CHAPTER IX

TOPOLOGICAL ENTROPY IV: PERIODIC POINTS

In this chapter we show how the entropy properties of compact group automorphisms relates to other dynamical properties.

Periodic points for automorphisms

Let X be a compact abelian group, and T an automorphism of X. The group of points with period n, for $n \geq 1$, is defined by

$$\mathrm{Fix}_n(T) = \{x \in X \mid T^n x = x\}.$$

Lemma 9.1. $\widehat{\mathrm{Fix}_n(T)} \cong \widehat{X}/(\hat{T}^n - I)\widehat{X}$.

Proof. Since T is a continuous automorphism, $\mathrm{Fix}_n(T)$ is a closed subgroup of X for each n. Let $\theta_n(x) = T^n(x) - x$, and consider the short exact sequence of topological groups

$$0 \longrightarrow \mathrm{Fix}_n(T) \longrightarrow X \xrightarrow{\theta_n} \theta_n(X) \longrightarrow 0.$$

The dual exact sequence is

$$0 \longrightarrow (\hat{T}^n - I)\widehat{X} \xrightarrow{\hat{\theta}_n} \widehat{X} \longrightarrow \widehat{\mathrm{Fix}_n(T)} \longrightarrow 0,$$

which proves the Lemma. □

Corollary 9.2. *The automorphism T has finitely many points of period n if and only if $\widehat{X}/(\hat{T}^n - I)\widehat{X}$ is finite, in which case*

$$|\mathrm{Fix}_n(T)| = |\widehat{X}/(\hat{T}^n - I)\widehat{X}|.$$

Proof. An abelian group is finite if and only if its dual is; also, any finite abelian group is isomorphic to its dual. □

Examples

Example 9.3. Let T be the automorphism of $\mathbb{T}^2$ dual to the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then the number of points with period n is given by

$$|\mathrm{Fix}_n(T)| = |\mathbb{Z}^2/(A^n - I)\mathbb{Z}^2| = |\det(A^n - I)| = |\rho_1^n - 1| \times |\rho_2^n - 1|, \tag{9.1}$$

where $\rho_1 > 1$ and ρ_2 are the eigenvalues of A. It follows that the number of points with period n has a growth rate,

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathrm{Fix}_n(T)| = \log \rho_1.$$

By Corollary 7.13, it follows that the growth rate of periodic points exists and coincides with the topological entropy of T. For later reference, notice that the calculation in (9.1) can be carried out in several ways. Firstly, the matrix $A^n - I$ defines a pair of relations in a two–generator abelian group, so that

$$\mathbb{Z}^2/(A^n - I)\mathbb{Z}^2 \cong \mathbb{Z}/a_n\mathbb{Z} \oplus \mathbb{Z}/b_n\mathbb{Z},$$

where $\begin{bmatrix} a_n & 0 \\ 0 & b_n \end{bmatrix}$ is the integer Smith canonical form for $A^n - I$. Then $a_n b_n = |\det(A^n - I)|$. A second method is the following: consider $J = [0,1) \times [0,1) \subset \mathbb{R}^2$ as a fundamental domain for $\mathbb{T}^2$. The matrix $A^n - I$, viewed as a linear map of $\mathbb{R}^2$, sends J onto a parallelogram with area $A^n - I$ and vertices on integer lattice points. It follows that this parallelogram is a disjoint union of regions R_i, each of which has the property that an integer translate of R_i falls into J, and this procedure covers J exactly $|\det(A^n - I)|$ times. It follows that $A^n - I$ is a $|\det(A^n - I)|$–to–1 map of $\mathbb{T}^2$, and so the kernel of $A^n - I$ has cardinality $|\det(A^n - I)|$ on the torus.

Example 9.4. Let T be the automorphism of $X = \widehat{\mathbb{Z}[\frac{1}{6}]}$ dual to multiplication by $\frac{2}{3}$. Recall that $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$ is a covering space for X, with fundamental domain $J = [0,1) \times \mathbb{Z}_2 \times \mathbb{Z}_3$. The linear map $\left(\frac{2}{3}\right)^n - 1$ sends J onto a region that covers X with multiplicity

$$|\left(\tfrac{2}{3}\right)^n - 1|_\infty \times |\left(\tfrac{2}{3}\right)^n - 1|_2 \times |\left(\tfrac{2}{3}\right)^n - 1|_3 = \left(1 - \tfrac{2^n}{3^n}\right) \times 1 \times 3^n = 3^n - 2^n.$$

It follows that

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathrm{Fix}_n(T)| = \log 3 = h(T).$$

Example 9.5. Let T be the automorphism of $\widehat{\mathbb{Q}}$ dual to multiplication by $\frac{2}{3}$. Then, by Lemma 9.1,

$$\widehat{\mathrm{Fix}_n(T)} \cong \mathbb{Q}/((\tfrac{2}{3})^n - 1)\mathbb{Q} = \{0\},$$

so there is only one point with period n for each $n \geq 1$.

Expansive automorphisms of connected groups

Recall that a homeomorphism T of a metric space X is *expansive* if there is a constant $\delta > 0$ with the property that, for any $x \neq y \in X$ there is an n for which $d(T^n x, T^n y) > \delta$. We are interested in automorphisms of compact connected metrizable groups that are expansive homeomorphisms.

A rich supply of automorphisms of connected groups may be constructed as follows. Let $A \in GL(d, \mathbb{Q})$ be a rational matrix with non–zero determinant. Let

$\pi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{T}^d$ denote the canonical projection map, and define a closed subgroup $H(A) \subset \mathbb{T}^d \times \mathbb{T}^d$ by setting

$$H(A) = \pi\left(\{(v, Av) \mid v \in \mathbb{R}^d\}\right). \tag{9.2}$$

Associated to $H(A)$ there is an automorphism $T^{(A)}$ of a compact connected abelian group $Y_{H(A)}$. The group is defined by

$$Y_{H(A)} = \{x \in \left(\mathbb{T}^d\right)^{\mathbb{Z}} \mid (x_k, x_{k+1}) \in H(A) \text{ for all } k \in \mathbb{Z}\} \tag{9.3}$$

and the map $T^{(A)}$ is the left shift on $Y_{H(A)}$.

THEOREM 9.6. *Let T be an expansive automorphism of a compact connected group X. Then there exists $d \geq 1$, a matrix $A \in GL(d, \mathbb{Q})$ none of whose eigenvalues has unit modulus, and an isomorphism of compact groups $\phi : X \to Y_{H(A)}$ such that $\phi(Tx) = T^{(A)}(\phi(x))$.*

PROOF. See Appendix B. □

THEOREM 9.7. *If T is an expansive automorphism of a compact connected group, then the growth rate of the number of periodic points of T exists and equals the topological entropy of T.*

PROOF. By Theorem 9.6, we may assume that $T = T^{(A)}$ on $Y_{H(A)}$. Let the characteristic polynomial of A be $p(\lambda) = \lambda^d - \alpha_{d-1}\lambda^{d-1} - \cdots - \alpha_0$, so by the Cayley–Hamilton theorem

$$A^d = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{d-1}A^{d-1}. \tag{9.4}$$

Let s be the least common denominator of the coefficients appearing in p. Define a group Z_A as follows: $Z_A = \{(x_0, \dots, x_{d-1}) \mid x_{j+1} = Ax_j \text{ for all } j = 0, \dots, d-1\}$; $\theta(Z_A) \supset \mathbb{Z}^d$, where $\theta(x_0, \dots, x_{d-1}) = x_0$, and Z_A is A–invariant. By (9.4), $\theta(Z_A) \subset \mathbb{Z}[\frac{1}{s}]^d$.

Now

$$\begin{aligned} |\mathrm{Fix}_n(T^{(A)})| &= \left|\frac{Y_{H(A)}}{(A^n - I)Y_{H(A)}}\right| \\ &= \left|\frac{Z_A}{(A^n - I)Z_A}\right| \\ &= \left|\frac{A^{-n}\mathbb{Z}^d}{(A^n - I)A^{-n}\mathbb{Z}^d}\right| \\ &= \left|\frac{A^{-n}\mathbb{Z}^d}{(A^n - I)\mathbb{Z}^d}\right| \\ &= s^n|\det(A^n - I)|, \end{aligned}$$

where $A^{-n}\mathbb{Z}^d$ is a subgroup of $\mathbb{Q}^n$ containing $\mathbb{Z}^d$ as an index s^n subgroup.

Let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of A, none of which has modulus one by the assumption of expansiveness. Then

$$\mathrm{Fix}_n(T^{(A)}) = s^n \prod_{i=1,\dots,d} |\lambda_i^n - 1|,$$

so that

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathrm{Fix}_n(T^{(A)})| = \log s + \sum_{i=1}^{d} \log^+ |\lambda_i|.$$

By Exercise 8.1, the last expression is the entropy of $T^{(A)}$. □

One may show that the set of periodic points (the union of all $\mathrm{Fix}_n(T^{(A)})$ over $n \geq 1$) is dense for maps of the form $T^{(A)}$ (see Exercise 9.4). We shall see this below as a corollary of a stronger statement.

Distribution of periodic points

Recall (see Exercise 7.3 and 7.4) that a sequence of Borel measures $\{\nu_n\}$ on X converge weakly to a measure ν if

$$\int_X f d\nu_n \to \int_X f d\nu$$

for every continuous function f on X.

Example 9.8. Let $X = [0, 1)$, the additive circle, and let $\nu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{j/n}$, where δ_x is the point mass at x. Then ν_n converges weakly to Lebesgue measure on X.

Let T be an automorphism of the compact abelian group X. For each $n \geq 1$, $\mathrm{Fix}_n(T)$ is a closed subgroup of X; denote by μ_n the normalized Haar measure on this compact group, and by μ the Haar measure on X. The periodic points of T are said to be *uniformly distributed with respect to Haar measure* if μ_n converges weakly to μ.

Recall from Chapter 2 the basic orthogonality relations enjoyed by characters: if Z is any compact abelian group, with Haar measure λ, then $\int \chi d\lambda = 0$ if and only if χ is non–trivial.

Lemma 9.9. *The periodic points of T acting on X are uniformly distributed with respect to T if and only if, for every non–trivial character $\chi \in \widehat{X}$, there is an N_χ with the property that χ restricted to $\mathrm{Fix}_n(T)$ is a non–trivial character for every $n \geq N_\chi$.*

Proof. Consider the following statements (each of which is a property of the automorphism T of X).

(1) $\mu_n \to \mu$ weakly.
(2) $\int f d\mu_n \to \int f d\mu$ for all $f \in C(X)$.
(3) $\int \chi d\mu_n \to \int \chi d\mu$ for all $\chi \in X$.
(4) for any $\chi \in X$, there is an N_χ such that $n \geq N_\chi \implies \chi \notin \mathrm{Fix}_n(T)^\perp$.

We claim that they are equivalent. That (1) is equivalent to (2) follows from the definition of weak convergence. (2) implies (3) since characters are continuous functions. The equivalence of (3) and (4) follows immediately from the orthogonality relations enjoyed by characters: a character χ on X restricts to a character $\chi|_{\mathrm{Fix}_n(T)}$ on $\mathrm{Fix}_n(T)$, and has or has not zero integral with respect to Haar measure on $\mathrm{Fix}_n(T)$ depending only on whether or not it is trivial.

The remaining implication is (3) $\implies$ (2). Assume (3) and let g be a continuous function on X with $\int g d\mu = 0$. For each $\epsilon > 0$, there is a finite trigonometric polynomial (a linear combination of finitely many characters) p on X with $|g(x) - p(x)| < \epsilon$ for all $x \in X$ (by the Stone–Weierstrass theorem). Then $|\int p d\mu_n - \int g d\mu_n| \le \int |p-g| d\mu_n < \epsilon$. On the other hand, by (3) we have that $\int p d\mu_n \to 0$ (it is in fact equal to 0 after some point). It follows that, for n large enough, $|\int g d\mu_n| < \epsilon$, showing (2). □

THEOREM 9.10. *Let T be an expansive automorphism of a compact connected abelian group. Then the periodic points of T are uniformly distributed with respect to Haar measure.*

PROOF. We use the notation of Lemma 9.9 and Theorem 9.7. By Theorem 9.6, we may assume that $T = T^{(A)}$ and $X = Y_{H(A)}$, where $A \in GL(d, \mathbb{Q})$.

Assume that the periodic points of $T^{(A)}$ are *not* uniformly distributed. By Lemma 9.7, it follows that there exists a non–trivial character χ and a sequence $n_j \to \infty$ with the property that

$$\chi \in (A^{n_j} - I)Z_A$$

for all $j \ge 1$. It follows that

$$\frac{Z_A}{(A^{n_j} - I)Z_A} \cong \frac{Z_A}{(A^{n_j} - I)Z_A + \langle\chi\rangle}, \tag{9.5}$$

where $\langle\chi\rangle$ denotes the smallest A–invariant subgroup of Z_A containing χ. Let

$$a(n_j) = \left|\frac{Z_A}{(A^{n_j} - I)Z_A}\right|; \quad b(n_j) = \left|\frac{Z_A}{(A^{n_j} - I)Z_A + \chi Z_A}\right|.$$

We know by Theorem 9.5 that

$$\lim_{j\to\infty} \frac{1}{n_j} a(n_j) = h(T^{(A)}) > 0, \tag{9.6}$$

(see Exercise 9.5). Now consider $b(n_j)$. Let S be the automorphism dual to multiplication by A on Y, where

$$\widehat{Y} = \frac{Z_A}{\langle\chi\rangle}.$$

Assume first that Y is connected: then $\widehat{Y}$ is torsion free, so $\widehat{Y} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \frac{\mathbb{Q}^d}{\chi\mathbb{Q}^d} = \mathbb{Q}^r$ for some $r < d$. It follows that the action induced by S on $\widehat{Y} \otimes_{\mathbb{Z}} \mathbb{Q}$ is given by some matrix in $GL(r, \mathbb{Q})$ which is (up to similarity) a block in A. It follows that $h(S) < h(T)$ by Yuzvinskii's formula. Notice that the entropy of S may be zero if $r = 0$.

If Y is not connected, then let $Y^{(0)}$ be the connected component of the identity: by the same argument, $\widehat{Y^{(0)}} \otimes \mathbb{Q}$ is isomorphic to $\mathbb{Q}^r$ for some $r < d$. On the other hand, $\widehat{\frac{Y}{Y^{(0)}}}$ is a torsion subgroup of a finitely generated group, hence finite. It follows that $h(S) = h(S|_{Y^{(0)}}) < h(T)$ again.

However, (9.5) and (9.6) together imply that $h(S) = h(T^{(A)})$, which contradicts the assumption that the periodic points of $T^{(A)}$ are not uniformly distributed. □

Corollary 9.11. *The set of periodic points for an expansive automorphism of a compact connected abelian group form a dense set.*

Proof. This follows at once from Theorem 9.8 and the observation that Haar measure is positive on non–empty open sets. □

Notes. The structure of compact connected groups admitting expansive automorphisms was first described by Lawton in [17]. More extensive references may be found in [13]. The growth rate of periodic points for expansive group automorphisms follows from more general results in Section 7 of [21], where the entropy and the growth rate of periodic points for expansive $\mathbb{Z}^d$ actions are computed. The connection between growth rate of periodic points and the distribution of periodic points follows from an observation of Klaus Schmidt. For expansive $\mathbb{Z}^d$ actions, the distribution of periodic points is discussed in [36]. In the non–expansive case, Doug Lind has shown Theorem 9.7 amd Theorem 9.10 for ergodic toral automorphisms that are not assumed to be expansive (see [19]).

Exercises

9.1 Using the notation of (9.3) and (9.4), prove that if $A \in GL(d, \mathbb{Z})$ then $Y_{H(A)}$ is a torus, and $T^{(A)}$ is the toral automorphism of that torus corresponding to the matrix A.

9.2 Using the notation of (9.3) and (9.4), prove that $T^{(A)}$ is an ergodic automorphism of $Y_{H(A)}$ if and only if A has no unit roots as eigenvalues.

9.3 Let s_A be the least common multiple of the denominators of the characteristic polynomial of the rational matrix A. Prove that $s_{A\oplus B} = s_A \times s_B$ and $s_{A^n} = s_A^n$.

9.4 Prove directly that the set of all periodic points for an expansive automorphism of a connected compact group form a dense set.

9.5 Let $A \in GL(d, \mathbb{Q})$, $d \geq 1$ define an expansive automorphism as in (9.3). Prove that $h(T^{(A)}) > 0$. (Hint: if A has an eigenvalue outside the unit circle, we are done. If not, what does that tell you about the number s?).

9.6 Consider the automorphism T dual to multiplication by $A \in GL(d, \mathbb{Q})$ on $\widehat{\mathbb{Q}}^d$. Prove that if T is ergodic, then $h(T) > 0$. Show by example that $h(T) > 0$ does not imply that T is ergodic. (Hint: you will need Kronecker's theorem).

9.7 For a homeomorphism T of a topological space X, with the property that $|\mathrm{Fix}_n(T)| < \infty$ for all $n \geq 1$, the *dynamical zeta function* is defined by $\zeta(s) = \exp\left(\sum_{n=1}^{\infty} \frac{s^n}{n} |\mathrm{Fix}_n(T)|\right)$. Prove that the dynamical zeta function of an expansive compact group automorphism $T^{(A)}$ is rational, compute it, and find the radius of convergence.

9.8 Prove that the dynamical zeta function is rational for ergodic toral automorphisms, and automorphisms of the full solenoid $\widehat{\mathbb{Q}}^d$. Find an example of a compact group automorphism with finitely many periodic points for each period and with irrational dynamical zeta function.

9.9 Show that expansive actions of $\mathbb{Z}^2$ do not have a uniform distribution of periodic points by the following steps. Define a $\mathbb{Z}^2$ action T on $X = \widehat{\mathbb{Z}[\frac{1}{6}]}$

by setting $\widehat{T_{(1,0)}}r = 2r$ and $\widehat{T_{(0,1)}}r = 3r$. Show that T is expansive: there is an open neighbourhood $U \subset X$ containing 0 with the property that $\bigcap_{n,m\in\mathbb{Z}} T_{(n,m)}(U) = \{0\}$. Let $\mu_{(n,m)}$ denote Haar measure on the group $\mathrm{Fix}_{(n,m)}(T) = \{x \in X \mid T_{(n,0)}(x) = T_{(0,m)}x = x\}$.
(1) Find sequences $p_n \to \infty$ and $q_n \to \infty$ with the property that $\mu_{(p_n,q_n)}$ converges weakly to Haar measure.
(2) Find sequences $r_n \to \infty$ and $s_n \to \infty$ with the property that $\mu_{(r_n,s_n)}$ is constant.
(3) Characterize the set of weak limit points of the set $\{\mu_{(n,m)}\}_{n,m\in\mathbb{Z}}$.