CHAPTER II

FOURIER ANALYSIS ON GROUPS

In this chapter we describe how the familiar theory of Fourier analysis for L^2 functions on the circle (as used in Theorem 1.11) may be extended to any compact abelian group. We shall also state formally the properties of Haar measure.

Introduction

DEFINITION 2.1. A *topological abelian group* is a Hausdorff topological space G which is also an abelian group, provided that the map $(g, h) \mapsto g - h$ is a continuous map from $G \times G$ (product topology) to G. If the topology of G is also locally compact (every point has an open neighbourhood whose closure is compact), then G is a *locally compact abelian group*, or an LCA group.

EXAMPLE 2.2. Almost every group you know is a topological group in some natural topology.

(1) $\mathbb{R}$, usual topology.
(2) $\mathbb{R}$, with the discrete topology.
(3) $\mathbb{Q}$, with the discrete topology.
(4) $\mathbb{S}^1$, the multiplicative circle; $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the additive circle.
(5) $\mathbb{Q}_p$, the field of p–adic numbers, in the p–adic topology.
(6) $\mathbb{Z}[x]$, in the discrete topology.
(7) any finite or countable group with the discrete topology.
(8) If k is a locally compact field, then $GL(n, k)$ is a non–abelian locally compact group with the topology obtained from the product topology on k^{n^2}.

A *homomorphism* $\phi : G \to H$ of LCA groups is a homomorphism of the groups which is also continuous. Most of the ways of constructing new groups from old preserve local compactness.

(1) If $\{G_i\}_{i \in I}$ are LCA groups, and each G_i is compact then the direct product $\prod_{i \in I} G_i$ is an LCA group (an infinite direct product of non–compact LCA groups provides an example of a topological group that is not locally compact).
(2) If $\{G_i\}_{i \in I}$ are LCA groups, then the direct sum $\sum_{i \in I} G_i$ is an LCA group (the direct sum consists of elements $(g_i)_{i \in I}$, with $g_i = 0$ for all but finitely many i.)
(3) If LCA groups G_n are joined by injective homomorphisms $\phi_n : G_n \to G_{n+1}$ then the direct limit $\varinjlim(G_n, \phi_n)$ is an LCA group.
(4) If LCA groups H_n are joined by surjective homomorphisms $\phi_n : H_{n+1} \to H_n$ then the projective limit $\varprojlim(H_n, \phi_n)$ is an LCA group.

(5) A generalization of (2) is the *restricted direct product*. Let $\{G_i\}_{i\in I}$ be a family of LCA groups, with the property that each G_i contains a compact subgroup H_i. Then the restricted direct product is defined to be the abstract subgroup $G \subset \prod_{i\in I} G_i$,

$$G = \{(g_i) \mid g_i \in H_i \text{ for all but finitely many } i \in I\}.$$

The group G is then an LCA group with respect to the coarsest topology making each subset of the form $\prod_{i\in J} G_i \times \prod_{i\notin J} H_i$ (J a finite subset of I) into an open subgroup of G. A non–trivial instance of this construction is described in Appendix A.

The most important property shared by LCA groups is the existence of Haar measure.

THEOREM 2.3. *If G is a LCA group, then there is a non–negative regular measure μ which is defined on the Borel sets of G and is translation invariant, $\mu(g + A) = \mu(A)$ for all measurable A. The measure is unique up to multiplication by a constant.*

(1) G is compact if and only if $\mu(G) < \infty$.
(2) If $\phi : G \to G$ is a surjective homomorphism and G is compact, then ϕ preserves the Haar measure, so ϕ is a measure preserving transformation of $(G, \mathcal{B}, \mu)$.
(3) The existence of Haar measure on G means we can define L^p spaces, integration and so on.
(4) The Haar measure connects the topology and the algebraic structure of G.

Dual groups, Fourier transforms

A complex valued function $\chi : G \to \mathbb{C}$ on a LCA group is a *character* on G if $|\chi(g)| = 1$ for all $g \in G$ and χ is a homomorphism. The set of all *continuous* characters of G forms a group, denoted $\widehat{G}$, called the *dual group* of G. The operation on $\widehat{G}$ is given by $(\chi_1 + \chi_2)(g) = \chi_1(g)\chi_2(g)$.

We will sometimes write $\langle g, \chi\rangle$ for $\chi(g)$, to emphasise that this is a pairing between G and $\widehat{G}$.

Introduce the UCC (uniform convergence on compact sets) topology on $\widehat{G}$ as follows. For $K \subset G$ compact and $r > 0$ let

$$N_{K,r} = \{\chi \mid |\chi(g) - 1| < r, g \in K\}.$$

Then these sets and their translates form a base for the topology of $\widehat{G}$. A subset E of $\widehat{G}$ that separates points (i.e. for $x \neq y$ in G, there is a character $\chi \in \widehat{G}$ with $\chi(x) \neq \chi(y)$) is dense in $\widehat{G}$.

THEOREM 2.4. *With the UCC topology, $\widehat{G}$ is an LCA group.*

If $f \in L^1(G)$, then the *Fourier transform* of f, denoted $\hat{f}$, is the function on $\widehat{G}$ defined by

$$\hat{f}(\chi) = \int_G f(g)\langle -g, \chi\rangle dg.$$

(1) The map $f \to \hat{f}(\chi)$ (for a fixed χ), is a complex homomorphism of $L^1(G)$, not identically 0. Conversely, every non–zero complex homomorphism of $L^1(G)$ is obtained in this way. Distinct characters give distinct homomorphisms.
(2) The image of the map $f \to \hat{f}$ is a separating self–adjoint algebra on $C_0(\widehat{G})$ (continuous functions vanishing at infinity), hence dense in $C_0(\widehat{G})$.
(3) The Fourier transform of $f * g$ is $\hat{f}\hat{g}$.
(4) $\|\hat{f}\|_\infty \leq \|f\|_1$ so the Fourier transform is continuous.
(5) If G is discrete, then $\widehat{G}$ is compact.
(6) If G is compact, then $\widehat{G}$ is discrete. (Outline proof: let f be the constant function 1 on G. By compactness, $f \in L^1(G)$. The Fourier transform of an L^1 function is continuous with the UCC topology on the dual group. By orthogonality (see below), $\widehat{f}(\chi) = 1$ if $\chi = 0$, and $\widehat{f}(\chi) = 0$ if not. Thus the fact that $\widehat{f}$ is continuous means that $\{0\}$ is an open subset of $\widehat{G}$, so $\widehat{G}$ is discrete.)

An application of (6) is to find the dual group of the infinite torus $G = \mathbb{T}^{\mathbb{N}}$. Denote elements of G by sequences $(g_0, g_1, \dots)$. First, any polynomial

$$f = a_0 + a_1 x + \cdots + a_n x^n$$

with integer coefficients determines a character χ_f on G by

$$\chi_f(g_0, g_1, \dots) = e^{2\pi i(a_0 g_0 + \cdots + a_n g_n)}.$$

Distinct points in G must differ on some co–ordinate. If they differ on co–ordinate n then some character of the form $\chi_{(ax^n)}$ will separate them. It follows that the group of polynomials $\mathbb{Z}[x]$ is a dense subgroup of $\widehat{G}$. By Tychonoff, the group G is compact, so by (6), the dual group $\widehat{G}$ is discrete. Thus $\mathbb{Z}[x]$ is a dense subgroup of a discrete set so $\mathbb{Z}[x] = \widehat{G}$.

The Fourier transform is defined on $L^1(G) \cap L^2(G)$, and it maps into a dense linear subspace of $L^2(\widehat{G})$ as an L^2 isometry; it therefore extends uniquely to an isometry $L^2(G) \to L^2(\widehat{G})$. This map is the *Plancherel transform*, which will also be denoted by $f \mapsto \hat{f}$. Note that every function in $L^2(\widehat{G})$ is the Plancherel transform of something in $L^2(G)$.

(1) [INVERSION THEOREM] The Haar measure on $\widehat{G}$ may be normalised to make:

$$f(g) = \int_{\widehat{G}} \hat{f}(\chi)\langle g, \chi\rangle d\chi$$

for a.e.(g), and $f \in L^2(G)$.
(2) [PARSEVAL FORMULA] Let f and g be L^2 functions on G. Then

$$(f, g)_G = \int_G f(x)\overline{g(x)}dx = \int_{\widehat{G}} \hat{f}(\chi)\overline{\hat{g}(\chi)}d\chi = (\hat{f}, \hat{g})_{\widehat{G}}.$$

Thus $f \mapsto \hat{f}$ is an isometry from the Hilbert space $L^2(G)$ onto the Hilbert space $L^2(\widehat{G})$. Note that if $\widehat{G}$ is countable, the integral over $\widehat{G}$ is a sum.

If G is compact, then normalize the Haar measure on G to have total mass 1. The elements of $\widehat{G}$ are all in $L^2(G)$ and they form an orthonormal basis for $L^2(G)$. The orthogonality is easy to check: let χ and η be distinct elements of $\widehat{G}$. Choose $h \in G$ with $(\chi\eta^{-1})(h) \neq 1$. Then

$$(\chi, \eta) = \int (\chi\eta^{-1})(g)dg = \int (\chi\eta^{-1})(g+h)dg$$

$$= (\chi\eta^{-1})(h) \int (\chi\eta^{-1})(g)dg = (\chi\eta^{-1})(h)(\chi, \eta)$$

so that $(\chi, \eta) = 0$.

Pontryagin duality

If G is a LCA group, then $\widehat{G}$ is also a LCA group. Let $\Gamma = \widehat{G}$; then Γ has a dual group $\widehat{\Gamma}$, which is again LCA. Any $g \in G$ defines a character on Γ by $\chi \mapsto \chi(g)$. This defines a canonical map $\alpha : G \to \widehat{\Gamma}$, defined by

$$\langle g, \chi \rangle = \langle \chi, \alpha(g) \rangle.$$

Theorem 2.5. *The map α is an isomorphism of topological groups.*

This is the celebrated duality theorem of Pontryagin and van Kampen. We now describe how it relates to the subgroup structure of LCA groups. If $H \subset G$ is a *closed* subgroup, then G/H is also an LCA group. The set $H^\perp = \{\chi \in \widehat{G} \mid \chi(h) = 1\ \forall h \in H\}$, the *annihilator* of H, is a closed subgroup of $\widehat{G}$.

(1) $\widehat{G/H} \cong H^\perp$.
(2) $\widehat{G}/H^\perp \cong \widehat{H}$.
(3) $H_1^\perp + H_2^\perp \cong \widehat{X}$, where $X = G/(H_1 \cap H_2)$.
(4) $H^{\perp\perp} \cong H$.

The dual of a homomorphism $\theta : G \to H$ is the map $\widehat{\theta} : \widehat{H} \to \widehat{G}$ defined by $\widehat{\theta}(\chi)(g) = \chi(\theta(g))$. The map $\widehat{\theta}$ is injective (resp. surjective) if and only if θ is surjective (resp. injective). This may be expressed as follows: the dual of the short exact sequence

$$0 \to K \to G \to H \to 0$$

is the short exact sequence

$$0 \to \widehat{H} \to \widehat{G} \to \widehat{K} \to 0.$$

Pontryagin duality changes topological properties into algebraic ones. If G is compact, then

(1) $\widehat{G}$ is torsion if and only if G is zero–dimensional.
(2) $\widehat{G}$ is torsion–free if and only if G is connected.

The assumption of compactness is essential for both of these results, as the example $G = \mathbb{Q}_p$ shows.

Examples

All the following can be proved by elementary methods (see Appendix A for more on (4)).

(1) $\widehat{\mathbb{T}} \cong \mathbb{Z}$ ($x \mapsto e^{2\pi i n x}$, $n \in \mathbb{Z}$.)
(2) $\widehat{\mathbb{R}} \cong \mathbb{R}$ ($x \mapsto e^{2\pi i t x}$, $t \in \mathbb{R}$.)
(3) $\widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$.
(4) More generally, any non–discrete locally compact field is isomorphic to its dual group. Prove this as follows: let k be a non–discrete locally compact field. Find a non–trivial character χ on k (this may take a little doing). Define a map $\eta : k \to \widehat{k}$ by $\eta(\xi)(x) = \chi(\xi x)$. Check that η is an isomorphism.
(5) If F is a finite abelian group, then $\widehat{F} \cong F$.

Groups like $\Gamma = \widehat{\mathbb{Q}}$ are a little harder to describe concretely. First, notice that any character on $\mathbb{R}$ defines a character on $\mathbb{Q}$ by restriction, and this assignment is injective because $\mathbb{Q}$ is dense in $\mathbb{R}$. It follows that a copy of $\mathbb{R}$ is embedded as a dense subgroup of Γ. On the other hand, $\mathbb{Q}$ is discrete so Γ is compact: it follows that the line $\mathbb{R}$ must be "wrapped" into the compact set Γ. Such groups are called *solenoids*. The group Γ can be described in many ways, one of which we describe here. Show that $\mathbb{Q}$ is the direct limit $\varinjlim(\frac{1}{n!}\mathbb{Z}, \phi_n)$ (where $\phi_n(x) = x$). It follows that

$$\Gamma = \varprojlim(\mathbb{T}, \theta_n)$$

where $\theta_n(t) = (n!)t$. A very neat description of Γ may be given in terms of adele groups (see Appendix A).

NOTES. Duality theory for LCA groups is due in this form originally to Pontryagin [27], [28], and to van Kampen [9]. Standard expositions are Rudin [29] and the first volume of Hewitt and Ross, [7]. The latter is old-fashioned but very complete. Exercise 2.11(5) comes from a paper by Lind [19].

Exercises

2.1 Generalise Theorem 1.11 to show that an automorphism T of a compact abelian group is ergodic if and only if the dual automorphism $\widehat{T}$ of $\widehat{G}$ is *aperiodic* ($\widehat{T}$ is aperiodic if $(\widehat{T}^n - I)$ is injective for all $n \geq 1$).
2.2 Show that a rotation $T : g \mapsto g+h$ of a compact metrizable group is ergodic if and only if $\{h^n\}$ is dense in the group.
2.3 Deduce from 2.2 that if G is a compact metrizable group with an ergodic rotation, then G is abelian.
2.4 Describe all *monothetic* groups (that is, abelian groups with an element that generates a dense subgroup) as follows. Prove that a compact group is monothetic if and only if its dual is a subgroup of the discrete circle (the circle group with the discrete topology). Prove that if G is monothetic and not compact, then $G \cong \mathbb{Z}$.
2.5 Characterize those compact abelian groups that admit ergodic rotations, and describe a compact abelian group that does *not* admit an ergodic rotation.
2.6 Let T be an invertible measure preserving transformation of the Lebesgue space $(X, \mathcal{B}, \mu)$. Recall that U_T is the operator on $L^2(\mu)$ defined by $U_T(f)(x) =$

$f(Tx)$. Check that U_T is a unitary operator. Prove the following theorems: (a) T is ergodic if and only if 1 is a simple eigenvalue of U_T. (b) If T is ergodic, then every eigenvalue of U_T is simple, and the set of all eigenvalues is a subgroup of $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. (c) Given any subgroup $H \subset \mathbb{S}^1$, there is an ergodic dynamical system $(X, \mathcal{B}, \mu, T)$ with the property that H is the group of eigenvalues of U_T.

2.7 Prove directly (using trigonometric polynomials) that an ergodic automorphism of a compact abelian group is mixing.

2.8 Let G be a compact abelian group. Prove that if every element of $\widehat{G}$ has finite order, then G is totally disconnected.

2.9 Let G be a compact abelian group. Prove that if $\widehat{G}$ contains an element of infinite order, then G contains a one–parameter subgroup.

2.10 Let G be a compact abelian group. Prove that G is connected if and only if $\widehat{G}$ is torsion–free.

2.11 Let T be the automorphism of the d–torus $\mathbb{T}^d$ dual to the automorphism A of $\mathbb{Z}^d$, where $A \in GL(d, \mathbb{Z})$. Let $F_m = \{x \in \mathbb{T}^d \mid T^m(x) = x\}$; show that this is a closed subgroup of $\mathbb{T}^d$ (the group of points with period m under T). Consider the sequence $0 \longrightarrow F_m \longrightarrow \mathbb{T}^d \stackrel{\theta}{\longrightarrow} \mathbb{T}^d \longrightarrow 0$, where $\theta(x) = T^m(x) - x$, and use it to prove the following.
(1) The sequence is exact for every $m \geq 1$ if and only if T is ergodic.
(2) T is ergodic if and only if F_m is finite for every $m \geq 1$.
(3) If T is ergodic, then the number of points with period m is given by $|F_m| = det(A^m - I) = \prod |\lambda^m - 1|$, where the product is taken over all eigenvalues of A.
(4) If A is *hyperbolic* (i.e. $|\lambda| \neq 1$ for every eigenvalue λ), prove that the number of periodic points has a growth rate:

$$\lim_{m \to \infty} \frac{1}{m} \log |F_m| = \sum \log^+ |\lambda|.$$

(5) If A is no longer assumed to be hyperbolic, but T is still ergodic, show that the existence of the limit (4) is equivalent to *Gelfond's theorem*: if λ is an algebraic integer with $|\lambda| = 1$, and λ is not a unit root, then for any $\epsilon > 0$, $|\lambda^m - 1| < e^{-\epsilon m}$ for only finitely many m.
(6) Show that the growth rate of periodic points of an ergodic toral automorphism is always strictly positive.

2.12 Prove that the periodic points of an ergodic toral automorphism are dense in the torus. (Hint: what characters are trivial on the set of periodic points?).

2.13 Let T be the automorphism of $\widehat{\mathbb{Z}[1/6]}$ dual to multiplication by $\frac{2}{3}$ on $\mathbb{Z}[1/6]$. Show that the set of periodic points for T is dense, and prove that the growth rate of the number of periodic points exists and is $\log 3$.

2.14 Let S be the automorphism of $\widehat{\mathbb{Q}}$ dual to multiplication by $\frac{2}{3}$ on $\mathbb{Q}$. Prove that the identity is the only periodic point for S.

2.15 Prove that the automorphism dual to multiplication by 2 on $\widehat{\mathbb{Z}[\frac{1}{2}]}$ is mixing of all orders in two different ways: firstly by proving that it is isomorphic to the full Bernoulli shift on two symbols with $(\frac{1}{2}, \frac{1}{2})$ measure, and secondly by using trigonometric polynomials.

2.16 A measure–preserving transformation T is *totally ergodic* if T^k is ergodic for every $k \geq 1$. Let T be a continuous automorphism of a compact abelian group X, and assume that $\widehat{X}$ is torsion–free. Show that T is ergodic if and only if T is totally ergodic.

2.17 Find an example of a compact abelian group with an ergodic automorphism that is not totally ergodic.

2.18 Let X be a compact abelian group, and assume that $\widehat{X}$ has a non–zero element of finite order. Does X have an ergodic automorphism that is not totally ergodic?