CHAPTER X

# FURTHER READING

What has been described in the previous chapters is a tiny part of the general study of dynamical systems. For an introduction to the study of ergodic theory and dynamical systems, see Walters [35]. In this final chapter we indicate two directions for further reading: one is a direct generalization of the topics treated here to higher–dimensional actions. The other is a discussion of the effect of removing the connectedness hypothesis.

## Higher–dimensional actions

Let $X$ be a compact abelian group, and consider a collection $T_1, \dots, T_d$ of $d$ commuting automorphisms of $X$. A fundamental obervation (see Kitchens and Schmidt [13]) is that the resulting $\mathbb{Z}^d$ action $T$, defined by $T_{(n_1,\dots,n_d)} = T_1^{n_1} \dots T_d^{n_d}$, gives $\widehat{X}$ the structure of a module over the ring $\mathfrak{R} = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm d}]$. This is defined as follows: $\mathfrak{M} = \widehat{X}$ is an additive group (or "$\mathbb{Z}$–module"), and the maps $\widehat{T}_1, \dots, \widehat{T}_d$ are $d$ commuting automorphisms of $\mathfrak{M}$. If multiplication by $u_i$ is identified with the action of the map $\widehat{T}_i$, then the structure of $\mathfrak{M}$ as a $\mathbb{Z}$–module extends to give $\mathfrak{M}$ the structure of an $\mathfrak{R}$–module.

Dynamical properties of the $\mathbb{Z}^d$ action $T$ may then be described in terms of the algebraic and geometric properties of the module $\mathfrak{M}$: ergodicity and mixing (Kitchens and Schmidt [13]), expansiveness and periodic points (Schmidt [31]), higher order mixing (Schmidt [30], Schmidt and Ward [32]), entropy and growth rate of periodic points for expansive actions (Lind, Schmidt and Ward [21]), and distribution of periodic points for expansive actions (Ward [36]). New phenomena appear in the $\mathbb{Z}^d$ setting, $d > 1$: in particular, ergodicity no longer implies positive entropy, and there are expansive actions on connected groups with non–uniformly distributed periodic points.

The entropy calculation in terms of $p$–adic contributions for finite $p$ takes on a different form for $\mathbb{Z}^d$ actions and is not completely understood: for expansive actions of $\mathbb{Z}^2$ there is a family of decomposition of the entropy into local contributions. This family is parametrized by directions in $\mathbb{Z}^2$ (see Chothi, Everest and Ward [5]).

## Disconnected groups

We have often assumed that the underlying compact group is connected. There is an interesting theory for automorphisms of disconnected groups (see Kitchens [12] and Kitchens and Schmidt [14]). The $p$–adic entropy calculations in Chapter 8 may be emulated for automorphisms of certain zero–dimensional groups, using the adelic machinery for function fields developed in Weil's book.

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