CHAPTER VI

TOPOLOGICAL ENTROPY I: DEFINITIONS

In this section we introduce an invariant of topological conjugacy for topological dynamical systems. The original definiton, due to Adler, Konheim and McAndrew, applies to continuous maps of compact topological spaces. We shall use first a later definition, due to Bowen and Dinaburg, which applies to uniformly continuous maps of metric spaces, not necessarily compact. It turns out that for our purposes the extra generality of allowing non–compact spaces is more important than allowing non–metric topological spaces.

Bowen's definition

Throughout, our example will be a uniformly continuous map $T : X \to X$, where $X = (X, d)$ is a metric space. Much of the material here is taken from Bowen (1971).

For $n \geq 1$, define a new metric d_n on X by $d_n(x, y) = \max_{0 \leq i \leq n-1} d(T^i x, T^i y)$.

A set $E \subset X$ is *(n, ϵ)–separated* if for any $x \neq y$ in E, $d_n(x, y) > \epsilon$. Dually, a set $F \subset X$ *(n, ϵ)–spans* another set K under T if for every $x \in K$ there is a $y \in F$ for which $d_n(x, y) \leq \epsilon$.

For a compact set $K \subset X$, let $r_n(\epsilon, K)$ be the smallest cardinality of any set which (n, ϵ) spans K under T, and let $s_n(\epsilon, K)$ be the largest cardinality of any (n, ϵ) separated set contained in K. Finally, define

$$r(\epsilon, K) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(\epsilon, K), \tag{6.1}$$

and

$$s(\epsilon, K) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(\epsilon, K). \tag{6.2}$$

Lemma 6.1. *(a) $r_n(\epsilon, K) \leq s_n(\epsilon, K) \leq r_n(\frac{1}{2}\epsilon, K) < \infty$. (b) If $\epsilon < \epsilon'$ then $r(\epsilon, K) \geq r(\epsilon', K)$ and $s(\epsilon, K) \geq s(\epsilon', K)$.*

Proof. (b) This is clear. (a) If E is an (n, ϵ) separated subset of K with maximum cardinality then E is an (n, ϵ) spanning set for K. For the second inequality, let E be an (n, ϵ) separated subset of K, and let F be an $(n, \epsilon/2)$ spanning set for K. Define a map $f : E \to F$ by selecting, for each $x \in E$, a point $f(x) \in F$ with the property that $d_n(x, f(x)) \leq \frac{1}{2}\epsilon$. Then f is injective, so $|E| \leq |F|$. □

It follows that the definition below makes sense.

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Definition 6.2. Let $T : X \to X$ be a uniformly continuous map. Then define

$$h(T,K) = h_d(T,K) = \lim_{\epsilon \to 0} r(\epsilon, K) = \lim_{\epsilon \to 0} s(\epsilon, K),$$

and the topological entropy of T to be

$$h(T) = h_d(T) = \sup_{K \subset X} h_d(T,K).$$

The supremum above is taken over all compact subsets of X. The following remarks are clear.
(1) If X is compact, then $h_d(T) = h_d(T,X)$. Also, in this case any continuous map is uniformly continuous, and any metric that defines a fixed topology will give the same entropy (see Exercise 6.1).
(2) If $K \subset K_1 \cup \cdots \cup K_n$ then $h_d(T,K) \le \max_i\{h_d(T,K_i)\}$.
(3) To compute $h_d(T)$, it is enough to take the supremum over all K with some bounded diameter.

Lemma 6.3. *$h(T^m) = mh(T)$ for any uniformly continuous T, and any $m > 0$.*

Proof. It is clear that $r_n(\epsilon, K, T^m) \le r_{mn}(\epsilon, K, T)$, so that $h_d(T^m) \le mh_d(T)$. Given $\epsilon > 0$, choose (by the uniform continuity) a $\delta > 0$ such that $d_m(x,y) \le \epsilon$ if $d(x,y) \le \delta$. It follows that an (n,δ)–spanning set for K with respect to T^m is automatically an (nm,ϵ)–spanning set for K with respect to T. So $r_{nm}(\epsilon, K, T) \le r_n(\delta, K, T^m)$, and therefore $mh_d(T) \le h_d(T^m)$. □

Lemma 6.4. *Let T_i be a uniformly continuous map on (X_i, d_i) for $i = 1, 2$. Then $T_1 \times T_2$ is a uniformly continuous map of $X_1 \times X_2$ with the maximum metric d, and*

$$h_d(T_1 \times T_2) \le h_{d_1}(T_1) + h_{d_2}(T_2).$$

If the limsup's in the definition of $r(\epsilon, K)$ or $s(\epsilon, K)$ are limits for either T_1 or T_2, then

$$h_d(T_1 \times T_2) = h_{d_1}(T_1) + h_{d_2}(T_2).$$

Proof. Let $K_1 \subset X_1$ and $K_2 \subset X_2$ be compact sets. If F_i is an (n,ϵ)–spanning set for K_i with respect to T_i, then $F_1 \times F_2$ is an (n,ϵ)–spanning set for $T_1 \times T_2$. It follows that

$$r_n(\epsilon, K_1 \times K_2, T_1 \times T_2) \le r_n(\epsilon, K_1, T_1) \times r_n(\epsilon, K_2, T_2),$$

so

$$h_d(T_1 \times T_2, K_1 \times K_2) \le h_{d_1}(T_1, K_1) + h_{d_2}(T_2, K_2).$$

Let π_i be the projection from $X_1 \times X_2$ onto X_i. If $K \subset X_1 \times X_2$ is compact, then $\pi_1(K)$, $\pi_2(K)$ are compact and $\pi_1(K) \times \pi_2(K) \supset K$. Therefore

$$h_d(T_1 \times T_2, K) \le h_d(T_1 \times T_2, \pi_1(K) \times \pi_2(K)),$$

and so

$$\begin{aligned} h_d(T_1 \times T_2) &= \sup_{K \subset X_1 \times X_2} h_d(T_1 \times T_2, K) \\ &= \sup_{K_1 \subset X_1, K_2 \subset X_2} h_d(T_1 \times T_2, K_1 \times K_2) \\ &\leq \sup_{K_1 \subset X_1} h_{d_1}(T_1, K_1) + \sup_{K_2 \subset X_2} h_{d_2}(T_2, K_2) \\ &= h_{d_1}(T_1) + h_{d_2}(T_2). \end{aligned}$$

Finally, assume that $\frac{1}{n} \log r_n(\epsilon, K_1, T_1)$ converges to $r(\epsilon, K_1, T_1)$ for every compact $K_1 \subset X_1$. Let $K_i \subset X_i$ be compact, and let $E_i \subset X_i$ be an (n, ϵ) separated set for T_i. Then $E_1 \times E_2 \subset K_1 \times K_2$ is (n, ϵ)–separated for $T_1 \times T_2$. Now any compact subset $K \subset X_1 \times X_2$ lies in some set of the form $K_1 \times K_2$, so

$$s_n(\epsilon, K_1 \times K_2, T_1 \times T_2) \geq s_n(\epsilon, K_1, T_1) \times s_n(\epsilon, K_2, T_2).$$

It follows that

$$\begin{aligned} s(\epsilon, K_1 \times K_2, T_1 \times T_2) &\geq \limsup_{n\to\infty} \frac{1}{n} \left(\log s_n(\epsilon, X_1, T_1) + \log s_n(\epsilon, X_2, T_2)\right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log s_n(\epsilon, X_1, T_1) + \limsup_{n\to\infty} \frac{1}{n} s_n(\epsilon, X_2, T_2) \\ &= s(\epsilon, K_1, T_1) + s(\epsilon, K_2, T_2) \end{aligned}$$

so

$$h_d(T_1 \times T_2) \geq h_{d_1}(T_1) + h_{d_2}(T_2).$$

□

Definition using open covers

In order to calculate some values, it is helpful to have several different definitions of topological entropy. In this section we very briefly describe the (earlier) definition due to Adler, Konheim and McAndrew. This definition will be seen to be very closely analogous to the measure–theoretic entropy.

Let now X be a compact topological space, with a continuous transformation T. Use letters $\alpha, \beta, \dots$ to denote open covers of X.

The *join* of two open covers, $\alpha \vee \beta$, is the cover by all sets of the form $A \cap B$, $A \in \alpha$, $B \in \beta$. An open cover β *refines* an open cover α, written $\alpha < \beta$ if every member of β is a subset of a member of α. If α is an open cover, then $T^{-1}\alpha$ is the open cover consisting of all sets of the form $T^{-1}A$, $A \in \alpha$. Notice that $T^{-1}(\alpha \vee \beta) = T^{-1}(\alpha) \vee T^{-1}(\beta)$ and $\alpha < \beta \implies T^{-1}(\alpha) < T^{-1}(\beta)$.

Definition 6.5. Let $N(\alpha)$ denote the number of sets in a finite subcover of α with smallest cardinality, and define the entropy of α to be $H(\alpha) = \log N(\alpha)$.

See Exercise 6.4.

Now fix an open cover α, and let $a_n = H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right)$. Then (by Exercise 6.4),

$$\begin{aligned} a_{n+k} &\le H\left(\bigvee_{i=0}^{n+k-1} T^{-i}\alpha\right) \\ &\le H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) + H\left(T^{-n}\bigvee_{j=0}^{k-1} T^{-i}\alpha\right) \\ &\le a_n + a_k. \end{aligned}$$

It follows (Exercise 6.5) that $\frac{1}{n}a_n$ converges, so the following definition makes sense.

Definition 6.6. The (topological) entropy of T with respect to α is defined to be

$$h^*(T,\alpha) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right).$$

Definiton 6.7. The *topological entropy* of T is

$$h^*(T) = \sup_{\alpha} h^*(T,\alpha).$$

Lemma 6.8. *If $T_i : X_i \to X_i$, $i = 1, 2$ are continuous maps, and $\theta : X_1 \to X_2$ is continuous and surjective, with $\theta T_1 = T_2\theta$, then $h^*(T_2) \le h^*(T_1)$. If θ is a homeomorphism, then $h^*(T_1) = h^*(T_2)$.*

Lemma 6.9. *If T is a homeomorphism, then $h^*(T) = h^*(T^{-1})$.*

These are clear (see Exercise 6.6). It is also clear that Lemma 6.9 cannot hold true for Bowen's definition (for example, the map $x \mapsto 2x$ on $\mathbb{R}$).

When the two definitions coincide

If X is a compact metric space, and T is a continuous map $X \to X$, then both of the above definitions apply. In this section we show that in this case, $h(T) = h^*(T)$.

Recall that the *diameter* of a cover α is given by $\operatorname{diam}(\alpha) = \sup_{A\in\alpha} \operatorname{diam}(A)$. Recall also the *Lebesgue Covering Lemma*: if α is an open cover of X then there is a $\delta > 0$ such that any subset of X with diameter $\le \delta$ lies in some member of α. Such a number δ is called a *Lebesgue number* for α. Notice that if α and β are open covers, and $\operatorname{diam}(\alpha)$ is less than a Lebesgue number for β, then $\beta < \alpha$.

Theorem 6.10. *If $\{\alpha_n\}$ is a sequence of open covers with* $\operatorname{diam}(\alpha_n) \to 0$, *then* $\lim_{n\to\infty} h^*(T,\alpha_n) = h^*(T)$. *(We allow the case $h^*(T) = \infty$).*

Proof. Assume that $h^*(T) < \infty$. Let $\epsilon > 0$ be given; choose an open cover γ with $h^*(T,\gamma) > h^*(T) - \epsilon$. Choose N so that $n \ge N$ implies that $\operatorname{diam}(\alpha_n) < \delta$. Then $\gamma < \alpha_n$ so $h^*(T,\gamma) \le h^*(T,\alpha_n)$ if $n \ge N$. It follows that $n \ge N$ implies that $h^*(T) \ge h^*(T,\alpha_n) > h^*(T) - \epsilon$, so $h^*(T,\alpha_n) \to h^*(T)$.

If $h^*(T) = \infty$, then for any R there is an open cover γ with $h^*(T,\gamma) > R$; the same argument as that above shows that $h^*(T,\alpha_n) \to \infty$. □

An immediate consequence of the above theorem is that covers of small diameter suffice to compute the entropy.

LEMMA 6.11. *(1) If α is an open cover with Lebesgue number δ then*

$$N\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) \le r_n(\delta/2, X) \le s_n(\delta/2, X).$$

(2) If $\epsilon > 0$, and $\operatorname{diam}(\gamma) \le \epsilon$ *then*

$$r_n(\epsilon, X) \le s_n(\epsilon, X) \le N\left(\bigvee_{i=0}^{n-1} T^{-i}\gamma\right).$$

PROOF. (1) The second inequality is in Lemma 6.1. Let F be an $(n, \delta/2)$ spanning set for X with cardinality $r_n(\delta/2, T)$. Then

$$X = \bigcup_{x\in F} \bigcap_{i=0}^{n-1} T^{-i}\bar{B}(T^i x, \delta/2).$$

Now for each i, the set $\bar{B}(T^i x, \delta/2)$ is contained in a member of α, so

$$N\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) \le r_n(\delta/2, X).$$

(2) Let E be an (n, ϵ) separated set of cardinality $s_n(\epsilon, X)$. No member of the cover $\bigvee_{i=0}^{n-1} T^{-i}\gamma$ can contain two elements of E, so

$$s_n(\epsilon, X) \le N\left(\bigvee_{i=0}^{n-1} T^{-i}\gamma\right).$$

□

We deduce that if α_ϵ is the cover by all open balls with radius 2ϵ, and γ_ϵ is any cover by open balls of radius $\epsilon/2$, then

$$N\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha_\epsilon\right) \le r_n(\epsilon, X) \le s_n(\epsilon, X) \le N\left(\bigvee_{i=0}^{n-1} T^{-i}\gamma_\epsilon\right). \tag{6.3}$$

THEOREM 6.12. *If $T : X \to X$ is a continuous map of a compact metric space (X, d) then $h_d(T) = h^*(T)$.*

PROOF. Given $\epsilon > 0$, and α_ϵ, γ_ϵ as in (6.3), we see that

$$h^*(T, \alpha_\epsilon) \le r(\epsilon, X, T) \le s(\epsilon, X, T) \le h^*(T, \gamma_\epsilon).$$

Let $\epsilon = 1/n$, and let $n \to \infty$, then the outer terms converge to $h^*(T)$ while the inner terms converge to $h(T)$. □

From now on, we will therefore use "$h(T)$" to denote either the Bowen definition or the definition using open covers.

Notes. Topological entropy was first introduced by Adler, Konheim and McAndrew [1] as a direct analogue of measure theoretic entropy. Bowen's later definition [3] in terms of spanning and separating sets has turned out to be more often useful.

Exercises

6.1 Let $T : X \to X$ be uniformly continuous, and let d_1, d_2 be uniformly equivalent metrics on X. Show that $h_{d_1}(T) = h_{d_2}(T)$.

6.2 Define $T : (0, \infty) \to (0, \infty)$ by $T(x) = 2x$. Let d_1 be the usual metric on $(0, \infty)$. Let d_2 be a metric on $(0, \infty)$ that coincides with d_1 on $[1, 2]$ but makes T into an isometry (use that fact that the orbit of $[1, 2]$ under T partitions $(0, \infty)$). Show that $h_{d_1}(T_1) > 0$ and $h_{d_2}(T_2) = 0$. Show directly that the metrics d_1 and d_2 are not uniformly equivalent.

6.3 Find an example of uniformly continuous maps T and S for which $h_d(T \times S) < h_{d_1}(T) + h_{d_2}(S)$ in the notation of Lemma 6.3.

6.4 If α and β are open covers of the compact topological space X, prove the following.
   a. $H(\alpha) \geq 0$, and $H(\alpha) = 0$ if and only if $N(\alpha) = 1$.
   b. If $\alpha < \beta$ then $H(\alpha) < H(\beta)$.
   c. $H(\alpha \vee \beta) \leq H(\alpha) + H(\beta)$.
   d. $H(T^{-1}\alpha) \leq H(\alpha)$, and equality does not hold in general.
   e. If T is surjective, then $H(T^{-1}\alpha) = H(\alpha)$.

6.5 Let $\{a_n\}_{n\geq 1}$ be a sequence of reals such that $a_{n+m} \leq a_n + a_m$ for all n, m. Prove that $\lim_{n\to\infty} \frac{1}{n} a_n$ exists and is equal to $\inf_n \frac{1}{n} a_n$. (In this problem the limit is allowed to be $-\infty$.)

6.6 Prove Lemma 6.8 and Lemma 6.9.