work in a chart, so we can assume without loss of generality that $M = \mathbb{R}^n$.

Let f be a C^{k+1} function and fix $p \in \mathbb{R}^n$. We need to prove that $(\theta f)(p) = 0$. For simplicity, we will show this for $p = 0$, the proof for general p following by centering the following arguments at p instead of 0. Replacing f by the difference between f and its Taylor polynomial of order $k+1$ about 0, we can assume $f(0) = 0$, and the first $k+1$ derivatives vanish at 0, since θ evaluated at the origin annihilates any polynomial. We shall prove that $f = g + h$, where g and h are two C^{k+1} functions, satisfying $g|U = 0$ and $h|V = 0$, where U, V are open sets such that $0 \in \mathrm{cl}(U) \bigcap \mathrm{cl}(V)$. Then, since θ is a local operator, $\theta g|U = 0$ and $\theta h|V = 0$, whence by continuity $\theta g|\mathrm{cl}(U) = 0$ and $\theta h|\mathrm{cl}(V) = 0$. Hence $(\theta f)(0) = (\theta g)(0) + (\theta h)(0) = 0$ and the theorem will be proved.

4.2.39 Lemma. *Let $\varphi : S^{n-1} \subset \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function and denote by $\pi : \mathbb{R}^n \backslash \{0\} \to S^{n-1}$, $\pi(x) = x/\|x\|$ the radial projection. Then for any positive integer r,*

$$\mathbf{D}^r(\varphi \circ \pi)(x) = \frac{(\psi \circ \pi)(x)}{\|x\|^r}$$

for some C^∞ function $\psi : S^{n-1} \to L^r_s(\mathbb{R}^n; \mathbb{R})$. In particular $\mathbf{D}^k(\varphi \circ \pi)(x) = O(\|x\|^{-r})$ as $\|x\| \to 0$.

Proof. For $r = 0$ choose $\varphi = \psi$. For $r = 1$, note that

$$\mathbf{D}\pi(x) \cdot v = -\frac{1}{\|x\|^2}\mathbf{D}\| \cdot \|(x) \cdot v + \frac{v}{\|x\|},$$

and so

$$\mathbf{D}(\varphi \circ \pi)(x) \cdot v = \frac{1}{\|x\|}\mathbf{D}\varphi(\pi(x)) \left(I - \frac{1}{\|x\|}\mathbf{D}\| \cdot \|(x) \right) \cdot v.$$

But the mapping $\ell'(x) = (1/\|x\|)\mathbf{D}\|\cdot\|(x)$ satisfies $\ell'(tx) = \ell'(x)$ for all $t > 0$ so that it is uniquely determined by $\ell = \ell'|S^{n-1}$. Hence

$$\mathbf{D}(\varphi \circ \pi)(x) = \frac{1}{\|x\|}(\psi \circ \pi)(x),$$

where $\psi(y) = \mathbf{D}\varphi(y) \cdot (I - \ell(y))$, $y \in S^{n-1}$. Now proceed by induction. ▼

Returning to the proof of the theorem, let f be as before, that is, of class C^{k+1} and $\mathbf{D}^i f(0) = 0$, $0 \leq i \leq k+1$, and let φ, π be as in the lemma. From Taylor's formula with remainder, we see that $\mathbf{D}^i f(x) = o(\|x\|^{k+1-i})$, $0 \leq i \leq k+1$, as $x \to 0$. Hence by the product rule and the lemma,

$$\mathbf{D}^i(f \cdot (\varphi \circ \pi))(x) = \sum_{j+\ell=i} o(\|x\|^{k+1-j}) O(\|x\|^{-\ell}) = o(\|x\|^{k+1-i})$$

so that $\mathbf{D}^i(f \cdot (\varphi \circ \pi))$, $0 \leq i \leq k+1$, can be continuously extended to 0, by making them vanish at 0. Therefore $f \cdot (g \circ \pi)$ is C^{k+1} for all $\mathbb{R}^n$.

Now choose the C^∞ function φ in the lemma to be zero on an open set O_1 and equal to 1 on an open set O_2 of S^{n-1}, $O_1 \cap O_2 = \varnothing$. Then the continuous extension g of $f \cdot (\varphi \circ \pi)$ to $\mathbb{R}^n$ is zero on $U = \pi^{-1}(O_1)$ and agrees with f on $V = \pi^{-1}(O_2)$. Let $h = f - g$ and thus f is the sum of two C^{k+1} functions, each of which vanishes in an open set having 0 in its closure. This completes the proof. ■

We do not know of an example of a derivation not given by a vector field on a C^1-manifold.

In infinite dimensions, the proof would require the norm of the model space to be C^∞ away from the origin and the function ψ in the lemma bounded with all derivatives bounded on the unit sphere. Unfortunately, this does not seem feasible under realistic hypotheses.

The foregoing proof is related to the method of "blowing-up" a singularity; see for example Takens [1974] and Buchner, Marsden, and Schecter [1983a]. There are also difficulties with this method in infinite dimensions in other problems, such as the Morse lemma (see Golubitsky and Marsden [1983] and Buchner, Marsden, and Schecter [1983b]).

4.2.40 Corollary. *Let M be a C^{k+1} finite-dimensional manifold. Then the only derivative from $\mathcal{F}^{k+1}(M)$ to $\mathcal{F}^k(M)$, where $l \leq k < \infty$, is zero.*

Proof. By the theorem, such a derivation is given by a C^{k-1} vector field X. If $X \neq 0$, then for some $f \in \mathcal{F}^{k+1}(M)$, $X[f]$ is only C^{k-1} but not C^k. ■

Next, we turn to the study of the relationship between tangent vectors and germ derivations. On $\mathcal{F}^k(M)$ consider the following equivalence relation: $f \sim_m g$ iff f and g agree on some neighborhood of $m \in M$. Equivalence classes of the relation $\sim_m$ are called ***germs*** at m; they form a vector space denoted by $\mathcal{F}^k_m(M)$. The differential $\mathbf{d}$ on functions clearly induces an $\mathbb{R}$-linear map, denoted by $\mathbf{d}_m$ on $\mathcal{F}^k_m(M)$ by $\mathbf{d}_m f = \mathbf{d}f(m)$, where we understand f on the left hand side as a germ. It is straightforward to see that $\mathbf{d}_m : \mathcal{F}^k_m(M) \to T^*_m M$ is $\mathbb{R}$-linear and satisfies the Leibniz rule. We say that an $\mathbb{R}$-linear map $\theta_m : \mathcal{F}^k_m(M) \to \mathbf{E}$, where $\mathbf{E}$ is a Banach space, is a ***germ derivation*** if θ_m satisfies the Leibniz rule. Thus, $\mathbf{d}_m$ is a $T^*_m M$-valued germ derivation.

Any tangent vector $v_m \in T_m M$ defines an $\mathbb{R}$-valued germ derivation by $v_m[f] = \langle \mathbf{d}f(m), v_m \rangle$. Conversely, localizing the statement and proof of Theorem 4.2.16(i) at m, we see that on a C^∞ finite-dimensional manifold, any $\mathbb{R}$-valued germ derivation at m defines a unique tangent vector, that is, *$T_m M$ is isomorphic to the vector space of $\mathbb{R}$-valued germ derivations on $\mathcal{F}_m(M)$*. The purpose of the rest of this supplement is to show that this result is false if M is a C^k-manifold. This is in sharp contrast to Theorem 4.2.38.

4.2.41 Theorem (Newns and Walker [1956]). *Let M be a finite dimensional C^k manifold, $l \leq k < \infty$. Then there are $\mathbb{R}$-valued germ derivations on $\mathcal{F}^k_m(M)$ which are not tangent vectors. In fact, the vector space of all $\mathbb{R}$-valued germ derivations on $\mathcal{F}^k_m(M)$ is* card($\mathbb{R}$)*-dimensional, where* card($\mathbb{R}$) *is the cardinality of the continuum.*

For the proof, we start with algebraic characterizations of T_mM and the vector space of all germ derivations.

4.2.42 Lemma. *Let*

$$\mathcal{F}^k_{m,0}(M) = \{\, f \in \mathcal{F}^k_m(M) \mid \mathbf{d}f(m) = 0 \,\}.$$

Then

$$\mathcal{F}^k_m(M)/\mathcal{F}^k_{m,0}(M) \text{ is isomorphic to } T^*_mM.$$

Therefore, since M is finite dimensional

$$(\mathcal{F}^k_m(M)/\mathcal{F}^k_{m,0}(M))^* \text{ is isomorphic to } T_mM.$$

Proof. The isomorphism of $\mathcal{F}^k_m(M)/\mathcal{F}^k_{m,0}(M)$ with T^*_mM is given by class of $(f) \mapsto \mathbf{d}f(m)$; this is a direct consequence of Corollary 4.2.14. ▼

4.2.43 Lemma. *Let $\mathcal{F}^k_{m,d}(M) = \operatorname{span}\{1, fg \perp f, g \in \mathcal{F}^k_m(M), f(m) = g(m) = 0\}$. Then the space of $\mathbb{R}$-linear germ derivations on $\mathcal{F}^k_m(M)$ is isomorphic to $(\mathcal{F}^k_m(M)/\mathcal{F}^k_{m,d}(M))^*$.*

Proof. Clearly, if θ_m is a germ derivation $\theta_m(1) = 0$ and $\theta_m(fg) = 0$ for any $f, g \in \mathcal{F}^k_m(M)$ with $f(m) = g(m) = 0$, so that θ_m defines a linear functional on $\mathcal{F}^k_m(M)$ which vanishes on the space $\mathcal{F}^k_{m,d}(M)$. Conversely, if λ is a linear functional on $\mathcal{F}^k_m(M)$ vanishing on $\mathcal{F}^k_{m,d}(M)$, then λ is a germ derivation, for if $f, g \in \mathcal{F}^k_m(M)$, we have

$$fg = (f - f(m))(g - g(m)) + f(m)g + g(m)f - f(m)g(m)$$

so that

$$\lambda(fg) = f(m)\lambda(g) + g(m)\lambda(f),$$

that is, the Leibniz rule holds. ▼

4.2.44 Lemma. *All germs in $\mathcal{F}^k_{m,d}(M)$ have $k+1$ derivatives at m (even though M is only of class C^k).*

Proof. Since any element of $\mathcal{F}^k_{m,d}(M)$ is of the form $a + bfg$, $f, g \in \mathcal{F}^k_m(M)$, $f(m) = g(m) = 0$, $a, b \in \mathbb{R}$, it suffices to prove the statement for fg. Passing to local charts, we have by the Leibniz rule

$$\mathbf{D}^k(fg) = (\mathbf{D}^k f)g + f(\mathbf{D}^k g) + \varphi,$$

for

$$\varphi = \sum_{i=1}^{k-1} (\mathbf{D}^i f)(\mathbf{D}^{k-i} g).$$

Clearly φ is C^1, since the highest order derivative in the expression of φ is $k-1$ and f, g are C^k. Moreover, since $\mathbf{D}^k f$ is continuous and $g(m) = 0$, using the definition of the derivative it follows that $\mathbf{D}[(\mathbf{D}^k f)g](m) = (\mathbf{D}^k f)(m)(\mathbf{D}g)(m)$. Therefore, fg has $k+1$ derivatives at m. ▼

Proof of Theorem 4.2.41. We clearly have

$$\mathcal{F}^k_{m,d}(M) \subset \mathcal{F}^k_{m,0}(M).$$

Choose a chart $(x^1, \ldots, x^n)$ at m, $x^i(m) = 0$, and consider the functions $|x^i|^{k+\varepsilon}$, $0 < \varepsilon < 1$. These functions are clearly in $\mathcal{F}^k_{m,0}(M)$, but are not in $\mathcal{F}^k_{m,d}(M)$ by Lemma 4.2.44, since they cannot be differentiated $k+1$

times at m. Therefore, $\mathcal{F}^k_{m,d}(M)$ is strictly contained in $\mathcal{F}^k_{m,0}(M)$ and thus T_mM is a strict subspace of the vector space of germ differentations on $\mathcal{F}^k_m(M)$ by Lemmas 4.2.42 and 4.2.43.

The functions $|x^i|^{k+\varepsilon}$, $0 < \varepsilon < 1$ are linearly independent in $\mathcal{F}^k_m M$ modulo $\mathcal{F}^k_{m,d}(M)$, because only a trivial linear combination of such functions has derivatives of order $k+1$ at m. Therefore, the dimension of $\mathcal{F}^k_m(M)/\mathcal{F}^k_{m,d}(M)$ is at least $\operatorname{card}(\mathbb{R})$. Since $\operatorname{card}(\mathcal{F}^k_m(M)) = \operatorname{card}(\mathbb{R})$, it follows that $\dim(\mathcal{F}^k_m(M)/\mathcal{F}^k_{m,d}(M)) = \operatorname{card}(\mathbb{R})$. Consequently, its dual, which by Lemma 4.2.43 coincides with the vector space of germ-derivations at m, also has dimension $\operatorname{card}(\mathbb{R})$. ■

Exercises

◇ **4.2-1.** (i) On $\mathbb{R}^2$, let $X(x, y) = (x, y; y, -x)$. Find the flow of X.

(ii) Solve the following for $f(t, x, y)$:

$$\frac{\partial f}{\partial t} = y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}$$

if $f(0, x, y) = y \sin x$.

◇ **4.2-2.** (i) Let X and Y be vector fields on M with complete flows F_t and G_t, respectively. If $[X, Y] = 0$, show that $X + Y$ has flow $H_t = F_t \circ G_t$. Is this true if X and Y are time dependent?

(ii) Show that if $[X, Y] = 0$ for all $Y \in \mathfrak{X}(M)$, then $X = 0$.

Hint: From the local formula conclude first that X is constant; then take for Y linear vector fields and apply the Hahn–Banach theorem. In infinite dimensions, assume the conditions hold locally or that the model spaces are C^∞.

◇ **4.2-3.** Show that, under suitable hypotheses, that the solution $f(x, t) = g(F_t(x))$ of problem (P) given in Proposition 4.2.11 is unique.
Hint: Consider the function

$$E(t) = \int_{\mathbb{R}^n} |f_1(x, t) - f_2(x, t)|^2 \, dx$$

where f_1 and f_2 are two solutions. Show that $dE/dt \leq \alpha E$ for a suitable constant α and conclude by Gronwall's inequality that $E = 0$. The "suitable hypotheses" are conditions that enable integration by parts to be performed in the computation of dE/dt.

Adapt this proof to get uniqueness of the solution in Proposition 4.2.28.

◇ **4.2-4.** Let $X, Y \in \mathfrak{X}(M)$ have flows F_t and G_t, respectively. Show that

$$[X, Y] = \left.\frac{d}{dt}\frac{d}{ds}\right|_{t,s=0} (F_{-t} \circ G_s \circ F_t).$$

Hint: The flow of F_t^*Y is $s \mapsto F_{-t} \circ G_s \circ F_t$.

◇ **4.2-5.** Show that $\mathrm{SO}(n)$ is parallelizable. See Exercise 3.5-19 for a proof that $\mathrm{SO}(n)$ is a manifold.
Hint: $\mathrm{SO}(n)$ is a group.

◇ **4.2-6.** Solve the following system of partial differential equations.

$$\frac{\partial Y^1}{\partial t} = (x+y)\frac{\partial Y^1}{\partial x} + (4x - 2y)\frac{\partial Y^1}{\partial y} - Y^1 - Y^2,$$
$$\frac{\partial Y^2}{\partial t} = (x+y)\frac{\partial Y^2}{\partial x} + (4x - 2y)\frac{\partial Y^2}{\partial y} - 4Y^1 + 2Y^2,$$

with initial conditions $Y^1(x, y, 0) = x + y$, $Y^2(x, y, 0) = x^2$.
HINT: The flow of the vector field $(x + y, 4x - 2y)$ is

$$(x, y) \mapsto \left(\frac{1}{5}(x - y)e^{-3t} + \frac{1}{5}(4x + y)e^{2t}, -\frac{4}{5}(x - y)e^{-3t} + \frac{1}{5}(4x + y)e^{2t} \right).$$

◇ **4.2-7.** Consider the following equation for $f(x, t)$ in ***divergence form***:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(H(f)) = 0$$

where H is a given function of f. Show that the characteristics are given by $\dot{x} = -H'(f)$. What does the transversality condition discussed in Supplement 4.2B become in this case?

◇ **4.2-8.** Let M and N be manifolds with N modeled on a Banach space which has a C^k norm away from the origin. Show that a given mapping $\varphi : M \to N$ is C^k iff $f \circ \varphi : M \to \mathbb{R}$ is C^k for all $f \in \mathcal{F}^k(N)$.

◇ **4.2-9.** Develop a product formula like that in Supplement 4.1A for the flow of $X + Y$ for time-dependent vector fields.
HINT: You will have to consider ***time-ordered*** products.

◇ **4.2-10** (Newns and Walker [1956]). In the terminology of Supplement 4.2D, consider a C^0-manifold modeled on $\mathbb{R}^n$. Show that any germ derivation is identically zero.
HINT: Write any $f \in \mathcal{F}^0_m(M)$, $f = f(m) + (f - f(m))^{1/3}(f - f(m))^{exc:2.3-27}$ and apply the derivation.

◇ **4.2-11** (More on the Lie bracket as a "commutator"). Let M be a manifold, $m \in M$, $v \in T_mM$. Recall that T_mM is a submanifold of $T(TM)$ and that $T_v(T_mM)$ is canonically isomorphic to T_mM. Also recall from Exercise 3.3-2 that on $T(TM)$ there is a canonical involution $s_M : T(TM) \to T(TM)$ satisfying $s_M \circ s_M =$ identity on $T(TM)$, $T\tau_M \circ s_M = \tau_{TM}$, and $\tau_{TM} \circ s_M = T\tau_M$, where $\tau_M : TM \to M$ and $\tau_{TM} : T(TM) \to TM$ are the canonical tangent bundle projections. Let $X, Y \in \mathfrak{X}(M)$ and denote by $TX, TY : TM \to T(TM)$ their tangent maps. Prove the following formulae for the Lie bracket:

$$\begin{aligned} [X, Y](m) &= s_M(T_mY(X(m))) - T_mX(Y(m)) \\ &= T_mY(X(m)) - s_M(T_mX(Y(m))), \end{aligned}$$

where the right hand sides, belonging to $T_{X(m)}(T_mM)$ and $T_{Y(m)}(T_mM)$ respectively, are thought of as elements of T_mM.
HINT: Show that $T_m\tau_M$ of the right hand sides is zero which proves that the right hand sides are not just elements of $T_{X(m)}(TM)$ and $T_{Y(m)}(TM)$ respectively, but of the indicated spaces. Then pass to a local chart.

4.3 An Introduction to Dynamical Systems

We have seen quite a bit of theoretical development concerning the interplay between the two aspects of vector fields, namely as differential operators and as ordinary differential equations. It is appropriate now to look a little more closely at geometric aspects of flows.

Much of the work in this section holds for infinite-dimensional as well as finite-dimensional manifolds. The reader who knows or is willing to learn some spectral theory from functional analysis can make the generalization.

This section is intended to link up the theory of this book with courses in ordinary differential equations that the reader may have taken. The section will be most beneficial if it is read with such a course in mind. We begin by introducing some of the most basic terminology regarding the stability of fixed points.

4.3.1 Definition. *Let X be a C^1 vector field on an n-manifold M. A point m is called a* ***critical point*** *(also called a* ***singular point*** *or an* ***equilibirum point****) of X if $X(m) = 0$. The* ***linearization*** *of X at a critical point m is the linear map*

$$X'(m) : T_mM \to T_mM$$

defined by

$$X'(x) \cdot v = \frac{d}{dt}(TF_t(m) \cdot v)\bigg|_{t=0}$$

where F_t is the flow of X. The eigenvalues (points in the spectrum) of $X'(m)$ are called ***characteristic exponents*** *of X at m.*

Some remarks will clarify this definition. F_t leaves m fixed: $F_t(m) = m$, since $c(t) \equiv m$ is the unique integral curve through m. Conversely, it is obvious that if $F_t(m) = m$ for all t, then m is a critical point. Thus T_mF_t is a linear map of T_mM to itself and so its t-derivative at 0, producing another linear map of T_mM to itself, makes sense.

4.3.2 Proposition. *Let m be a critical point of X and let (U, φ) be a chart on M with $\varphi(m) = x_0 \in \mathbb{R}^n$. Let $x = (x^1, \ldots, x^n)$ denote coordinates in $\mathbb{R}^n$ and $X^i(x^1, \ldots, x^n)$, $i = 1, \ldots, n$, the components of the local representative of X. Then the matrix of $X'(m)$ in these coordinates is*

$$\left[\frac{\partial X^i}{\partial x^j}\right]_{x=x_0}.$$

Proof. This follows from the equations

$$X^i(F_t(x)) = \frac{d}{dt}F_t^i(x)$$

after differentiating in x and setting $x = x_0$, $t = 0$. ■

The name "characteristic exponent" arises as follows. We have the linear differential equation

$$\frac{d}{dt}T_mF_t = X'(x) \circ T_mF_t$$

and so

$$T_mF_t = e^{tX'(m)}.$$

Here the exponential is defined, for example, by a power series. The actual computation of these exponentials is learned in differential equations courses, using the Jordan canonical form. (See Hirsch and Smale [1974], for instance.) In particular, if $\mu_1, \ldots, \mu_n$ are the characterisitc exponents of X at m, the eigenvalues of T_mF_t are

$$e^{t\mu_1}, \ldots, e^{t\mu_n}.$$

The characteristic exponents will be related to the following notion of stability of a critical point.

4.3.3 Definition. *Let m be a critical point of X. Then*

(i) *m is* ***stable*** *(or* ***Liapunov stable****) if for any neighborhood U of m, there is a neighborhood V of m such that if $m' \in V$, then m' is $+$ complete and $F_t(m') \in U$ for all $t \geq 0$. (See Figure* 4.3.1(a).)

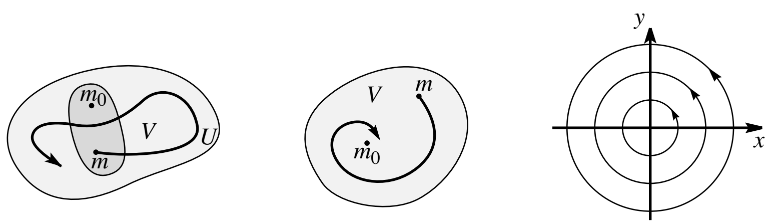
(ii) *m is **asymptotically stable** if there is a neighborhood V of m such that if $m' \in V$, then m is + complete, $F_t(V) \subset F_s(V)$ if $t > s$ and*

$$\lim_{t \to +\infty} F_t(V) = \{m\},$$

that is, for any neighborhood U of m, there is a T such that $F_t(V) \subset U$ if $t \geq T$. (See Figure 4.3.1(b).)

It is obvious that asymptotic stability implies stability. The harmonic oscillator $\ddot{x} = -x$ giving a flow in the plane shows that stability need not imply asymptotic stability (Figure 4.3.1(c)).

4.3.4 Theorem (Liapunov's Stability Criterion)**.** *Suppose X is C^1 and m is a critical point of X. Assume the spectrum of $X'(m)$ is strictly in the left half plane.* (*In finite dimensions, the characteristic exponents of m have negative real parts.*) *Then m is asymptotically stable.* (*In a similar way, if* $\mathrm{Re}(\mu_i) > 0$, *m is asymptotically unstable, that is, asymptotically stable as $t \to -\infty$.*)



(a) Stable (b) Asymptotically stable (c) Harmonic oscillator

FIGURE 4.3.1. Stability of equilibria

The proof we give requires some spectral theory that we shall now review. For the finite dimensional case, consult the exercises. This proof in fact can be adapted to work for many partial differential equations (see Marsden and Hughes [1983, Chapters 6, 7, and p. 483]).

Let $T : \mathbf{E} \to \mathbf{E}$ be a bounded linear operator on a Banach space $\mathbf{E}$ and let $\sigma(T)$ denote its spectrum; that is,

$$\sigma(T) = \{\, \lambda \in \mathbb{C} \mid T - \lambda I \text{ is not invertible on the complexification of } \mathbf{E} \,\}.$$

Then $\sigma(T)$ is non-empty, is compact, and for $\lambda \in \sigma(T)$, $|\lambda| \leq \|T\|$. Let $r(T)$ denote its ***spectral radius***, defined by $r(T) = \sup\{\, |\lambda| \mid \lambda \in \sigma(T) \,\}$.

4.3.5 Theorem (Spectral Radius Formula)**.**

$$r(t) = \lim_{n \to \infty} \|T^n\|^{1/n}.$$

The proof is analogous to the formula for the radius of convergence of a power series and can be supplied without difficulty; cf. Rudin [1973, p. 355]. The following lemma is also not difficult and is proved in Rudin [1973] and Dunford and Schwartz [1963].

4.3.6 Theorem (Spectral Mapping Theorem)**.** *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function and define

$$f(T) = \sum_{n=0}^{\infty} a_n T^n.$$

Then $\sigma(f(T)) = f(\sigma(T))$.

4.3.7 Lemma. *Let $T : \mathbf{E} \to \mathbf{E}$ be a bounded linear operator on a Banach space $\mathbf{E}$. Let r be any number greater than $r(T)$, the spectral radius of T. Then there is a norm $|\cdot|$ on $\mathbf{E}$ equivalent to the original norm such that $|T| \leq r$.*

Proof. From the spectral radius formula, we get $\sup_{n\geq 0}(\|T^n\|/r^n) < \infty$, so if we define

$$|x| = \sup_{n\geq 0} \frac{\|T^n(x)\|}{r^n},$$

then $|\ |$ is a norm and

$$\|x\| \leq |x| \leq \left(\sup_{n\geq 0} \frac{\|T^n\|}{r^n}\right) \|x\|.$$

Hence

$$|T(x)| = \sup_{n\geq 0} \frac{\|T^{n+1}(x)\|}{r^n} = r \sup_{n\geq 0} \frac{\|T^{n+1}(x)\|}{r^{n+1}} \leq r|x|. \qquad \blacksquare$$

4.3.8 Lemma. *Let $A : \mathbf{E} \to \mathbf{E}$ be a bounded operator on $\mathbf{E}$ and let $r > \sigma(A)$ (i.e., if $\lambda \in \sigma(A)$, $\mathrm{Re}(\lambda) > r$). Then there is an equivalent norm $|\cdot|$ on $\mathbf{E}$ such that for $t \geq 0$, $|e^{tA}| \leq e^{rt}$.*

Proof. Using Theorem 4.3.6, e^{rt} is $\geq$ spectral radius of e^{tA}; that is, $e^{rt} \geq lim_{n\to\infty}\|e^{ntA}\|^{1/n}$. Set

$$|x| = \sup_{n\geq 0, t\geq 0} \frac{\|e^{ntA}(x)\|}{e^{rnt}}$$

and proceed as in Lemma 4.3.7. ■

There is an analogous lemma if $r < \sigma(A)$, giving $|e^{tA}| \geq e^{rt}$.

4.3.9 Lemma. *Let $T : \mathbf{E} \to \mathbf{E}$ be a bounded linear operator. Let $\sigma(T) \subset \{ z \mid \mathrm{Re}(z) < 0 \}$ (resp., $\sigma(T) \subset \{ z \mid \mathrm{Re}(z) > 0 \}$). Then the origin is an attracting (resp., repelling) fixed point for the flow $\varphi_t = e^{tT}$ of T.*

Proof. If $\sigma(T) \subset \{ z \mid \mathrm{Re}(z) < 0 \}$, there is an $r < 0$ with $\sigma(T) < r$, as $\sigma(T)$ is compact. Thus by Lemma 4.3.8, $|e^{tA}| \leq e^{rt} \to 0$ as $t \to +\infty$. ■

Proof of Liapunov's Stability Criterion Theorem 4.3.4. Without loss of generality, we can assume that M is a Banach space $\mathbf{E}$ and that $m = 0$. As above, renorm $\mathbf{E}$ and find $\varepsilon > 0$ such that $\|e^{tA}\| \leq e^{-\varepsilon t}$, where $A = X'(0)$.

From the local existence theory, there is an r-ball about 0 for which the time of existence is uniform if the initial condition x_0 lies in this ball. Let

$$R(x) = X(x) - \mathbf{D}X(0) \cdot x.$$

Find $r_2 \leq r$ such that $\|x\| \leq r_2$ implies $\|R(x)\| \leq \alpha\|x\|$, where $\alpha = \varepsilon/2$.

Let D be the open $r_2/2$ ball about 0. We shall show that if $x_0 \in D$, then the integral curve starting at x_0 remains in D and $\to 0$ exponentially as $t \to +\infty$. This will prove the result. Let $x(t)$ be the integral curve of X starting at x_0. Suppose $x(t)$ remains in D for $0 \leq t < T$. The equation

$$x(t) = x_0 + \int_0^t X(x(s))\, ds = x_0 + \int_0^t [Ax(s) + R(x(s))]\, ds$$

gives, by the variation of constants formula (Exercise 4.1-5),

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} R(x(s))\, ds,$$

and so

$$\|x(t)\| \le e^{-t\varepsilon}\|x_0\| + \alpha \int_0^t e^{-(t-s)\varepsilon}\|x(s)\|\, ds.$$

Letting $f(t) = e^{t\varepsilon}\|x(t)\|$, the previous inequality becomes

$$f(t) \le \|x_0\| + \alpha \int_0^1 f(s)\, ds,$$

and so, by Gronwall's inequality, $f(t) \le \|x_0\| e^{\alpha t}$. Thus

$$\|x(t)\| \le \|x_0\| e^{(\alpha-\varepsilon)t} = \|x_0\| e^{-\varepsilon t/2}.$$

Hence $x(t) \in D$, $0 \le t < T$, so as in Proposition 4.1.19, $x(t)$ may be indefinitely extended in t and the foregoing estimate holds. ■

One can also show that if M is finite dimensional and m is a stable equilibrium, then no eigenvalue of $X'(m)$ has strictly positive real part; see Hirsch and Smale [1974, pp. 187–190] and the remarks below on invariant manifolds. See Hille and Phillips [1957], and Curtain and Pritchard [1977] for the infinite dimensional linear case.

Another method of proving stability is to use Liapunov functions.

4.3.10 Definition. *Let $X \in \mathfrak{X}^r(M)$, $r \ge 1$, and let m be an equilibrium solution for X, that is, $X(m) = 0$. A **Liapunov function** for X at m is a continuous function $L : U \to \mathbb{R}$ defined on a neighborhood U of m, differentiable on $U \backslash \{m\}$, and satisfying the following conditions:*

(i) *$L(m) = 0$ and $L(m') > 0$ if $m' \ne m$;*

(ii) *$X[L] \le 0$ on $U \backslash \{m\}$;*

(iii) *there is a connected chart $\varphi : V \to \mathbf{E}$ where $m \in V \subset U$, $\varphi(m) = 0$, and an $\varepsilon > 0$ satisfying $B_\varepsilon(0) = \{ x \in \mathbf{E} \mid \|x\| \le \varepsilon \} \subset \varphi(V)$, such that for all $0 < \varepsilon' \le \varepsilon$,*

$$\inf\{ L(\varphi^{-1}(x)) \mid \|x\| = \varepsilon' \} > 0.$$

*The Liapunov function L is said to be **strict**, if* (ii) *is replaced by* (ii)$'$ *$X[L] < 0$ in $U \backslash \{m\}$.*

Conditions (i) and (iii) are called the ***potential well hypothesis***. In finite dimensions, (iii) follows automatically from compactness of the sphere of radius ε' and (i). By the Lie derivative formula, condition (ii) is equivalent to the statement: L is decreasing along integral curves of X.

4.3.11 Theorem. *Let $X \in \mathfrak{X}^r(M)$, $r \ge 1$, and m be an equilibrium of X. If there exists a Liapunov function for X at m, then m is stable.*

Proof. Since the statement is local, we can assume M is a Banach space $\mathbf{E}$ and $m = 0$. By Lemma 4.1.8, there is a neighborhood U of 0 in $\mathbf{E}$ such that all solutions starting in U exist for time $t \in [-\delta, \delta]$, with δ depending only on X and U, but not on the solution. Now fix $\varepsilon > 0$ as in (iii) such that the open ball $D_\varepsilon(0)$ is included in U. Let $\rho(\varepsilon) > 0$ be the minimum value of L on the sphere of radius ε, and define the open set $U' = \{ x \in D_\varepsilon(0) \mid L(x) < \rho(\varepsilon) \}$. By (i), $U' \ne \varnothing$, $0 \in U'$, and by (ii), no solution starting in U' can meet the sphere of radius ε (since L is decreasing on integral curves of X). Thus all solutions starting in U' never leave $D_\varepsilon(0) \subset U$ and therefore by uniformity of time of existence, these solutions can be extended indefinitely in steps of δ time units. This shows 0 is stable. ■

Note that if $\mathbf{E}$ is finite dimensional, the proof can be done without appeal to Lemma 4.1.8: since the closed ε-ball is compact, solutions starting in U' exist for all time by Proposition 4.1.19.

4.3.12 Theorem. *Let $X \in \mathfrak{X}^r(M)$, $r \geq 1$, m be an equilibrium of X, and L a strict Liapunov function for X at m. Then m is asymptotically stable if any one of the following conditions hold:*

(i) *M is finite dimensional*;

(ii) *solutions starting near m stay in a compact set* (*i.e., trajectories are precompact*);

(iii) *in a chart $\varphi : V \to \mathbf{E}$ satisfying* (iii) *in Definition* 4.3.10 *the following inequality is valid for some constant $a > 0$*

$$X[L](x) \leq -a\|X(x)\|.$$

Proof. We can assume $M = \mathbf{E}$, and $m = 0$. By Theorem 4.3.11, 0 is stable, so if t_n is an increasing sequence, $t_n \to \infty$, and $x(t)$ is an integral curve of X starting in U' (see the proof of Theorem 4.3.11), the sequence $\{x(t_n)\}$ in $\mathbf{E}$ has a convergent subsequence in cases (i) and (ii). Let us show that under hypothesis (iii), $\{x(t_n)\}$ is Cauchy, so by completeness of $\mathbf{E}$ it is convergent. For $t > s$, the inequality

$$L(x(t)) - L(x(s)) = \int_s^t X[L](x(\lambda))\,d\lambda \leq -a\int_s^t \|X(x(\lambda))\| < 0,$$

implies that

$$\begin{aligned} L(x(s)) - L(x(t)) &\geq a\int_s^t \|X(x(\lambda))\|\,d\lambda \\ &\geq a\left\|\int_s^t X(x(\lambda))\,d\lambda\right\| \\ &= a\|x(t) - x(s)\|, \end{aligned}$$

which together with the continuity of $\lambda \mapsto L(x(\lambda))$ shows that $\{x(t_n)\}$ is a Cauchy sequence. Thus, in all three cases, there is a sequence $t_n \to +\infty$ such that $x(t_n) \to x_0 \in D_\varepsilon(0)$, $D_\varepsilon(0)$ being given in the proof of Theorem 4.3.11. We shall prove that $x_0 = 0$. Since $L(x(t))$ is a *strictly* decreasing function of t by (ii)$'$, $L(x(t)) > L(x_0)$ for all $t > 0$. If $x_0 \neq 0$, let $c(t)$ be the solution of X starting at x_0, so that $L(c(t)) < L(x_0)$, again since $t \mapsto L(x(t))$ is strictly decreasing. Thus, for any solution $\tilde{c}(t)$ starting close to x_0, $L(\tilde{c}(t)) < L(x_0)$ by continuity of L. Now take $\tilde{c}(0) = x(t_n)$ for n large to get the contradiction $L(x(t_n+t)) < L(x_0)$. Therefore $x_0 = 0$ is the only limit point of $\{\, x(t) \mid t \geq 0 \,\}$ if $x(0) \in U'$, that is, 0 is asymptotically stable. ■

The method of Theorem 4.3.12 can be used to detect the instability of equilibrium solutions.

4.3.13 Theorem. *Let m be an equilibrium point of $X \in \mathfrak{X}^r(M)$, $r \geq 1$. Assume there is a continuous function $L : U \to M$ defined in a neighborhood of U of m, which is differentiable on $U\backslash\{m\}$, and satisfyies $L(m) = 0$, $X[L] \geq a > 0$* (*respectively, $\leq a < 0$*) *on $U\backslash\{m\}$. If there exists a sequence $m_k \to m$ such that $L(m_k) > 0$* (*respectively, < 0*), *then m is unstable.*

Proof. We need to show that there is a neighborhood W of m such that for any neighborhood V of m, $V \subset U$, there is a point m_V whose integral curve leaves W. Since m is an equilibrium, by Corollary 4.1.25, there is a neighborhood $W_1 \subset U$ of m such that each integral curve starting in W_1 exists for time at least $1/a$. Let $W = \{\, m \in W_1 \mid L(m) < 1/2 \,\}$. We can assume $M = \mathbf{E}$, and $m = 0$. Let $c_n(t)$ denote the integral curve of X with initial condition $m_n \in W$. Then

$$L(c_n(t)) - L(m_n) = \int_0^t X[L](c_n(\lambda))\,d\lambda \geq at$$

so that

$$L(c_n(1/a)) \geq 1 + L(m_n) > 1,$$

that is, $c_n(1/a) \notin W$. Thus all integral curves starting at the points $m_n \in W$ leave W after time at most $1/a$. Since $m_n \to 0$, the origin is unstable. ■

Note that if M is finite dimensional, the condition $X[L] \geq a > 0$ can be replaced by the condition $X[L] > 0$; this follows, as usual, by local compactness of M.

4.3.14 Examples.

A. The vector field

$$X(x,y) = (-y - x^5)\frac{\partial}{\partial x} + (x - 2y^3)\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$$

has the origin as an isolated equilibrium. The characteristic exponents of X at $(0,0)$ are $\pm i$ and so Liapunov's Stability Criterion (Theorem 4.3.4) does not give any information regarding the stability of the origin. If we suspect that $(0,0)$ is asymptotically stable, we can try searching for a Liapunov function of the form $L(x,y) = ax^2 + by^2$, so we need to determine the coefficients $a, b, \neq 0$ in such a way that $X[L] < 0$. We have

$$X[L] = 2ax(-y - x^5) + 2by(x - 2y^3) = 2xy(b - a) - 2ax^6 - 4by^4,$$

so that choosing $a = b = 1$, we get $X[L] = -2(x^6 + 2y^4)$ which is strictly negative if $(x,y) \neq (0,0)$. Thus the origin is asymptotically stable by Theorem 4.3.12.

B. Consider the vector field

$$X(x,y) = (-y + x^5)\frac{\partial}{\partial x} + (x + 2y^3)\frac{\partial}{\partial y}$$

with the origin as an isolated critical point and characteristic exponents $\pm i$. Again Liapunov's Stability Criterion cannot be applied, so that we search for a function $L(x,y) = ax^2 + by^2$, $a, b \neq 0$ in such a way that $X[L]$ has a definite sign. As above we get

$$X[L] = 2ax(-y + x^5) + 2by(x + 2y^3) = 2xy(b - a) + 2ax^6 + 4by^4,$$

so that choosing $a = b = 1$, it follows that $X[L] = 2(x^6 + y^4) > 0$ if $(x,y) \neq (0,0)$. Thus, by Theorem 4.3.13, the origin is unstable.

These two examples show that if the spectrum of X lies on the imaginary axis, the stability nature of the equilibrium is determined by the nonlinear terms.

C. Consider Newton's equations in $\mathbb{R}^3$, $\ddot{\mathbf{q}} = -(1/m)\nabla V(\mathbf{q})$ written as a first order system $\ddot{\mathbf{q}} = \mathbf{v}, \ddot{\mathbf{v}} = -(1/m)\nabla V(\mathbf{q})$ and so define a vector field X on $\mathbb{R}^3 \times \mathbb{R}^3$. Let $(\mathbf{q}_0, \mathbf{v}_0)$ be an equilibrium of this system, so that $\mathbf{v}_0 = \mathbf{0}$ and $\nabla V(\mathbf{q}_0) = \mathbf{0}$. In Example 4.1.23B we saw that the total energy

$$E(\mathbf{q}, \mathbf{v}) = \frac{1}{2}m\|\mathbf{v}\|^2 + V(\mathbf{q})$$

is conserved, so we try to use E to construct a Liapunov function L. Since $L(\mathbf{q}_0, \mathbf{0}) = 0$, define

$$L(\mathbf{q}, \mathbf{v}) = E(\mathbf{q}, \mathbf{v}) - E(\mathbf{q}_0, \mathbf{0}) = \frac{1}{2}m\|\mathbf{v}\|^2 + V(\mathbf{q} - V(\mathbf{q}_0),$$

which satisfies $X[L] = 0$ by conservation of energy. If $V(\mathbf{q}) > V(\mathbf{q}_0)$ for $\mathbf{q} \neq \mathbf{q}_0$, then L is a Liapunov function. Thus we have proved

Lagrange's Stability Theorem: *an equilibrium point $(\mathbf{q}_0, \mathbf{0})$ of Newton's equations for a particle of mass m, moving under the influence of a potential V, which has a local absolute minimum at $\mathbf{q}_0$, is stable.*

D. *Let* $\mathbf{E}$ *be a Banach space and* $L : \mathbf{E} \to \mathbb{R}$ *be* C^2 *in a neighborhood of* 0. *If* $\mathbf{D}L(0) = 0$ *and there is a constant* $c > 0$ *such that* $\mathbf{D}^2L(0)(e, e) > c\|e\|^2$ *for all* e, *then* 0 *lies in a potential well for* L (*i.e.,* (ii) *and* (iii) *of Definition* 4.3.10 *hold*). Indeed, by Taylor's theorem 2.4.15,

$$L(h) - L(0) = \frac{1}{2}\mathbf{D}^2L(0)(h, h) + o(h^2) \geq c\frac{\|h\|^2}{2} + o(h^2).$$

Thus, if $\delta > 0$ is such that for all $\|h\| < \delta$, $|o(h^2)| \leq c\|h\|^2/4$, then $L(h) - L(0) > c\|h\|^2/4$, that is,

$$\inf_{\|h\|=\varepsilon}[L(h) - L(0)] \geq \frac{c\varepsilon}{4}$$

for $\varepsilon < \delta$. ♦

In many basic infinite dimensional examples, some technical sharpening of the preceding ideas is necessary for them to be applicable. We refer the reader to LaSalle [1976], Marsden and Hughes [1983, Section 6.6], Hale, Magalhaes and Oliva [1984], and Holm, Marsden, Ratiu, and Weinstein [1985] for more information.

Next we turn to cases where the equilibirum need not be stable.

A critical point is called ***hyperbolic*** or ***elementary*** if none of its characteristic exponents has zero real part. A generalization of Liapunov's theorem called the ***Hartman–Grobman theorem*** shows that near a hyperbolic critical point the flow looks like that of its linearization. (See Hartman [1973, Chapter 9] and Nelson [1969, Chapter 3], for proofs and discussions.) In the plane, the possible hyperbolic flows near a critical point are summarized in the table below and shown in Figure 4.3.2. (Remember that for real systems, the characteristic exponents occur in conjugate pairs.)

Eigenvalues	**Real Jordan form**	**Name**	**Part of Fig. 4.3.2**
$\lambda_1 < 0 < \lambda_2$	$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	saddle	(a)
$\lambda_1 < \lambda_2 < 0$		stable focus	(b)
$\lambda_1 = \lambda_2 < 0$		stable node	(c)
$\lambda_1 = \lambda_2 < 0$	$\begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix}$	stable improper node	(d)
$\lambda_1 = a + ib,\ a < 0$ $\lambda_2 = a - ib,\ b \neq 0$	$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$	stable spiral sink	(e)
$\lambda_1 = ib,\ \lambda_2 = -ib,$ $b \neq 0$	$\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$	center	(f)

In cases 1 to 5, all arrows in the phase portraits are reversed and "stable" is replaced by "unstable," if the signs of λ_1, λ_2, are changed. In the original coordinate system (x^1, x^2) all phase portraits in Figure 4.3.2 are rotated and sheared.

To understand the higher dimensional case, a little more spectral theory is required.

4.3.15 Lemma. *Suppose* $\sigma(T) = \sigma_1 \cup \sigma_2$ *where* $d(\sigma_1, \sigma_2) > 0$. *Then there are unique* T*-invariant subspaces* $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *such that* $\mathbf{E} = \mathbf{E}_1 \oplus \mathbf{E}_2$, $\sigma(T_i) = \sigma_i$, *where* $T_i = T|\mathbf{E}_i$; $\mathbf{E}_i$ *is called the* ***generalized eigenspace*** *of* σ_i.

The basic idea of the proof is this: let γ_j be a closed curve with σ_j in its interior and σ_k, $k \neq j$, in its exterior; then

$$T_j = \frac{1}{2\pi i}\int_{\gamma_j} \frac{dz}{zI - T}.$$

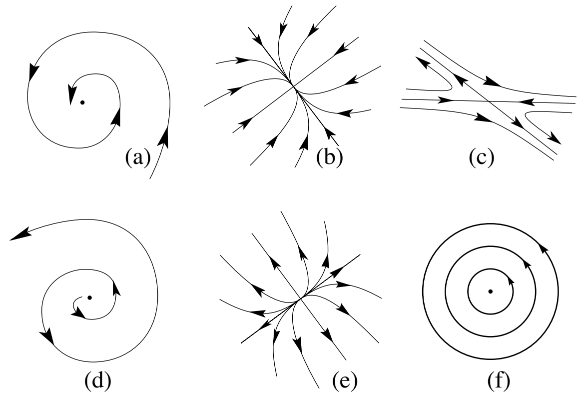


FIGURE 4.3.2. Phase portraits for two dimensional equilibria

Note that the eigenspace of an eigenvalue λ is not always the same as the generalized eigenspace of λ. In the finite dimensional case, the generalized eigenspace of T is the subspace corresponding to all the Jordan blocks containing λ in the Jordan canonical form.

4.3.16 Lemma. *Let T, σ_1, and σ_2 be as in Lemma* 4.3.15*; assume*

$$d(\exp(\sigma_1), \exp(\sigma_2)) > 0.$$

Then for the operator $\exp(tT)$*, the generalized eigenspace of* $\exp(tT_i)$ *is* $\mathbf{E}_i$*.*

Proof. Write, according to Lemma 4.3.15, $\mathbf{E} = \mathbf{E}_1 \oplus \mathbf{E}_2$. Thus

$$\begin{aligned} e^{tT}(e_1, e_2) &= \sum_{n=0}^{\infty} \frac{t^n T^n}{n!}(e_1, e_2) = \sum_{n=0}^{\infty} \left(\frac{t^n T^n}{n!} e_1, \frac{t^n T^n}{n!} e_2 \right) \\ &= \left(\sum_{n=0}^{\infty} \frac{t^n T^n}{n!} e_1, \sum_{n=0}^{\infty} \frac{t^n T^n}{n!} e_2 \right) = (e^{tT_1} e_1, e^{tT_2} e_2). \end{aligned}$$

From this the result follows easily. ■

Now we discuss the generic nonlinear case; that is, let m be a hyperbolic equilibrium of the vector field X and let F_t be its flow. Define the ***inset*** of m by

$$\operatorname{In}(m) = \{ m' \in M \mid F_t(m') \to m \text{ as } t \to +\infty \}$$

and similarly, the ***outset*** is defined by

$$\operatorname{Out}(m) = \{ m' \in M \mid F_t(m') \to m \text{ as } t \to -\infty \}.$$

In the case of a linear system, $\dot{x} = Ax$, where A has no eigenvalue on the imaginary axis (so the origin is a hyperbolic critical point), $\operatorname{In}(0)$ is the generalized eigenspace of the eigenvalues with negative real parts, while $\operatorname{Out}(0)$ is the generalized eigenspace corresponding to the eigenvalues with positive real parts. Clearly, these are complementary subspaces. The dimension of the linear subspace $\operatorname{In}(0)$ is called the ***stability index*** of the critical point. The ***Hartman–Grobman linearization theorem*** states that the phase portrait of X near m is ***topologically conjugate*** to the phase portrait of the linear system $\dot{x} = Ax$, near the origin, where $A = X'(m)$, the linearized vector field at m. This means there is a homeomorphism of the two domains, preserving the oriented trajectories of the respective flows. Thus in this nonlinear hyperbolic case, the inset and outset are C^0 submanifolds. Another important theorem of dynamical systems theory, the

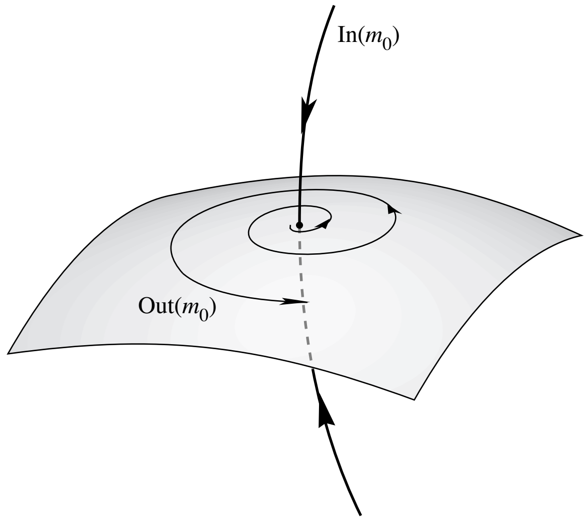


FIGURE 4.3.3. Insets and outsets

stable manifold theorem (Smale [1967]) says that in addition, these are smooth, injectively immersed submanifolds, intersecting transversally at the critical point m. See Figure 4.3.3 for an illustration showing part of the inset and outset near the critical point.

It follows from these important results that there are (up to topological conjugacy) only a few essentially different phase portraits, near hyperbolic critical points. These are classified by the dimension of their insets, called the ***stability index***, which is denoted by $S(X, m)$ for m an equilibrium, as in the linear case.

The word ***index*** comes up in this context with another meaning. If M is finite dimensional and m is a critical point of a vector field X, the ***topological index*** of m is $+1$ if the number of eigenvalues (counting multiplicities) with negative real part is even and is -1 if it is odd. Let this index be denoted $I(X, m)$, so that $I(X, m) = (-1)^{S(X,m)}$. ***The Poincaré–Hopf index theorem*** states that if M is compact and X only has (isolated) hyperbolic critical points, then

$$\sum_{\substack{m \text{ is a critical} \\ \text{point of } X}} I(X, m) = \chi(M)$$

where $\chi(M)$ is the Euler–Poincaré characteristic of M. For isolated nonhyperbolic critical points the index is also defined but requires degree theory for its definition—a kind of generalized winding number; see §7.5 or Guillemin and Pollack [1974, p. 133].

We now illustrate these basic concepts about critical points with some classical applications.

4.3.17 Examples.

A. The ***simple pendulum with linear damping*** is defined by the second-order equation

$$\ddot{x} + c\dot{x} + k \sin x = 0 \quad (c > 0).$$

This is equivalent to the following dynamical system whose phase portrait is shown in Figure 4.3.4:

$$\dot{x} = v, \quad \dot{v} = -cv - k \sin x.$$

The stable focus at the origin represents the motionless, hanging pendulum. The saddle point at $(k\pi, 0)$ corresponds to the motionless bob, balanced at the top of its swing.

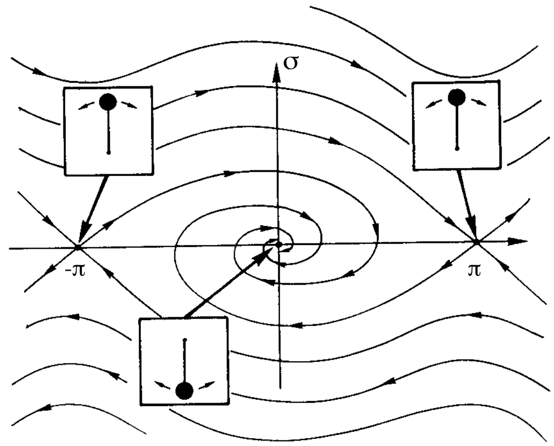


FIGURE 4.3.4. The pendulum with linear damping

B. Another classical equation models the buckling column (see Stoker [1950, Chapter 3, Section 10]):

$$m\ddot{x} + c\dot{x} + a_1 x + a_3 x^3 = 0 \quad (a_1 < 0,\ a_3, c > 0),$$

or equivalently, the planar dynamical system

$$\dot{x} = v, \quad \dot{v} = -\frac{cv}{m} - \frac{a_1 x}{m} - \frac{a_3 x^3}{m}$$

with the phase portrait shown in Figure 4.3.5. This has *two stable foci* on the horizontal axis, denoted m_1 and m_2, corresponding to the column buckling (due to a heavy weight on the top) to either side. The saddle at the origin corresponds to the unstable equilibrium of the straight, unbuckled column.

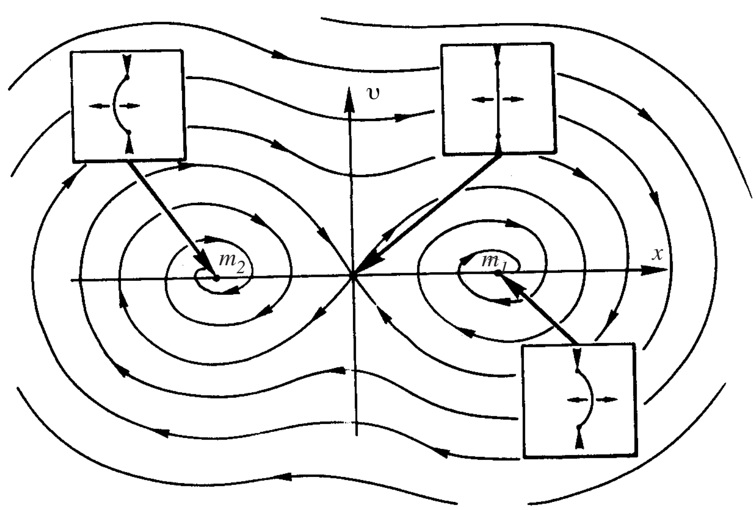


FIGURE 4.3.5. Phase portrait of the buckling column

Note that in this phase portrait, some initial conditions tend toward one stable focus, while some tend toward the other. The two tendencies are divided by the curve, $\mathrm{In}(0, 0)$, the inset of the saddle at the origin.

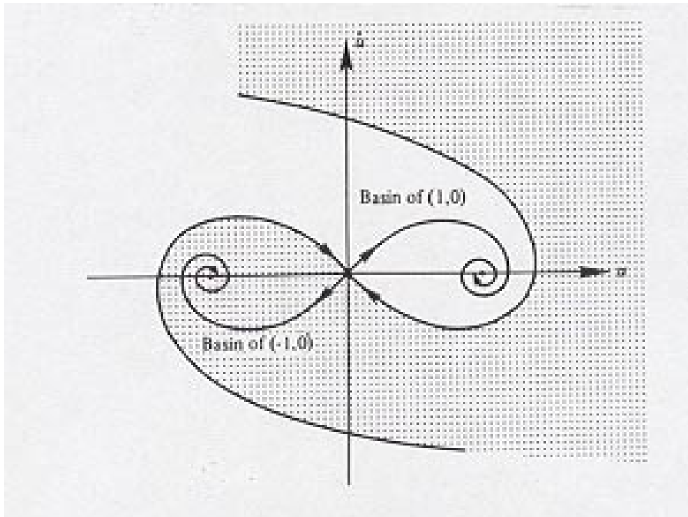


FIGURE 4.3.6. Basins of attraction of $\ddot{x} + \dot{x} - x + x^3 = 0$

This is called the ***separatrix***, as it separates the domain into the two disjoint open sets, $\text{In}(m_0)$ and $\text{In}(m_1)$. The stable foci are called ***attractors***, and their insets are called their basins. See Figure 4.3.6 for the special case $\ddot{x}+\dot{x}-x+x^3 = 0$. This is a special case of a general theory, which is increasingly important in dynamical systems applications. The attractors are regarded as the principal features of the phase portrait; the size of their basins measures the probability of observing the attractor, and the separatrices help find them. ♦

Another basic ingredient in the qualitative theory is the notion of a ***closed orbit***, also called a ***limit cycle***.

4.3.18 Definition. *An orbit $\gamma(t)$ for a vector field X is called* ***closed*** *when $\gamma(t)$ is not a fixed point and there is a $\tau > 0$ such that $\gamma(t + \tau) = \gamma(t)$ for all t. The* ***inset*** *of γ, $\text{In}(\gamma)$, is the set of points $m \in M$ such that $F_t(m) \to \gamma$ as $t \to +\infty$ (i.e., the distance between $F_t(m)$ and the (compact) set $\{ \gamma(t) \mid 0 \le t \le \tau \}$ tends to zero as $t \to \infty$). Likewise, the* ***outset****, $\text{Out}(\gamma)$, is the set of points tending to γ as $t \to -\infty$.*

4.3.19 Example. One of the earliest occurrences of an attractive closed orbit in an important application is found in Baron ***Rayleigh's model for the violin string*** (see Rayleigh [1887, Volume 1, Section 68a]),

$$\ddot{u} + k_1\dot{u} + k_3\dot{u}^3 + \omega^2 u = 0, \quad k_1 < 0 < k_3,$$

or equivalently,

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -k_1 v - k_3 v^3 - \omega^2 u, \end{aligned}$$

with the phase portrait shown in Figure 4.3.7.

This phase portrait has an unstable focus at the origin, with an attractive closed orbit around it. That is, the closed orbit γ is a limiting set for every point in its basin (or inset) $\text{In}(\gamma)$, which is an open set of the domain. In fact the entire plane (excepting the origin) comprises the basin of this closed orbit. Thus every trajectory tends asymptotically to the limit cycle γ and winds around closer and closer to it. Meanwhile this closed orbit is a periodic function of time, in the sense of Definition 4.3.18. Thus the eventual (asymptotic) behavior of every trajectory (other than the unstable constant trajectory at the origin) is periodic; it is an oscillation.

This picture thus models the *sustained oscillation* of the violin string, under the influence of the moving bow. Related systems occur in electrical engineering under the name van der Pol equation. (See Hirsch and Smale [1974, Chapter 10] for a discussion.) ♦

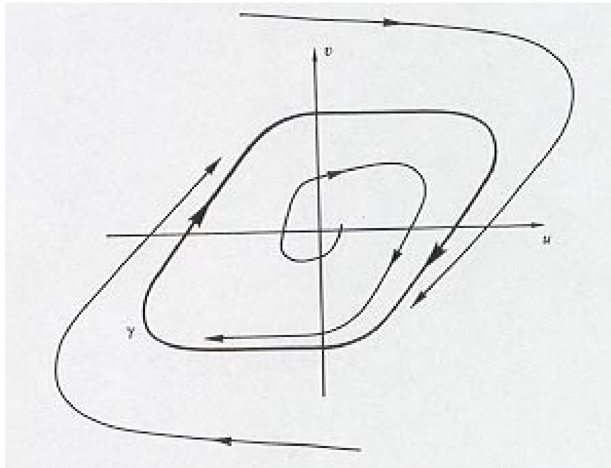


FIGURE 4.3.7. Rayleigh equation

We now proceed toward the analog of Liapunov's theorem for the stability of closed orbits. To do this we need to introduce Poincaré maps and characteristic multipliers.

4.3.20 Definition. *Let X be a C^r vector field on a manifold M, $r \geq 1$. A* ***local transversal section*** *of X at $m \in M$ is a submanifold $S \subset M$ of codimension one with $m \in S$ and for all $s \in S$, $X(s)$ is not contained in T_sS.*

Let X be a C^r vector field on a manifold M with C^r flow $F : \mathcal{D}_X \subset M \times \mathbb{R} \to M$, γ a closed orbit of X with period τ, and S a local transversal section of X at $m \in \gamma$. A ***Poincaré map*** *of γ is a C^r mapping $\Theta : W_0 \to W_1$ where:*

PM1. *$W_0, W_1 \subset S$ are open neighborhoods of $m \in S$, and Θ is a C^r diffeomorphism;*

PM2. *there is a C^r function $\delta : W_0 \to \mathbb{R}$ such that for all $s \in W_0$, $(s, \tau - \delta(s)) \in \mathcal{D}_X$, and $\Theta(s) = F(s, \tau - \delta(s))$; and finally,*

PM3. *if $t \in [0, \tau - \delta(s)]$, then $F(s,t) \notin W_0$ (see Figure 4.3.8).*

4.3.21 Theorem (Existence and Uniqueness of Poincaré Maps)**.**

(i) *If X is a C^r vector field on M, and γ is a closed orbit of X, then there exists a Poincaré map of γ.*

(ii) *If $\Theta : W_0 \to W_1$ is a Poincaré map of γ (in a local transversal section S at $m \in \gamma$) and Θ' also (in S' at $m' \in \gamma$), then Θ and Θ' are locally conjugate. That is, there are open neighborhoods W_2 of $m \in S$, W_2' of $m' \in S'$, and a C^r-diffeomorphism $H : W_2 \to W_2'$, such that*

$$W_2 \subset W_0 \cap W_1, \quad W_2' \subset W_0' \cap W_1'$$

and the following diagram commutes:

$$\begin{array}{ccc} \Theta^{-1}(W_2) \cap W_2 & \xrightarrow{\ \Theta\ } & W_2 \cap \Theta(W_2) \\ \Big\downarrow H & & \Big\downarrow H \\ W_2' & \xrightarrow[\ \Theta\ ]{} & S' \end{array}$$

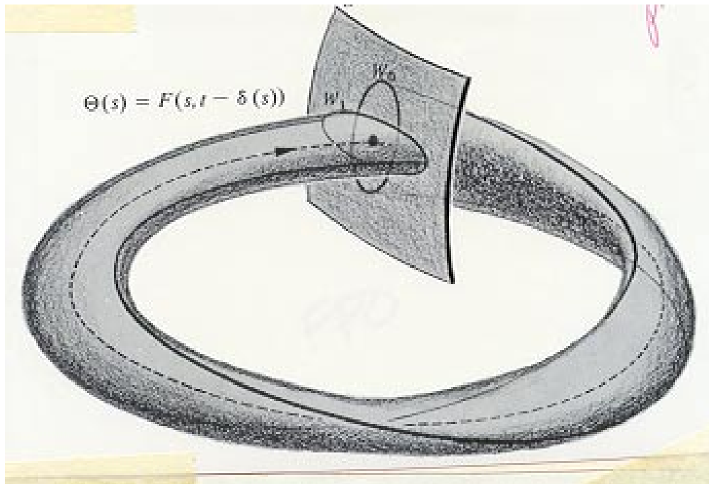


FIGURE 4.3.8. Poincaré maps

Proof. (i) At any point $m \in \gamma$ we have $X(m) \neq 0$, so there exists a flow box chart (U, φ) at m with $\varphi(U) = V \times I \subset \mathbb{R}^{n-1} \times \mathbb{R}$. Then $S = \varphi^{-1}(V \times \{0\})$ is a local transversal section at m. If $F : \mathcal{D}_X \subset M \times \mathbb{R} \to M$ is the integral of X, $\mathcal{D}_X$ is open, so we may suppose $U \times [-\tau, \tau] \subset \mathcal{D}_X$, where τ is the period of γ. As $F_\tau(m) = m \in M$ and F_τ is a homeomorphism, $U_0 = F_\tau^{-1}(U) \cap U$ is an open neighborhood of $m \in M$ with $F_\tau(U_0) \subset U$. Let $W_0 = S \cap U_0$ and $W_2 = F_\tau(W_0)$. Then W_2 is a local transversal section at $m \in M$ and $F_\tau : W_0 \to W_2$ is a diffeomorphism (see Figure 4.3.9).

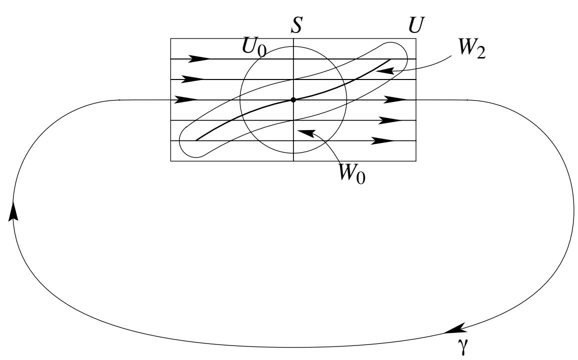


FIGURE 4.3.9. Coordinates adapted to a periodic orbit

If $U_2 = F_\tau(U_0)$, then we may regard U_0 and U_2 as open submanifolds of the vector bundle $V \times \mathbb{R}$ (by identification using φ) and then $F_\tau : U_0 \to U_2$ is a C^r diffeomorphism mapping fibers into fibers, as φ identifies orbits with fibers, and F_τ preserves orbits. Thus W_2 is a section of an open subbundle. More precisely, if $\pi : V \times I \to V$ and $\rho : V \times I \to I$ are the projection maps, then the composite mapping is C^r and

$$W_0 \xrightarrow{F_\tau} W_2 \xrightarrow{\varphi} V \times I \xrightarrow{\pi} V \xrightarrow{\varphi^{-1}} S$$

has a tangent that is an isomorphism at each point, and so by the inverse mapping theorem, it is a C^r diffeomorphism onto an open submanifold. Let W_1 be its image, and Θ the composite mapping.

We now show that $\Theta : W_0 \to W_1$ is a Poincaré map. Obviously **PM1** is satisfied. For **PM2**, we identify U and $V \times I$ by means of φ to simplify notations. Then $\pi : W_2 \to W_1$ is a diffeomorphism, and its inverse

$(\pi|W_2)^{-1} : W_1 \to W_2 \subset W_1 \times \mathbb{R}$ is a section corresponding to a smooth function $\sigma : W_1 \to \mathbb{R}$. In fact, σ is defined implicitly by

$$F_\tau(w_0) = (\pi \circ F_\tau(w_0), \rho \circ F_\tau(w_0)) = (\pi \circ F_\tau(w_0), \sigma \circ \pi F_\tau(w_0))$$

or $\rho \circ F_\tau(w_0) = \sigma \circ \pi F_\tau(w_0)$. Now let $\delta : W_0 \to \mathbb{R}$ be given by $w_0 \mapsto \sigma \circ F_t(w_0)$ which is C^r. Then we have

$$\begin{aligned} F_{\tau-\delta(w_0)}(w_0) &= (F_{-\delta(w_0)} \circ F_\tau) \\ &= (\pi \circ F_\tau(w_0), \rho \circ F_\tau(w_0) - \delta(w_0)) \\ &= (\pi \circ F_\tau(w_0), 0) \\ &= \Theta(w_0). \end{aligned}$$

Finally, **PM3** is obvious since (U, φ) is a flow box.

(ii) The proof is burdensome because of the notational complexity in the definition of local conjugacy, so we will be satisfied to prove uniqueness under additional simplifying hypotheses that lead to global conjugacy (identified by italics). The general case will be left to the reader.

We consider first the special case $m = m'$. Choose a flow box chart (U, φ) at m, and *assume* $S \cup S' \subset U$, *and that* S *and* S' *intersect each orbit arc in* U *at most once, and that they intersect exactly the same sets of orbits*. (These three conditions may always be obtained by shrinking S and S'.) Then let $W_2 = S$, $W_2' = S'$, and $H : W_2 \to W_2'$ the bijection given by the orbits in U. As in (i), this is seen to be a C^r diffeomorphism, and $H \circ \Theta = \Theta' \circ H$.

Finally, suppose $m \neq m'$. Then $F_a(m) = m'$ for some $a \in]0, \tau[$, and as $\mathcal{D}_X$ is open there is a neighborhood U of m such that $U \times \{a\} \subset \mathcal{D}_X$. Then $F_a(U \cap S) = S''$ is a local transversal section of X at $m' \in \gamma$, and $H = F_a$ effects a conjugacy between Θ and $\Theta'' = F_a \circ \Theta \circ F_a^{-1}$ on S''. By the preceding paragraph, Θ'' and Θ' are locally conjugate, but conjugacy is an equivalence relation. This completes the argument. ■

If γ is a closed orbit of $X \in \mathfrak{X}(M)$ and $m \in \gamma$, the behavior of nearby orbits is given by a Poincaré map Θ on a local transversal section S at m. Clearly $T_m\Theta \in L(T_mS, T_mS)$ is a linear approximation to Θ at m. By uniqueness of Θ up to local conjugacy, $T_{m'}\Theta'$ is similar to $T_m\Theta$, for any other Poincaré map Θ' on a local transversal section at $m' \in \gamma$. Therefore, the spectrum of $T_m\Theta$ is independent of $m \in \gamma$ and the particular section S at m.

4.3.22 Definition. *If γ is a closed orbit of $X \in \mathfrak{X}(M)$,* ***the characteristic multipliers*** *of X at γ are the points in the spectrum of $T_m\Theta$, for any Poincaré map Θ at any $m \in \gamma$.*

Another linear approximation to the flow near γ is given by $T_mF_\tau \in L(T_mM, T_mM)$ if $m \in \gamma$ and τ is the period of γ. Note that $F_\tau^*(X(m)) = X(m)$, so T_mF_τ always has an eigenvalue 1 corresponding to the eigenvector $X(m)$. The $(n-1)$ remaining eigenvalues (if $\dim(M) = n$) are in fact the characteristic multipliers of X at γ.

4.3.23 Proposition. *If γ is a closed orbit of $X \in \mathfrak{X}(M)$ of period τ and c_γ is the set of characteristic multipliers of X at γ, then $c_\gamma \cup \{1\}$ is the spectrum of T_mF_τ, for any $m \in \gamma$.*

Proof. We can work in a chart modeled on $\mathbf{E}$ and assume $m = 0$. Let V be the span of $X(m)$ so $\mathbf{E} = T_mM \oplus V$. Write the flow $F_t(x, y) = (F_t^1(x, y), F_t^2(x, y))$. By definition, we have

$$\mathbf{D}_1F_t^1(m) = T_m\Theta \quad \text{and} \quad \mathbf{D}_2F_\tau^2(m) \cdot X(m) = X(m).$$

Thus the matrix of T_mF_τ is of the form

$$\begin{bmatrix} T_m\Theta & 0 \\ A & 1 \end{bmatrix}$$

where $A = \mathbf{D}_1F_\tau^2(m)$. From this it follows that the spectrum of T_mF_τ is $\{1\} \cup c_\gamma$. ■

If the characteristic exponents of an equilibrium point lie (strictly) in the left half-plane, we know from Liapunov's theorem that the equilibrium is stable. For closed orbits we introduce stability by means of the following definition.

4.3.24 Definition. *Let X be a vector field on a manifold M and γ a closed orbit of X. An orbit $F_t(m_0)$ is said to* ***wind toward*** *γ if m_0 is + complete and for any local transversal section S to X at $m \in \gamma$ there is a $t(0)$ such that $F_{t(0)}(m_0) \in S$ and successive applications of the Poincaré map yield a sequence of points that converges to m. If the closed orbit γ has a neighborhood U such that for any $m_0 \in U$, the orbit through m_0 winds towards γ, then γ is called* ***asymptotically stable****.*

In other words, orbits starting "close" to γ, "converge" to γ; see Figure 4.3.10.

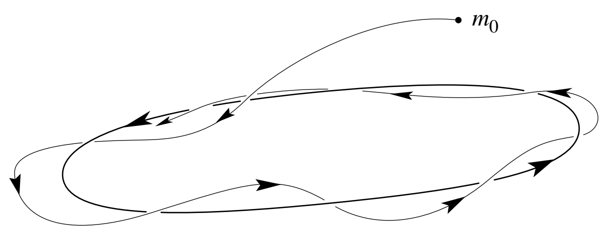


FIGURE 4.3.10. Stable periodic orbit

4.3.25 Proposition. *If γ is an asymptotically stable periodic orbit of the vector field X and $m_0 \in U$, the neighborhood given in Definition 4.3.24, then for any neighborhood V of γ, there exists $t_0 > 0$ such that for all $t \geq t_0$, $F_t(m) \in V$.*

Proof. Define $m_k = \Theta^k(m_0)$, where Θ is a Poincaré map for a local transversal section at m to γ containing m_0. Let $t(n)$ be the "return time" of n, that is, $t(n)$ is defined by $F_{t(n)}(n) \in S$. If τ denotes the period of γ and $\tau_k = t(m_k)$, then since $m_k \to m$, it follows that $\tau_k \to \tau$ since $t(n)$ is a smooth function of n by Theorem 4.3.21. Let M be an upper bound for the set $\{\, |\tau_k| \mid k \in \mathbb{N} \,\}$. By smoothness of the flow, $F_s(m_k) \to F_s(m)$ as $k \to \infty$, uniformly in $s \in [0, M]$. Now write for any $t > 0$, $F_t(m_0) = F_{T(t)}(m_{k(t)})$, for $T(t) \in [0, M]$ and observe that $k(t) \to \infty$ as $t \to \infty$. Therefore, if W is any neighborhood of $F_{T(t)}(m_0)$ contained in V, since $F_t(m_0) = F_{T(t)}(m_{k(t)})$ converges to $F_{T(t)}(m)$ as $t \to \infty$ it follows that there exists $t_0 > 0$ such that for all $t \geq t_0$, $F_t(m_0) \in W \subset V$. ■

4.3.26 Theorem (Liapunov Stability Theorem for Closed Orbits). *Let γ be a closed orbit of $X \in \mathfrak{X}(M)$ and let the characteristic multipliers of γ lie strictly inside the unit circle. Then γ is asymptotically stable.*

The proof relies on the following lemma.

4.3.27 Lemma. *Let $T : \mathbf{E} \to \mathbf{E}$ be a bounded linear operator. Let $\sigma(T)$ lie strictly inside the unit circle. Then $\lim_{n\to\infty} T^n e = 0$ for all $e \in \mathbf{E}$.*

Proof. By Lemma 4.3.7 and compactness of $\sigma(T)$, there is a norm $|\ |$ on $\mathbf{E}$ equivalent to the original norm on $\mathbf{E}$ such that $|T| \leq r < 1$. Therefore $|T^n e| \leq r^n |e| \to 0$ as $n \to \infty$. ▼

4.3.28 Lemma. *Let $f : S \to S$ be a smooth map on a manifold S with $f(s) = s$ for some s. Let the spectrum of $T_s f$ lie strictly inside the unit circle. Then there is a neighborhood U of s such that if $s' \in U$, $f(s') \in U$ and $f^n(s') \to s$ as $n \to \infty$, where $f^n = f \circ f \circ \cdots \circ f$ (n times).*

Proof. We can assume that S is a Banach space $\mathbf{E}$ and that $s = 0$. As above, renorm $\mathbf{E}$ and find $0 < r < 1$ such that $|T| \leq r$, where $T = \mathbf{D}f(0)$. Let $\varepsilon > 0$ be such that $r + \varepsilon < 1$. Choose a neighborhood V of 0 such that for all $x \in V$

$$|f(x) - Tx| \leq \varepsilon |x|$$

which is possible since f is smooth. Therefore,

$$|f(x)| \leq |Tx| + \varepsilon|x| \leq (r+\varepsilon)|x|.$$

Now, choose $\delta > 0$ such that the ball U of radius δ at 0 lies in V. Then the above inequality implies $|f^n(x)| \leq (r+\varepsilon)^n|x|$ for all $x \in U$ which shows that $f^n(x) \in U$ and $f^n(x) \to 0$. ▼

Proof of Theorem 4.3.26. The previous lemma applied to $f = \Theta$, a Poincaré map in a transversal slice S to γ at m, implies that there is an open neighborhood V of m in S such that the orbit through every point of V winds toward γ. Thus, the orbit through every point of $U = \{ F_t(m) \mid t \geq 0 \} \supset \gamma$ winds toward γ. U is a neighborhood of γ since by the straightening out theorem 4.1.14, each point of γ has a neighborhood contained in $\{ F_t(\Theta(U)) \mid t > -\varepsilon, \varepsilon > 0 \}$. ■

4.3.29 Definition. *If $X \in \mathfrak{X}(M)$ and γ is a closed orbit of X, γ is called* ***hyperbolic*** *if none of the characteristic multipliers of X at γ has modulus* 1.

Hyperbolic closed orbits are isolated (see Abraham and Robbin [1967, Chapter 5]). The local qualitative behavior near an hyperbolic closed orbit, γ, may be visualized with the aid of the Poincaré map, $\Theta : W_0 \subset S \to W_1 \subset S$, as shown in Figure 4.3.8. The qualitiative behavior of this map, under iterations, determines the asymptotic behavior of the trajectories near γ. Let $m \in \gamma$ be the base point of the section, and $s \in S$. Then $\mathrm{In}(\gamma)$, the inset of γ, intersects S in the inset of m under the iterations of Θ. That is, $s \in \mathrm{In}(\gamma)$ if the trajectory $F_t(s)$ winds towards γ, and this is equivalent to saying that $\Theta^k(s)$ tends to m as $k \to +\infty$.

The inset and outset of $m \in S$ are classified by linear algebra, as there is an analogue of the linearization theorem for maps at hyperbolic critical points. The ***linearization theorem for maps*** says that there is a C^0 coordinate chart on S, in which the local representative of Θ is a linear map. Recall that in the hyperbolic case, the spectrum of this linear isomorphism avoids the unit circle. The eigenvalues inside the unit circle determine the generalized eigenspace of contraction that is, the inset of $m \in S$ under the iterates of Θ. The eigenvalues outside the unit circle similarly determine the outset of $m \in S$. Although this argument provides only local C^0 submanifolds, the ***global stable manifold theorem*** improves this: the inset and outset of a fixed point of a diffeomorphism are smooth, injectively immersed submanifolds meeting transversally at m.

Returning to closed orbits, the inset and outset of $\gamma \subset M$ may be visualized by choosing a section S_m at every point $m \in \gamma$. The inset and outset of γ in M intersect each section S_m in submanifolds of S_m, meeting transversally at $m \in \gamma$. In fact, $\mathrm{In}(\gamma)$ is a cylinder over γ, that is, a bundle of injectively immersed disks. So, likewise, is $\mathrm{Out}(\gamma)$. And these two cylinders intersect transversally in γ, as shown in Figure 4.3.11. These bundles need not be trivial.

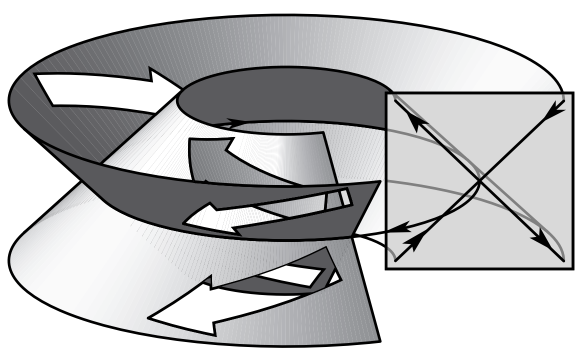


FIGURE 4.3.11. Insets and outsets for periodic orbits

Another argument is sometimes used to study the inset and outset of a closed orbit, in place of the Poincaré section technique described before, and is originally due to Smale [1967]. The flow F_t leaves the closed orbit invariant. A special coordinate chart may be found in a neighborhood of γ. The neighborhood is a disk bundle over γ, and the flow F_t, is a bundle map. On each fiber, F_t is a linear map of the form $Z_t e^{Rt}$, where Z_t is a constant, and R is a linear map. Thus, if $s = (m, x)$ is a point in the chart, the local representative of F_t is given by the expression

$$F_t(m, x) = (m_t, Z_t e^{Rt} \cdot x)$$

called the ***Floquet normal form***. This is the ***linearization theorem for closed orbits***. A related result, the ***Floquet theorem***, eliminates the dependence of Z_t on t, by making a further (time-dependent) change of coordinates (see Hartman [1973, Chapter 4, Section 6], or Abraham and Robbin [1967]). Finally, linear algebra applied to the linear map R in the exponent of the Floquet normal form, establishes the C^0 structure of the inset and outset of γ.

To get an overall picture of a dynamical system in which all critical elements (critical points and closed orbits) are hyperbolic, we try to draw or visualize the insets and outsets of each. Those with open insets are ***attractors***, and their open insets are their ***basins***. The domain is divided into basins by the ***separatrices***, which includes the insets of all the nonattractive (saddle-type) critical elements (and possible other, more complicated limit sets, called ***chaotic attractors***, not described here.)

We conclude with an example of sufficient complexity, which has been at the center of dynamical system theory for over a century.

4.3.30 Example. The simple pendulum equation may be "simplified" by approximating $\sin x$ by two terms of its MacLaurin expansion. The resulting system is a model for a ***nonlinear spring with linear damping***,

$$\dot{x} = v, \quad \dot{v} = -cv - kx + \frac{k}{3}x^3.$$

Adding a periodic forcing term, we have

$$\dot{x} = v, \quad \dot{v} = -cv - kx + \frac{k}{3}x^4 + F\cos\omega t.$$

This time-dependent system in the plane is transformed into an autonomous system in a solid ring by adding an angular variable proportional to the time, $\theta = \omega t$. Thus,

$$\dot{x} = v, \quad \dot{x} = -cv - kx + \frac{k}{3}x^3 + F\cos\theta, \quad \dot{\theta} = \omega$$

Although this was introduced by Baron Rayleigh to study the resonance of tuning forks, piano strings, and so on, in his classic 1877 book, *Theory of Sound* , this system is generally named the ***Duffing equation*** after Duffing who obtained the first important results in 1908 (see Stoker [1950] for additional information). ♦

Depending on the values of the three parameters (c, k, F) various phase portraits are obtained. One of these is shown in Figure 4.3.12, adapted from the experiments of Hayashi [1964]. There are three closed orbits: two attracting, one of saddle type. The inset of the saddle is a cylinder topologically, but the whole cylinder revolves around the saddle-type closed orbit. This cylinder is the separatrix between the two basins. For other parameter values the dynamics can be chaotic (see for example, Holmes [1979a, 1979b] and Ueda [1980]).

For further information on dynamical systems, see, for example, Guckenheimer and Holmes [1983].

Exercises

◇ **4.3-1.** (i) Let $E \to M$ be a vector bundle and $m \in M$ an element of the zero section. Show that $T_m E$ is isomorphic to $T_m M \oplus E_m$ in a natural, chart independent way.

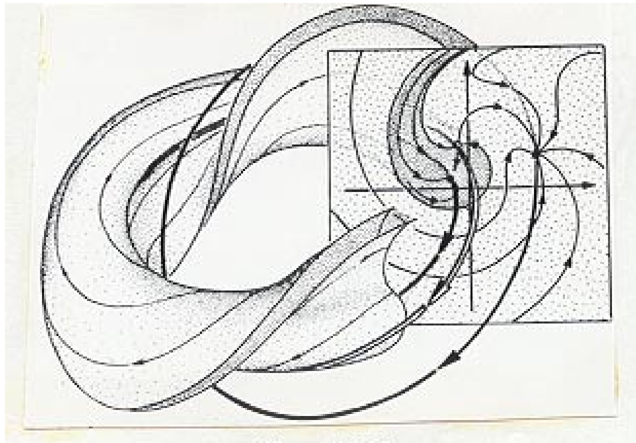


FIGURE 4.3.12. Phase portrait for the nonlinear spring with linear damping and forcing.

(ii) If $\xi : M \to E$ is a section of E, and $\xi(m) = 0$, define $\xi'(m) : T_m M \to E_m$ to be the projection of $T_m \xi$ to E_m. Write out $\xi'(m)$ relative to coordinates.

(iii) Show that if X is a vector field, then $X'(m)$ defined this way coincides with Definition 4.3.1.

◇ **4.3-2.** Prove that the equation $\ddot{\theta} + 2k\dot{\theta} - q \sin\theta = 0$ $(q > 0,\ k > 0)$ has a saddle point at $\theta = 0$, $\dot{\theta} = 0$.

◇ **4.3-3.** Consider the differential equations $\dot{r} = ar^3 - br$, $\dot{\theta} = 1$ using polar coordinates in the plane.

(i) Determine those a, b for which this system has an attractive periodic orbit.

(ii) Calculate the eigenvalues of this system at the origin for various a, b.

◇ **4.3-4.** Let $X \in \mathfrak{X}(M)$, $\varphi : M \to N$ be a diffeomorphism, and $Y = \varphi_* X$. Show that

(i) $m \in M$ is a critical point of X iff $\varphi(m)$ is a critical point of Y and the characteristic exponents are the same for each;

(ii) $\gamma \subset M$ is a closed orbit of X iff $\varphi(\gamma)$ is a closed orbit of Y and their characteristic multipliers are the same.

◇ **4.3-5.** The energy for a symmetric heavy top is

$$H(\theta, p_\theta) = \frac{1}{2I \sin^2\theta} \left\{ p_\psi^2 (b - \cos\theta)^2 + p_\theta^2 \sin^2\theta \right\} + \frac{p_\psi^2}{J} Mg\ell \cos\theta$$

where $I, J > 0$, b, p_ψ, and $Mg\ell > 0$ are constants. The dynamics of the top is described by the differential equations $\dot{\theta} = \partial H / \partial p_\theta$, $\dot{p_\theta} = -\partial H / \partial \theta$.

(i) Show that $\theta = 0$, $p_\theta = 0$ is a saddle point if

$$0 < p_\psi < 2(Mg\ell I)^{1/2}$$

(a slow top).

(ii) Verify that $\cos\theta = 1 - \gamma \operatorname{sech}^2((\beta\gamma)^{1/2}/2)$, where $\gamma = 2 - b^2/\beta$ and $\beta = 2Mg\ell/I$ describe both the outset and inset of this saddle point. (This is called a ***homoclinic orbit***.)

(iii) Is $\theta = 0$, $p_\theta = 0$ stable if $p_\psi > (Mg\ell I)^{1/2}$?

HINT: Use the fact that H is constant along the trajectories.

◇ **4.3-6.** Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ and suppose that $a < \mathrm{Re}(\lambda_i) < b$, for all eigenvalues $\lambda_i, i = 1, \ldots, n$ of A. Show that $\mathbb{R}^n$ admits an inner product $\langle\langle\, , \rangle\rangle$ with associated norm $|||\cdot|||$ such that

$$a|||x|||^2 \leq \langle\langle Ax, x\rangle\rangle \leq b|||x|||^2.$$

Prove this by following the outline below.

(i) If A is diagonalizable over $\mathbb{C}$, then find a basis in $\mathbb{R}^n$ in which the matrix of A has either the entries on the diagonal, the real eigenvalues of A, or 2×2 blocks of the form

$$\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}, \quad \text{for } \lambda_j = a_j + ib_j, \text{ if } b_j \neq 0$$

Choose the inner product $\langle\langle\, , \rangle\rangle$ on $\mathbb{R}^n$ such that the one and two-dimensional invariant subspaces of A defined by this block-matrix are mutually orthogonal; pick the standard $\mathbb{R}^2$-basis in the associated two-dimensional spaces.

(ii) If A is not diagonalizable, then pass to the real Jordan form. There are two kinds of Jordan $k_j \times k_j$ blocks:

$$\begin{bmatrix} \lambda_j & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_j & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_j & \cdots & 0 & 0 \\ \cdots & & & & \cdots & \cdots \\ 0 & \cdots & \cdots & & 1 & \lambda_j \end{bmatrix},$$

if $\lambda_j \in \mathbb{R}$, or

$$\begin{bmatrix} \Delta_j & 0 & 0 & \cdots & 0 & 0 \\ I_2 & \Delta_j & 0 & \cdots & 0 & 0 \\ 0 & I_2 & \Delta_j & \cdots & 0 & 0 \\ \cdots & & & & \cdots & \cdots \\ 0 & \cdots & \cdots & & I_2 & \Delta_j \end{bmatrix}$$

where

$$\Delta_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

if $\lambda_j = a_j + ib_j$, $a_j, b_j \in \mathbb{R}$, $b_j \neq 0$. For the first kind of block, choose the basis $e'_1, \ldots, e'_{k(j)}$ of eigenvectors of the diagonal part, which we call D. Then, for $\varepsilon > 0$ small, put $e^\varepsilon_r = e'_r/e^{r-1}$, $r = 1, \ldots, k(j)$ and define $\langle\, , \rangle_\varepsilon$ on the subspace $\mathrm{span}\{e^\varepsilon_1, \ldots, e^\varepsilon_{k(j)}\} \subset \mathbb{R}^n$ to be the Euclidean inner product given by this basis. Compute the marix of A in this basis and show that

$$\frac{\langle Ax, x\rangle_\varepsilon}{\langle x, x\rangle_\varepsilon} \to \frac{Dx \cdot x}{\|x\|^2}, \quad \text{as} \quad \varepsilon \to 0.$$

Conclude that for ε small the statement holds for the first kind of block. do the same for the second kind of block.

◇ **4.3-7.** Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$. Show that the following are equivalent.

(i) All eigenvalues of A have strictly negative real part (the origin is called a sink in this case).

(ii) For any norm $|\cdot|$ on $\mathbb{R}^n$, there exist constants $k > 0$ and $\varepsilon > 0$ such that for all $t \geq 0$, $|e^{tA}| \leq ke^{-t\varepsilon}$.

(iii) There is a norm $|||\cdot|||$ on $\mathbb{R}^n$ and a constant $\delta > 0$ such that for all $t \geq 0$, $|||e^{tA}||| \leq e^{-t\delta}$.

HINT: (ii) $\Rightarrow$ (i) by using the real Jordan form: if every solution of $\dot{x} = Ax$ tends to zero as $t \to +\infty$, then every eigenvalue of A has strictly negative real part. For (i) $\Rightarrow$ (iii) use Exercise 4.3-5 and observe that if $x(t)$ is a solution of $\dot{x} = Ax$, then we have

$$\frac{d}{dt}|||x(t)||| = \frac{\langle\langle x(t), Ax(t)\rangle\rangle}{|||x(t)|||},$$

so that we get the following inequality: $at \leq \log|||x(t)|||/\log|||x(0)||| \leq bt$, where $a = \min\{\operatorname{Re}(\lambda_i) \mid i = 1, \ldots, n\}$, and $b = \max\{\operatorname{Re}(\lambda_i) \mid i = 1, \ldots, n\}$. Then let $-\varepsilon = b$.

Prove a similar theorem if all eigenvalues of A have strictly positive real part; the origin is then called a ***source***.

◇ **4.3-8.** Give a proof of Theorem 4.3.4 in the finite dimensional case without using the variation of constants formula (Exercise 4.1-5) and using Exercise 4.3-6.
HINT: If $A = X'(0)$ locally show

$$\lim_{x \to 0} \frac{\langle\langle X(x) - Ax, x\rangle\rangle}{|||x|||^2} = 0.$$

Since $\langle\langle Ax, x\rangle\rangle \leq -\varepsilon|||x|||^2$, $\varepsilon = \max \operatorname{Re}\{\lambda_i \mid i = 1, \ldots, n, \lambda_i \text{ eigenvalues of } A\}$, there exists $\delta > 0$ such that if $|||x||| \leq \delta$, then $\langle\langle X(x), x\rangle\rangle \leq -C|||x|||^2$, for some $C > 0$. Show that if $x(t)$ is a solution curve in the closed δ-ball, $t \in [0, T]$, then

$$\frac{d|||x(t)|||}{dt} \leq -C|||x(t)|||.$$

Conclude $|||x(t)||| \leq \delta$ for all $t \in [0, T]$ and thus by compactness of the δ-ball, $x(t)$ exists for all $t \geq 0$. Finally, show that $|||x(t)||| \leq e^{-t\varepsilon}|||x(0)|||$.

◇ **4.3-9.** An equilibrium point m of a vector field $X \in \mathfrak{X}(M)$ is called a ***sink***, if there is a $\delta > 0$ such that all points in the spectrum of $X'(m)$ have real part $< -\delta$.

(i) Show that in a neighborhood of a sink there is no other equilibrium of X.

(ii) If $M = \mathbb{R}^n$ and X is a linear vector field, Exercise 4.3-7 shows that provided $\lim_{t\to\infty} m(t) = 0$ for every integral curve $m(t)$ of X, then the eigenvalues of X have all strictly negative real part. Show that this statement is false for general vector fields by finding an example of a non-linear vector field X on $\mathbb{R}^n$ whose integral curves tend to zero as $t \to \infty$, but is such that $X'(0)$ has at least one eigenvalue with zero real part.

HINT : See Exercise 4.3-3.

◇ **4.3-10** (Hyperbolic Flows)**.** An operator $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ is called ***hyperbolic*** if no eigenvalue of A has zero real part. The linear flow $x \mapsto e^{tA}x$ is then called a ***hyperbolic flow***.

(i) Let A be hyperbolic. Show that there is a direct sum decomposition $\mathbb{R}^n = \mathbf{E}^s \oplus \mathbf{E}^u$, $A(\mathbf{E}^s) \subset \mathbf{E}^s$, $A(\mathbf{E}^u) \subset \mathbf{E}^u$, such that the origin is a sink on $\mathbf{E}^s$ and a source on $\mathbf{E}^u$; $\mathbf{E}^s$ and $\mathbf{E}^u$ are called the ***stable*** and ***unstable subspaces*** of $\mathbb{R}^n$. Show that the decomposition is unique.

HINT: $\mathbf{E}^s$ is the sum of all subspaces defined by the real Jordan form for which the real part of the eigenvalues is negative. For uniqueness, if $\mathbb{R}^n = \mathbf{E}'^s \oplus \mathbf{E}'^u$ and $v \in \mathbf{E}'^s$, then $v = x + y$, $x \in \mathbf{E}^s$, $y \in \mathbf{E}^u$ with $e^{tA}v \to 0$ as $t \to \infty$, so that $e^{tA}x \to 0$, $e^{tA}y \to 0$ as $t \to \infty$. But since the origin is a source on $\mathbf{E}'^u$, $\|e^{tA}y\| \geq e^{t\varepsilon}\|y\|$ for some $\varepsilon > 0$ by the analogue of Exercise 4.3-7(ii).

(ii) Show that A is hyperbolic iff for each $x \neq 0$, $\|e^{tA}x\| \to \infty$ as $t \to \pm\infty$.

(iii) Conclude that hyperbolic flows have no periodic orbits.

◇ **4.3-11** (Gradient flows; continuation of Exercise 4.1-8)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^1 and let $X = (\partial f/\partial x^1, \ldots, \partial f/\partial x^n)$. Show that at regular points of f, the integral curves of X cross the level surfaces of f orthogonally and that every singular point of f is an equilibrium of X. Show that isolated maxima of f are asymptotically stable. HINT: If x_0 is the isolated maximum of f, then $f(x_0) - f(x)$ is a strict Liapunov function. Draw the level sets of f and the integral curves of X on the same diagram in $\mathbb{R}^2$, when f is defined by $f(x^1, x^2) = (x^1 - 1)^2 + (x^2 - 2)^2(x^2 - 3)^2$.

◇ **4.3-12.** Consider $\ddot{u} + \dot{u} + u^3 = 0$. Show that solutions converge to zero like $C/\sqrt{t}$ as $t \to \infty$ by considering $H(u, \dot{u}) = (u + \dot{u})^2 + u^3 + u^4$.

◇ **4.3-13.** Use the method of Liapunov functions to study the stability of the origin for the following vector fields:

(i) $X(x, y) = -(3y + x^3)(\partial/\partial x) + (2x - 5y^3)(\partial/\partial y)$ (asymptotically stable);

(ii) $X(x, y) = -xy^4(\partial/\partial x) + x^6 y(\partial/\partial y)$ (stable; look for L of the form $x^4 + ay^6$);

(iii) $X(x, y) = (xy - x^3 + y)(\partial/\partial x) + (x^4 - x^2 y + x^3)(\partial/\partial y)$ (stable; look for L of the form $ax^4 + by^2$);

(iv) $X(x, y) = (y + x^7)(\partial/\partial x) + (y^9 - x)(\partial/\partial y)$ (unstable);

(v) $X(x, y, z) = 3y(z + 1)(\partial/\partial x) + x(z + 1)(\partial/\partial y) + yz(\partial/\partial z)$ (stable);

(vi)

$$\begin{aligned} X(x, y, z) = {} & (-x^5 + 5x^6 + 2y^3 + xz^2 + xyz)(\partial/\partial x) \\ & + (-y - 2z + 3x^6 + 4yz + xz + xy^2)(\partial/\partial y) \\ & + (2y - z - 2x^8 - y^2 + xz^2 + xy^3)(\partial/\partial z) \end{aligned}$$

(asymptotically stable; use $L(x, y, z) = (1/2)x^2 + 5(y^2 + z^2)$).

◇ **4.3-14.** Consider the following vector field on $\mathbb{R}^{n+1}$;

$$X(s, x) = (as^N + f(s) + g(s, x), Ax + F(x) + h(s, x))$$

where $s \in \mathbb{R}$, $x \in \mathbb{R}^n$, $f(0) = \cdots = f^{(N)}(0) = 0$, $g(s, x)$, has all derivatives of order ≤ 2 zero at the origin, and $F(x), h(s, x)$ vanish together with their first derivative at the origin. Assume the $n \times n$ matrix A has all eigenvalues distinct with strictly negative real part. Prove the following theorem of Liapunov: *if N is even or N is odd and $a > 0$, then the origin is unstable; if N is odd and $a < 0$, the origin is asymptotically stable.* HINT: for N even, use $L(s, x) = s - a\|x\|^2/2$ and for N odd, $L(s, x) = (s^2 - a\|x\|^2)/2$; show that in both cases the sign of $X[L]$ near the origin is given by the sign of a.

◇ **4.3-15.** Let $\mathbf{E}$ be a Banach space and $A : \mathbb{R} \to L(\mathbf{E}, \mathbf{E})$ a continuous map. Let $F_{t,s}$ denote the evolution operator of the time-dependent vector field $X(t, x) = A(t)x$ on $\mathbf{E}$.

(i) Show that $F_{t,s} \in \mathrm{GL}(\mathbf{E})$.

(ii) Show that $\|F_{t,s}\| \leq e^{(t-s)\alpha}$, where $\alpha = \sup_{\lambda \in [s,t]} \|A(\lambda)\|$. Conclude that the vector field $X(t, x)$ is complete.

HINT: Use Gronwall's inequality and the time dependent version of Proposition 4.1.22.

Next assume that A is periodic with period T, that is, $A(t + T) = A(t)$ for all $t \in \mathbb{R}$.

(iii) Show that $F_{t+T,s+T} = F_{t,s}$ for any $t, s \in \mathbb{R}$.

HINT: Show that $t \mapsto F_{t+T,s+T}(x)$ satisfies the differential equation $\dot{x} = A(t)x$.

(iv) Define the ***monodromy operator*** by $M(t) = F_{t+T,t}$. Show that if A is independent of t, then $M(t) = e^{TA}$ is also independent of t. Show that $M(s) = F_{t,s} \circ M(t) \circ F_{s,t}$ for any continuous $A : \mathbb{R} \to L(\mathbf{E}, \mathbf{E})$. Conclude that all solutions of $\dot{x} = A(t)x$ are of period T if and only if there is a t_0 such that $M(t_0) =$ identity. Show that the eigenvalues of $M(t)$ are independent of t.

(v) (Floquet). Show that each real eigenvalue λ of $M(t_0)$ determines a solution, denoted $c(t; \lambda, t_0)$ of $\dot{x} = A(t)x$ satisfying $c(t+T; \lambda, t_0) = \lambda c(t; \lambda, t_0)$, and also each complex eigenvalue $\lambda = a + ib$, of $M(t_0)$, where $a, b \in \mathbb{R}$, determines a pair of solutions denoted $c^r(t; \lambda, t_0)$ and $c^i(t; \lambda, t_0)$ satisfying

$$\begin{bmatrix} c^r(t+T;\lambda,t_0) \\ c^i(t+T;\lambda,t_0) \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c^r(t;\lambda,t_0) \\ c^i(t;\lambda,t_0) \end{bmatrix}.$$

HINT: Let $c(t; \lambda, t_0)$ denote the solution of $\dot{x} = A(t)x$ with the initial condition $c(t_0; \lambda, t_0) = e$, where e is an eigenvector of $M(t_0)$ corresponding to λ; if λ is complex, work on the complexification of $\mathbf{E}$. Then show that $c(t+T; \lambda, t_0) - \lambda c(t; \lambda, t_0)$ satisfies the same differential equation and its value at t_0 is zero since $c(t_0 + T; \lambda, t_0) = M(t_0)c(t_0; \lambda, t_0) = \lambda c(t_0; \lambda, t_0)$.)

(vi) (Floquet). Show that there is a nontrivial periodic solution of period T of $\dot{x} = A(t)x$ if and only if 1 is an eigenvalue of $M(t_0)$ for some $t_0 \in \mathbb{R}$.

HINT: If $c(t)$ is such a periodic solution, then $c(t_0) = c(t_0 + T) = M(t_0)c(t_0)$.

(vii) (Liapunov). Let $P : \mathbb{R} \to \mathrm{GL}(\mathbf{E})$ be a C^1 function which is periodic with period T. Show that the change of variable $y = P(t)x$ transforms the equation $\dot{x} = A(t)x$ into the equation $\dot{y} = B(t)y$, where $B(t) = (P'(t) + P(t)A(t))P(t)^{-1}$. If $N(t)$ is the monodromy operator of $\dot{y} = B(t)y$, show that $N(t) = P(t)M(t)P(t)^{-1}$.

◇ **4.3-16.** (i) (Liapunov). Let $\mathbf{E}$ be a finite dimensional *complex* vector space and let $A : \mathbb{R} \to L(\mathbf{E}, \mathbf{E})$ be a continuous function which is periodic with period T. Let $M(s)$ be the monodromy operator of the equation $\dot{x} = A(t)x$ and let $B \in L(\mathbf{E}, \mathbf{E})$ be such that $M(s) = e^{TB}$ (see Exercise 4.1-15). Define $P(t) = e^{tB}F_{s,t}$ and put $y(t) = P(t)x(t)$. Use (vii) in the previous exercise to show that $y(t)$ satisfies $\dot{y} = By$. Prove that $P(t)$ is a periodic C^1-function with period T.

HINT: Use (iii) of the previous exercise and $e^{TB} = M(s) = F_{s+T,s}$. Thus, for complex finite dimensional vector spaces, the equation $\dot{x} = A(t)x$, where $A(t+T) = A(t)$, can be transformed via $y(t) = P(t)x(t)$ into the constant coefficient linear equation $\dot{y} = By$.

(ii) Since the general solution of $\dot{y} = By$ is a linear combination of vectors $\exp(t\lambda_i)t^{\ell(i)}u_i$, where $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of B, $u_i \in \mathbf{E}$, and $1 \le \ell(i) \le$ multiplicity of λ_i, conclude that the general solution of $\dot{x} = A(t)x$ is a linear combination of vectors $\exp(t\lambda_i)t^{\ell(i)}P(t)^{-1}u_i$ where $P(t+T) = P(t)$. Show that the eigenvalues of $M(s) = e^{TB}$ are $\exp(T\lambda_i)$. Show that $\mathrm{Re}(\lambda_i) < 0$ (> 0) for all $i = 1, \ldots, m$ if and only if all the solutions of $\dot{x} = A(t)x$ converge to the origin as $t \to +\infty$ $(-\infty)$, that is, if and only if the origin is asympotically stable (unstable).

4.4 Frobenius' Theorem and Foliations

The main pillars supporting differential topology and calculus on manifolds are the implicit function theorem, the existence theorem for ordinary differential equations, and Frobenius' theorem, which we discuss briefly here. First some definitions.

4.4.1 Definition. *Let M be a manifold and let $E \subset TM$ be a subbundle of its tangent bundle; that is, E is a **distribution** (or a **plane field**) on M.*

(i) *We say E is **involutive** if for any two vector fields X and Y defined on open sets of M and which take values in E, $[X, Y]$ takes values in E as well.*

(ii) *We say E is **integrable** if for any $m \in M$ there is a (local) submanifold $N \subset M$, called a **(local) integral manifold** of E at m containing m, whose tangent bundle is exactly E restricted to N.*

The situation is shown in Figure 4.4.1.

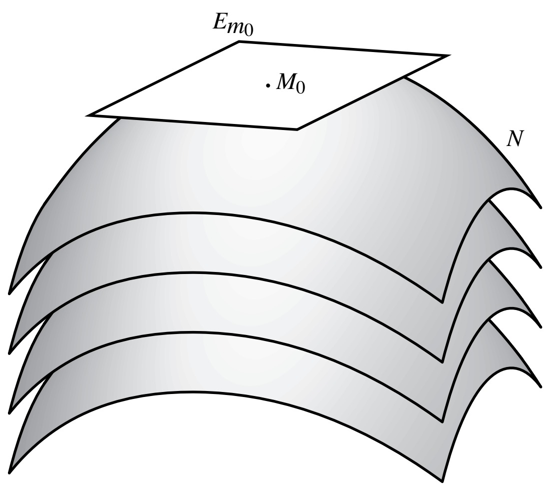


FIGURE 4.4.1. Local integrable manifolds

4.4.2 Examples.

A. Any subbundle E of TM with one dimensional fibers is involutive; E is also integrable, which is seen in the following way. Using local bundle charts for TM at $m \in M$ with the subbundle property for E, find in an open neighborhood of m, and a vector field that never vanishes and has values in E. Its local integral curves through m have as their tangent bundles E restricted to these curves. If the vector field can be found globally and has no zeros, then through any point of the manifold there is exactly one maximal integral curve of the vector field, and this integral curve never reduces to a point.

B. Let $f : M \to N$ be a submersion and consider the bundle $\ker Tf \subset TM$. This bundle is involutive since for any $X, Y \in \mathfrak{X}(M)$ which take values in $\ker Tf$, we have $Tf([X, Y]) = 0$ by Proposition 4.2.25. The bundle is integrable since for any $m \in M$ the restriction of $\ker Tf$ to the submanifold $f^{-1}(f(m))$ coincides with the tangent bundle of this submanifold (see §3.5).

C. Let $\mathbb{T}^n$ be the n-dimensional torus, $n \geq 2$. Let $1 \leq k \leq n$ and consider $E = \{(v_1, \ldots, v_n) \in T\mathbb{T}^n \mid v_{k+1} = \cdots = v_n = 0\}$. This distribution is involutive and integrable; the integral manifold through $(t_1, \ldots, t_n)$ is $\mathbb{T}^k \times (t_{k+1}, \ldots, t_n)$.

D. $E = TM$ is involutive and integrable; the integral submanifold through any point is M itself.

E. An example of a noninvolutive distribution is as follows. Let $M = \mathrm{SO}(3)$, the rotation group (see Exercise 3.5-19). The tangent space at $I = identity$ consists of the 3×3 skew symmetric matrices. Let

$$E_I = \left\{ A \in T_I \mathrm{SO}(3) \;\middle|\; A = \begin{bmatrix} 0 & 0 & -q \\ 0 & 0 & p \\ q & -p & 0 \end{bmatrix} \text{ for some } p, q \in \mathbb{R} \right\}$$

a two dimensional subspace. For $Q \in \mathrm{SO}(3)$, let

$$E_Q = \{ B \in T_Q \mathrm{SO}(3) \mid Q^{-1}B \in E_I \}.$$

Then $E = \bigcup\{ E_Q \mid Q \in \mathrm{SO}(3) \}$ is a distribution but is not involutive. In fact, one computes that the two vector fields with $p = 1$, $q = 0$ and $p = 0$, $q = 1$ have a bracket that does not lie in E. Further insight into this example is gained after one studies Lie groups (a supplementary chapter).

F. Let E be a distribution on M. Suppose that a collection $\mathcal{E}$ of smooth sections of E spans E in the sense that for each section X of E there are vector fields $X_1, \ldots, X_k$ in $\mathcal{E}$ and smooth functions $a^1, \ldots, a^k$ on M such that $X = a^i X_i$. Suppose $\mathcal{E}$ is closed under bracketing; that is, if X and Y have values in $\mathcal{E}$, so does $[X, Y]$. We claim that E is *involutive*.

To prove this assertion, let X and Y be sections of E and write $X = a^i X_i$ and $Y = b^j Y_j$, where a^i and b^j are smooth functions on M and X_i and Y_j belong to $\mathcal{E}$. We calculate

$$[X, Y] = B^j Y_j - A^i X_i + a^i b^j [X_i, Y_j],$$

where

$$B^j = a^i X_i[b^j] \quad \text{and} \quad A^i = b^j Y_j[a^i].$$

Thus $[X, Y]$ is a section of E, so E is involutive. ♦

Frobenius' theorem asserts that the two conditions in Definition 4.4.1 are equivalent.

4.4.3 Theorem (The Local Frobenius Theorem). *A subbundle E of TM is involutive if and only if it is integrable.*

Proof. Suppose E is integrable. Let X and Y be sections of E and let N be a local integral manifold through $m \in M$. At points of N, X and Y are tangent to N, so define restricted vector fields $X|_N, Y|_N$ on N. By Proposition 4.2.25 (on φ-relatedness of brackets) applied to the inclusion map, we have

$$[X|_N, Y|_N] = [X, Y]|_N.$$

Since N is a manifold, $[X|_N, Y|_N]$ is a vector field on N, so $[X, Y]$ is tangent to N and hence in E.

Conversely, suppose that E is involutive. By choosing a vector bundle chart, one is reduced to this local situation: E is a model space for the fibers of E, F is a complementary space, and $U \times V \subset E \times F$ is an open neighborhood of $(0, 0)$, so $U \times V$ is a local model for M. We have a map $f : U \times V \to L(E, F)$ such that the fiber of E over (x, y) is

$$E_{(x,y)} = \{ (u, f(x, y) \cdot u) \mid u \in E \} \subset E \times F,$$

and we can assume we are working near $(0, 0)$ and $f(0, 0) = 0$. Let us express involutivity of E in terms of f.

For fixed $u \in E$, let $X_u(x, y) = (u, f(x, y) \cdot u)$. Using the local formula for the Lie bracket (see formula (4.2.5)) one finds that

$$\begin{aligned}[X_{u_1}, X_{u_2}](x, y) = (0, &\mathbf{D}f(x, y) \cdot (u_1, f(x, y) \cdot u_1) \cdot u_2 \\ &-\mathbf{D}f(x, y) \cdot (u_2, f(x, y) \cdot u_2) \cdot u_1).\end{aligned} \tag{4.4.1}$$

By the involution assumption, this lies in $E_{(x,y)}$. Since the first component vanishes, the local description of $E_{(x,y)}$ above shows that the second must as well; that is, we get the following identity:

$$\mathbf{D}f(x,y)\cdot(u_1, f(x,y)\cdot u_1)\cdot u_2 = \mathbf{D}f(x,y)\cdot(u_2, f(x,y)\cdot u_2)\cdot u_1. \tag{4.4.1'}$$

Consider the time-dependent vector fields

$$X_t(x,y) = (0, f(tx,y)\cdot x) \quad \text{and} \quad X_{t,u}(x,y) = (u, f(tx,y)\cdot tu),$$

so that by the local formula for the Jacobi–Lie bracket,

$$\begin{aligned}[X_t, X_{t,u}](x,y) &= (0, t\mathbf{D}_2 f(tx,y)\cdot(f(tx,y)\cdot x)\cdot u \\ &\qquad -t\mathbf{D}f(tx,y)\cdot(u, f(tx,y)\cdot u)\cdot x - f(tx,y)\cdot u) \\ &= (0, -t\mathbf{D}_1 f(tx,y)\cdot x\cdot u - f(tx,y)\cdot u),\end{aligned} \tag{4.4.2}$$

where the last equality follows from (4.4.1′). But $\partial X_{t,u}/\partial t$ equals the negative of the right hand side of equation (4.4.2), that is,

$$[X_t, X_{t,u}] + \frac{\partial X_{t,u}}{\partial t} = 0,$$

which by Theorem 4.2.31 is equivalent to

$$\frac{d}{dt} F_t^* X_{t,u} = 0, \tag{4.4.3}$$

where $F_t = F_{t,0}$ and $F_{t,s}$ is the evolution operator of the time dependent vector field X_t. Since $X_t(0,0) = 0$, it follows that F_t is defined for $0 \leq t \leq 1$ by Corollary 4.1.25.

Since F_0 is the identity, relation (4.4.3) implies that

$$F_t^* X_{t,u} = X_{0,u}, \quad \text{i.e.,} \quad TF_1 \circ X_{0,u} = X_{1,u} \circ F_1. \tag{4.4.4}$$

Let $N = F_1(E \times \{0\})$, a submanifold of $E \times F$, the model space of M. If $(x,y) = F_1(e,0)$, the tangent space at (x,y) to N equals

$$\begin{aligned}T_{(x,y)}N &= \{\, T_{(e,0)}F_1(u,0) \mid u \in E \,\} = \{\, T_{(e,0)}F_1(X_{0,u}(e,0)) \mid u \in E \,\} \\ &= \{\, X_{1,u}(F_1(e,0)) \mid u \in E \,\} \qquad\qquad \text{(by (4.4.4))} \\ &= \{\, (u, f(x,y)\cdot u) \mid u \in E \,\} = E_{(x,y)}.\end{aligned}$$ ■

The method of using the time-one map of a time-dependent flow to provide the appropriate coordinate change is useful in a number of situations and is called the ***method of Lie transforms***. An abstract version of this method is given later in Example 5.4.7; we shall use this method again in Chapter 6 to prove the Poincaré lemma and in Chapter 8 to prove the Darboux theorem.

NOTE: The method of Lie transforms is also used in singularity theory and bifurcation theory (see Golubitsky and Schaeffer [1985]). For a proof of the Morse lemma using this method, see Proposition 5.5.8, which is based on Palais [1969] and Golubitsky and Marsden [1983]. For a proof of the Frobenius theorem from the implicit function theorem using manifolds of maps in the spirit of Supplement 4.1C, see Penot [1970]. See Exercise 4.4-6 for another proof of the Frobenius theorem in finite dimensions.

The Frobenius theorem is intimately connected to the global concept of foliations. Roughly speaking, the integral manifolds N can be glued together to form a "nicely stacked" family of submanifolds filling out M (see Figure 4.4.1 or Example 4.4.2A).

4.4.4 Definition. *Let M be a manifold and $\Phi = \{\pounds_\alpha\}_{\alpha \in A}$ a partition of M into disjoint connected sets called* ***leaves****. The partition Φ is called a* ***foliation*** *if each point of M has a chart (U, φ), $\varphi : U \to U' \times V' \subset E \oplus F$ such that for each $\pounds_\alpha$ the connected components $(U \cap \pounds_\alpha)^\beta$ of $U \cap \pounds_\alpha$ are given by $\varphi((U \cap \pounds_\alpha)^\beta) = U' \times \{c_\alpha^\beta\}$, where $c_\alpha^\beta \in F$ are constants f or each $\alpha \in A$ and β. Such charts are called* ***foliated*** *(or* ***distinguished****) by Φ. The* ***dimension*** *(respectively,* ***codimension****)* ***of the foliation*** *Φ is the dimension of E (resp., F). See Figure* 4.4.2.

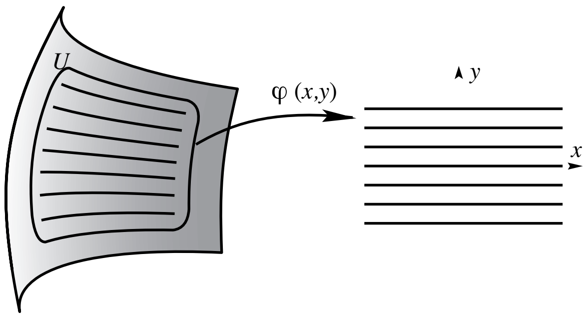


FIGURE 4.4.2. Chart for a foliation

Note that each leaf $\pounds_\alpha$ is a connected immersed submanifold. In general, this immersion is *not* an embedding; that is, the induced topology on $\pounds_\alpha$ from M does not necessarily coincide with the topology of $\pounds_\alpha$ (the leaf $\pounds_\alpha$ may accumulate on itself, for example). A differentiable structure on $\pounds_\alpha$ is induced by the foliated charts in the following manner. If (U, φ), $\varphi : U \to U' \times V' \subset E \oplus F$ is a foliated chart on M, and $\chi : E \oplus F \to F$ is the canonical projection, then $\chi \circ \varphi$ restricted to $(U \cap \pounds_\alpha)^\beta$ defines a chart on $\pounds_\alpha$.

4.4.5 Examples.

A. The ***trivial foliation*** of a connected manifold M has only one leaf, M itself. It has codimension zero. If M is finite dimensional, both M and the leaf have the same dimension. Conversely, on a finite-dimensional connected manifold M, a foliation of dimension equal to $\dim(M)$ is the trivial foliation.

B. The ***discrete foliation*** of a manifold M is the only zero-dimensional foliation; its leaves are all points of M. If M is finite dimensional, the dimension of M is the codimension of this foliation.

C. A vector field X that never vanishes on M determines a foliation; its leaves are the maximal integral curves of the vector field X. The fact that this is a foliation is the straightening out theorem (see §4.1).

D. Let $f : M \to N$ be a submersion. It defines a foliation on M (of codimension equal to $\dim(N)$ if $\dim(N)$ is finite) by the collection of all connected components of $f^{-1}(n)$ when n varies throughout N. The fact that this is a foliation is given by Theorem 3.5.4. In particular, we see that $E \oplus F$ is foliated by the family $\{E \times \{f\}\}_{f \in F}$.

E. In the preceding example, let $M = \mathbb{R}^3$, $N = \mathbb{R}$, and $f(x^1, x^2, x^3) = \varphi(r^2)\exp(x^3)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a C^∞ function satisfying $\varphi(0) = 0$, $\varphi(1) = 0$, and $\varphi'(s) < 0$ for $s > 0$, and where $r^2 = (x^1)^2 + (x^2)^2$. Since

$$\mathbf{d}f(x^1, x^2, x^3) = \exp(x^3)[2\varphi'(r^2)x^1\mathbf{d}x^1 + 2\varphi'(r^2)x^2\mathbf{d}x^2 + \varphi(r^2)\mathbf{d}x^3]$$

and $\varphi(r^2)$ is a strictly decreasing function of r^2, f is submersion, so its level sets define a codimension one foliation on $\mathbb{R}^3$. Since the only zero of $\varphi(r^2)$ occurs for $r = 1$, $f^{-1}(0)$ equals the cylinder $\{\,(x^1, x^2, x^3) \mid (x^1)^2 + (x^2)^2 = 1\,\}$. Since $\varphi(r^2)$ is a positive function for $r \in [0, 1[$, it follows that if $c > 0$, then $f^{-1}(c) = \{\,(x^1, x^2, \log(c/\varphi(r^2)) \mid 0 \le (x^1)^2 + (x^2)^2 < 1\,\}$, which is diffeomorphic to the open unit ball in (x^1, x^2)-space via the projection $(x^1, x^2, \log(c/\varphi(r^2)) \mapsto (x^1, x^2)$. Note that the leaves $f^{-1}(c)$, $c > 0$, are asymptotically tangent to the cylinder $f^{-1}(0)$. Finally, since $\varphi(r^2) < 0$ if $r > 1$, for $c < 0$ the leaves given by $f^{-1}(c) = \{ (x^1, x^2, \log(c/\varphi(r^2)) \mid (x^1)^2 + (x^2)^2 > 1 \}$ are diffeomorphic to the plane minus the closed unit disk, that is, they are diffeomorphic to cylinders. As before, note that the cylinders $f^{-1}(c)$ are symptotically tangent to $f^{-1}(0)$; see Figure 4.4.3.

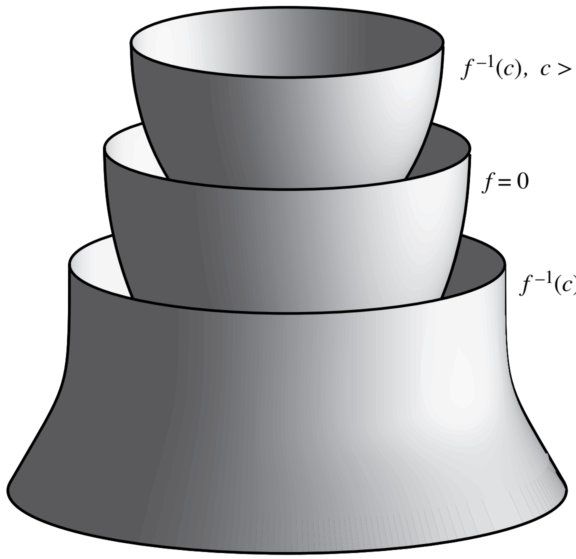


FIGURE 4.4.3. A change in the topology of level sets

F. (The Reeb Foliation on the Solid Torus and the Klein Bottle; Reeb [1952]). We claim that in the previous example, the leaves are in some sense translation invariant. The cylinder $f^{-1}(0)$ is invariant; if $c \neq 0$, invariance is in the sense $f^{-1}(c) + (0, 0, \log t) = f^{-1}(tc)$, for any $t > 0$. Consider the part of the foliation within the solid cylinder $(x^1)^2 + (x^2)^2 \leq 1$ and form the solid torus from this cylinder: identify $(a^1, a^2, 0)$ with $(b^1, b^2, 1)$ iff $a^i = b^i$, $i = 1, 2$. The foliation of the solid torus so obtained is called the ***orientable Reeb foliation***. Out of the cylinder one can form the Klein bottle (see Figure 1.4.2) by considering the equivalence relation which identifies $(a^1, a^2, 0)$ with $(b^1, b^2, 1)$ iff $a^1 = b^1, a^2 = -b^2$. In this way one obtains the ***nonorientable Reeb foliation***. (This terminology regarding orientability will be explained in §6.5.)

G. (The Reeb Foliation on S^3; Reeb [1952]). Two orientable Reeb foliations on the solid torus determine a foliation on S^3 in the following way. The sphere S^3 is the union of two solid tori which are identified along their common boundary, the torus $\mathbb{T}^2$, by the diffeomorphism taking meridians of one to parallels of the other and vice-versa. This foliation is called the ***Reeb foliation of*** S^3; it has one leaf diffeomorphic to the torus $\mathbb{T}^2$ and all its other leaves are diffeomorphic to $\mathbb{R}^2$ and accumulate on the torus. Below we describe, pictorially, the decomposition of S^3 in two solid tori. Remove the north pole, $(1, 0, 0, 0)$, of S^3 and stereographically project the rest of S^3 onto $\mathbb{R}^3$. In the plane (x^2, x^4) draw two equal circles centered on the x^2-axis at the points a and b, where $-a = b$. Rotating about the x^4-axis yields the solid torus in $\mathbb{R}^3$. Now draw all of the circles in the (x^2, x^4) plane minus the two discs, centered on the x^4-axis and passing through a and b. Each such circle yields two connected arcs joining the two discs. In addition, consider the two portions of the x^2-axis: the line joining the two discs and the two rays going off from each disk separately. (See Figure 4.4.4.) Now rotate this figure about the x^4-axis. All arcs joining the disks generate smooth surfaces diffeomorphic to $\mathbb{R}^2$ and each such surface meets the solid torus along a parallel. Only the two rays emanating from the disks generate a surface diffeomorphic to the cylinder. Now add the north pole back to S^3 and pull back the whole structure via stereographic projection from $\mathbb{R}^3$ to S^3: the cylinder becomes a torus and all surfaces diffeomorphic to $\mathbb{R}^2$ intersect this torus in meridians. Thus S^3 is the union of two solid tori glued along their common boundary by identifying parallels of one with meridians of the other and vice-versa. ♦

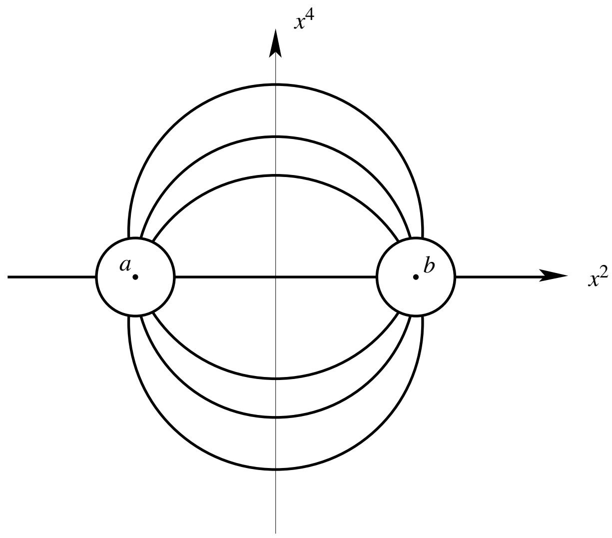


FIGURE 4.4.4. Construction for the Reeb foliation.

4.4.6 Proposition. *Let M be a manifold and $\Phi = \{\mathcal{L}_\alpha\}_{\alpha \in A}$ be a foliation on M. The set*

$$T(M, \Phi) = \bigcup_{\alpha \in A} \bigcup_{m \in \mathcal{L}_\alpha} T_m \mathcal{L}_\alpha$$

*is a subbundle of TM called the **tangent bundle to the foliation**. The quotient bundle, denoted $\nu(\Phi) = TM/T(M, \Phi)$, is called the **normal bundle** to the foliation Φ. Elements of $T(M, \Phi)$ are called **vectors tangent to the foliation** Φ.*

Proof. Let (U, φ), $\varphi : U \to U' \times V' \to E \oplus F$ be a foliated chart. Since $T_u\varphi(T_u\mathcal{L}_\alpha) = E \times \{0\}$ for every $u \in U \cap \mathcal{L}_\alpha$, we have

$$T\varphi(TU \cap T(M, \Phi)) = (U' \times V') \times (E \times \{0\}).$$

Thus, the standard tangent bundle charts induced by foliated charts of M have the subbundle property and naturally induce vector bundle charts by mapping $v_m \in T_m(M, \Phi)$ to $(\varphi(m), T_m\varphi(v_m)) \in (U' \times V') \times (E \times \{0\})$. ■

4.4.7 Theorem (The Global Frobenius Theorem). *Let E be a subbundle of TM. The following are equivalent:*

(i) *There exists a foliation Φ on M such that $E = T(M, \Phi)$.*

(ii) *E is integrable.*

(iii) *E is involutive.*

Proof. The equivalence of (ii) and (iii) was proved in Theorem 4.4.3. Let (i) hold. Working with a foliated chart, E is integrable by Proposition 4.4.6, the integral submanifolds being the leaves of Φ. Thus (ii) holds. Finally, we need to show that (ii) implies (i). Consider on M the family of (local) integral manifolds of E, each equipped with its own submanifold topology. It is straightforward to verify that the family of finite intersections of open subsets of these local integral submanifolds defines a topology on M, finer in general than the original one. Let $\{\pounds_\alpha\}_{\alpha \in A}$ be its connected components. Then, denoting by $(\pounds_\alpha \cap U)^\beta$ the connected components of $\pounds_\alpha \cap U$ in U, we have by definition $E|(U \cap \pounds_\alpha)^\beta = (E$ restricted to $(U \cap \pounds_\alpha)^\beta)$ equals $T((U \cap \pounds_\alpha)^\beta)$. Let $(\tau^{-1}(U), \psi)$, $U \subset M$ be a vector bundle chart of TM with the subbundle property for E, and let $\varphi : U \to U_1$ be the induced chart on the base. This means, shrinking U if necessary, that $\varphi : U \to U' \times V' \subset E \oplus F$ and

$$\psi(\tau^{-1}(U) \cap E) = (U' \times V') \times (E \times \{0\}).$$

Thus, $\varphi((U \cap \pounds_\alpha)^\beta) = U' \times \{c_\alpha^\beta\}$ and so M is foliated by the family $\Phi = \{\pounds_\alpha\}_{\alpha \in A}$. Because $E|(U \cap \pounds_\alpha)^\beta = T((U \cap \pounds_\alpha)^\beta)$, we also have $T(M, \Phi) = E$. ■

There is an important global topological condition that integrable subbundles must satisfy that was discovered by Bott [1970]. The result, called the ***Bott Vanishing Theorem***, can be found, along with related results, by readers with background in algebraic topology, in Lawson [1977].

The leaves of a foliation are characterized by the following property.

4.4.8 Proposition. *Let Φ be a foliation on M. Then x and y are in the same leaf if and only if x and y lie on the same integral curve of a vector field X defined on an open set in M and which is tangent to the foliation Φ.*

Proof. Let X be a vector field on M with values in $T(M, \Phi)$ and assume that x and y lie on the same integral curve of X. Let $\pounds$ denote the leaf of Φ containing x. Since X is tangent to the foliation, $X|\pounds$ is a vector field on $\pounds$ and thus any integral curve of X starting in $\pounds$ stays in $\pounds$. Since y is on such an integral curve, it follows that $y \in \pounds$.

Conversely, let $x, y \in \pounds$ and let $c(t)$ be a smooth non-intersecting curve in $\pounds$ such that $c(0) = x$, $c(1) = y$, $c'(t) \neq 0$. (This can always be done on a connected manifold by showing that the set of points that can be so joined is open and closed.) Thus $c : [0, 1] \to \pounds$ is an immersion, and hence by compactness of $[0, 1], c$ is an embedding. Using Definition 4.4.4, there is a neighborhood of the curve c in M which is diffeomorphic to a neighborhood of $[0, 1] \times \{0\} \times \{0\}$ in $\mathbb{R} \times F \times G$ for Banach spaces F and G such that the leaves of the foliation have the local representation $\mathbb{R} \times F \times \{w\}$, for fixed $w \in G$, and the image of the curve c has the local representation $[0, 1] \times \{0\} \times \{0\}$. Thus we can find a vector field X which is defined by $c'(t)$ along c and extends off c to be constant in this local representation. ■

Let R denote the following equivalence relation in a manifold M with a given folation Φ : xRy if x, y belong to the same leaf of Φ. The previous proposition shows that R is an open equivalence relation. It is of interest to know whether M/R is a manifold. Foliations for which R is a regular equivalence relation are called ***regular foliations***. (See §3.5 for a discussion of regular equivalence relations.) The following is a useful criterion.

4.4.9 Proposition. *Let Φ be a foliation on a manifold M and R the equivalence relation in M determined by Φ. R is regular iff for every $m \in M$ there exists a local submanifold Σ_m of M such that Σ_m intersects every leaf in at most one point (or nowhere) and $T_m\Sigma_m \oplus T_m(M, \Phi) = T_mM$. (Sometimes Σ_m is called a **slice** or a **local cross-section** for the foliation.)*

Proof. Assume that R is regular and let $\pi : M \to M/R$ be the canonical projection. For Σ_m choose the submanifold using the following construction. Since π is a submersion, in appropriate charts $(U, \varphi), (V, \psi)$, where $\varphi : U \to U' \times V'$ and $\psi : V \to V'$, the local representative of π, $\pi_{\varphi\psi} : U' \times V' \to V'$, is the projection onto the second factor, and every leaf $\pi^{-1}(v) \subset U$, $v \in V$, is represented in these charts as $U' \times \{v'\}$ where $v' = \psi(v)$. Thus if $\Sigma_m = \psi^{-1}(\{0\} \times V')$, we see that Σ_m satisfies the two required conditions.

Conversely, assume that each point $m \in M$ admits a slice Σ_m. Working with a foliated chart, we are reduced to the following situation: let U, V be open balls centered at the origin in Banach spaces E and F, respectively, let Σ be a submanifold of $U \times V$, $(0,0) \in \Sigma$, such that $T_{(0,0)}\Sigma = F$, and $\Sigma \cap (U \times \{v\})$ is at most one point for all $v \in V$. If $p_2 : E \oplus F \to F$ is the second projection, since $p_2|\Sigma$ has tangent map at $(0,0)$ equal to the identity, it follows that for V small enough, $p_2|\Sigma : \Sigma \to V$ is a diffeomorphism. Shrinking Σ and V if necessary we can assume that $\Sigma \cap (U \times \{v\})$ is exactly one point. Let $q : V \to \Sigma$ be the inverse of p_2 and define the smooth map $s : U \times V \to \Sigma$ by $s(u,v) = q(v)$. Then $\Sigma \cap (U \times \{v\}) = \{q(v)\}$, thus showing that Σ is a slice in the sense of Lemma 3.5.26. Pulling everything back to M by the foliated chart, the prior argument shows that for each point $m \in M$ there is an open neighborhood U, a submanifold Σ_m of U, and a smooth map $s : U \to \Sigma_m$ such that $\pounds_u \cap \Sigma_m = \{s(u)\}$, where $\pounds_u$ is the leaf containing $u \in U$. By the argument following Lemma 3.5.26, the equivalence relation R is locally regular, that is, $R_U = R \cap (U \times U)$ is regular. If $U' = \pi^{-1}(\pi(U))$ where $\pi : M \to M/R$ is the projection, the argument at the end of Step 1 in the proof of Theorem 3.5.25 shows that R is regular. Thus, all that remains to be proved is that U can be chosen to equal U'. But this is clear by defining $s' : U' \to \Sigma_m$ by $s'(u') = s(u)$, where $u \in U \bigcap \pounds_{u'}$, $\pounds_{u'}$ being the leaf containing u'; smoothness of s' follows from smoothness of s by composing it locally with the flow of a vector field given by Proposition 4.4.8. ■

To get a feeling for the foregoing condition we will study the linear flow on the torus.

4.4.10 Example. On the two-torus $\mathbb{T}^2$ consider the global flow $F : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{T}^2$ defined by $F(t,(s_1,s_2)) = (s_1 e^{2\pi i t}, s_2 e^{2\pi i \alpha t})$ for a fixed number $\alpha \in [0,1[$. By Example 4.4.5C this defines a foliation on $\mathbb{T}^2$. If $\alpha \in \mathbb{Q}$, notice that every integral curve is closed and that all integral curves have the same period. The condition of the previous theorem is easily verified and we conclude that in this case the equivalence relation R is regular; $\mathbb{T}^2/R = S^1$. If α is irrational, however, the situation is completely different. Let $\varphi(t) = (e^{2\pi i t}, e^{2\pi i \alpha t})$ denote the integral curve through $(1,1)$. The following argument shows that $\mathrm{cl}\,(\varphi(\mathbb{R})) = \mathbb{T}^2$; that is, $\varphi(\mathbb{R})$ is *dense* in $\mathbb{T}^2$. Let $p = (e^{2\pi i x}, e^{2\pi i y}) \in \mathbb{T}^2$; then for all $m \in \mathbb{Z}$,

$$\varphi(x+m) - p = (0, e^{2\pi i \alpha x}(e^{2\pi i m \alpha} - e^{2\pi i z}))$$

where $y = \alpha x + z$. It suffices to show that $C = \{ e^{2\pi i m\alpha} \in S^1 \mid m \in \mathbb{Z} \}$ is dense in S^1 because then there is a sequence $m_k \in \mathbb{Z}$ such that $\exp(2\pi i m_k \alpha)$ converges to $e^{2\pi i z}$. Hence, $\varphi(x + m_k)$ converges to p. If for each $k \in \mathbb{Z}_+$ we divide S^1 into k arcs of length $2\pi/k$, then, because $\{ e^{2\pi i m\alpha} \in S^1 \mid m = 1,2,\ldots,k+1 \}$ are distinct for some $1 \le n_k < m_k \le k+1$, $\exp(2\pi i m_k \alpha)$ and $\exp(2\pi i n_k \alpha)$ belong to the same arc. Therefore,

$$|\exp(2\pi i m_k \alpha) - \exp(2\pi i n_k \alpha)| < \frac{2\pi}{k},$$

which implies $|\exp(2\pi i q_k \alpha) - 1| < 2\pi/k$, where $q_k = m_k - n_k$. Because

$$\bigcup_{j \in \mathbb{Z}_+} \{ e^{2\pi i \alpha s} \in S^1 \mid s \in [jq_k, (j+1)q_k] \} = S^1,$$

every arc of length less than $2\pi/k$ contains some $\exp(2\pi i j q_k)$, which proves $\mathrm{cl}(C) = S^1$. Thus any submanifold $\Sigma_m m \in \mathbb{T}^2$ not coinciding with the integral curve through m will have to intersect $\varphi(\mathbb{R})$ infinitely many times; the condition in the previous theorem is violated and so R is *not* regular.

♦

Remark. Novikov [1965] has shown that the Reeb foliation is in some sense typical. A foliation Φ on M is said to be ***transversally orientable*** if $TM = T(M,\Phi) \oplus E$, where E is an orientable subbundle of TM (see Exercise 6.5-14 for the definition). A foliation on a three dimensional manifold M is said to have a ***Reeb component*** if it has a compact leaf diffeomorphic to $\mathbb{T}^2$ or $\mathbb{K}$ and if the foliation within this torus or Klein bottle is diffeomorphic to the orientable or non-orientable Reeb foliation in the solid torus or the solid Klein bottle. Novikov has proved the following remarkable result. *Let Φ be a transversally orientable C^2 codimension one foliation of a compact three dimensional manifold M. If $\pi_1(M)$ is finite, then Φ has a Reeb component, which may or may not be orientable. If $\pi_2(M) \neq 0$ (with no hypotheses on $\pi_1(M)$) and Φ has no Reeb components, then all the leaves of Φ are compact with finite fundamental group*. We refer the reader to Camacho and Neto [1985] for a proof of this result and to this reference and Lawson [1977] for a study of foliations in general. ♦

Even though foliations encompass "nice" partitions of a manifold into submanifolds, there are important situations when foliations are inappropriate because they are not regular or the leaves jump in dimension from point to point. Consider, for example, $\mathbb{R}^2$ as a union of concentric circles centered at the origin. As this example suggests, one would like to relax the condition that M/R be a manifold, provided that M/R turns out to be a union of manifolds that fit "nicely" together. *Stratifications*, another concept allowing us to "stack" manifolds, turn out to be the natural tool to describe the topology of orbit spaces of compact Lie group actions or non-compact Lie group actions admitting a slice (see, for instance, Bredon [1972], Burghelea, Albu, and Ratiu [1975], Fischer [1970], and Bourguigon [1975]). We shall limit ourselves to the definition in the finite-dimensional case and some simple remarks.

4.4.11 Definition. *Let M be a locally compact topological space. A **stratification** of M is a partition of M into manifolds $\{M_a\}_{a\in A}$ called **strata**, satisfying the following conditions*:

- **S1.** *M_a are manifolds of constant dimension; they are submanifolds of M if M is itself a manifold.*
- **S2.** *The family $\{M_a^\alpha\}$ of connected components of all the M_a is a locally finite partition of M; that is, for every $m \in M$, there exists an open neighborhood U of m in M intersecting only finitely many M_b^β.*
- **S3.** *If $M_a^\alpha \cap \mathrm{cl}(M_b^\beta) \neq \varnothing$ for $(a,\alpha) \neq (b,\beta)$, then $M_a^\alpha \subset M_b^\beta$ and* $\dim(M_a) < \dim(M_b)$.
- **S4.** $\mathrm{cl}(M_a)\backslash M_a$ *is a disjoint union of strata of dimension strictly less than* $\dim(M_a)$.

From the definition it follows that if $M_a^\alpha \cap \mathrm{cl}(M_b^\beta) \neq \varnothing$ and if $m \in M_a^\alpha \subset \mathrm{cl}(M_b^\beta)$ has an open neighborhood U in the topology of M_a^α such that $U \subset M_a^\alpha \cap \mathrm{cl}(M_b^\beta)$, then necessarily $M_a^\alpha \subset M_b^\beta$ and thus $\dim(M_a) < \dim(M_b)$. To see this, it is enough to note that the given hypothesis makes $M_a^\alpha \cap \mathrm{cl}(M_b^\beta)$ open in M_a^α. Since it is also closed (by definition of the relative topology) and M_a^α is connected, it must equal M_a^α itself, whence $M_a^\alpha \subset \mathrm{cl}(M_b^\beta)$ and by **S3**, $M_a^\alpha \subset M_b^\beta$ and $\dim(M_a) < \dim(M_b)$.

For nonregular equivalence relations R, M/R is often a stratified space. The intuitive idea is that it is often possible to group together equivalence classes of the same dimension, and this grouping is parametrized by a manifold, which will be a stratum in M/R. A simple example is $\mathbb{R}^2$ partitioned by circles (the equivalence classes for R). The circles of positive radius are parametrized by the interval $]0,\infty[$. Thus M/R is the stratified set $[0,\infty[$ consisting of the two strata $\{0\}$ and $]0,\infty[$.

Exercises

◇ **4.4-1.** Let M be an n- manifold such that $TM = E_1 \oplus \cdots \oplus E_p$, where $E_i, i = 1, \ldots, p$ is an involutive subbundle of TM. Show that there are subspaces $\mathbf{E}_i \subset \mathbb{R}^n$, $i = 1, \ldots, p$ such that $\mathbb{R}^n = \mathbf{E}_1 \oplus \ldots \oplus \mathbf{E}_p$ and local charts $\varphi : U \subset M \to V \subset \mathbb{R}^n$, such that $T\varphi$ maps each fiber of E_i onto $\mathbf{E}_i$.

◇ **4.4-2.** In $\mathbb{R}^4$ consider the family of surfaces given by $x^2 + y^2 + z^2 - t^2 = \text{const}$. Show that these surfaces define a stratification. What part of $\mathbb{R}^4$ should be thrown out to obtain a regular foliation?

◇ **4.4-3.** Let $f : M \to N$ be a C^∞ map and Φ a foliation on N. The map f is said to be ***transversal*** to Φ, denoted $f \pitchfork \Phi$, if for every $m \in M$,

$$T_m f(T_m M) + T_{f(m)}(N,\Phi) = T_{f(m)}N \quad \text{and} \quad (T_m f)^{-1}(T_{f(m)}(N,\Phi))$$

splits in T_mM. Show that if $\{\mathcal{L}_\alpha\}_{\alpha\in A}$ are the leaves of Φ, the connected components of $f^{-1}(\mathcal{L}_\alpha)$ are leaves of a foliation (denoted by $f^*(\Phi)$) on M, and if Φ has finite codimension in N, then so does the foliation $f^*(\Phi)$ on M and the two codimensions coincide.

◇ **4.4-4** (Bourbaki [1971])**.** Let M be a manifold and denote by M' the manifold with underlying set M but with a different differentiable structure. Show that the collection of connected components of M' defines a foliation of M iff for every $m \in M$, there exists an open set U in M, $m \in U$, a manifold N, and a submersion $\rho : U \to N$ such that the submanifold $\rho^{-1}(n)$ of U is open in M' for all $n \in N$.
Hint: For the "if" part use Lemma 3.3.5 and for the "only if" part use Exercise 3.2-6 to define a manifold structure on the leaves; the charts of the second structure are $(U \cap \mathcal{L}_\alpha)^\beta \to U'$.

◇ **4.4-5.** On the manifold SO(3), consider the partition $\mathcal{L}_A = \{\, QA \mid Q$ is an arbitrary rotation about the z-axis in $\mathbb{R}^3\,\}$, $A \in \mathrm{SO}(3)$. Show that $\Phi = \{\, \mathcal{L}_A \mid A \in \mathrm{SO}(3)\,\}$ is a regular foliation and that the quotient manifold $\mathrm{SO}(3)/\mathbb{R}$ is diffeomorphic to S^2.

◇ **4.4-6** (Hirsch and Weinstein)**.** Give another proof of the Frobenius theorem as follows:
Step 1. Prove it for the Abelian case in which all sections of E satisfy $[X, Y] = 0$ by choosing a local basis $X_1, \ldots, X_k$ of sections and successively flowing out by the commuting flows of $X_1, \ldots, X_k$.
Step 2 Given a k-dimensional plane field, locally write it as a "graph" over $\mathbb{R}^k$. Choose k commuting vector fields on $\mathbb{R}^k$ and lift them to the plane field. If E is involutive, the bracket of two of them lies in E and, moreover, since the bracket "pushes down" to $\mathbb{R}^k$ (by "relatedness"), it is zero. (This is actually demonstrated in formula (4.4.1).) Now use Step 1.