Models and Examples

The procedure followed in this work has been to start with some general, physically based, ideas about space (the axioms) and deduce the properties of the geometry they entail. It has not been to start with some known and familiar mathematical objects and to inductively find axioms which describe the characteristics which they have in common. Altho, of course, the physical ideas about space, and the Euclidean geometry (that has been developing for millennia) which has substantially informed those ideas, do originate in a process of induction from human experience.

This kind of procedure is open to the possibility that it may be vacuous. That is, there may be nothing which satisfies the axioms; the axioms may be inconsistent with each other. And the consequences so laboriously deduced may not be the properties of anything. That this is not the case is established by the fact that Euclidean geometry satisfies the axioms. That is, it is a mathematical model for physical geometry. The axioms of physical geometry are, therefore, as consistent as Euclidean geometry.

This being the case, it might be suggested that Euclidean geometry is the only thing which satisfies the axioms and therefore physical geometry is not interesting. Either this is true or it is not. If it is true then the axioms of physical geometry constitute a new axiom system for Euclidean geometry. And that would be very interesting. If it is not then there might be novel and interesting possibilities in physical geometry worth exploring. The former is not the case, of course. This is demonstrated by the existence of mathematical models of physical geometry different from the Euclidean one. Non-Euclidean Riemannian geometry (E.g. non-positively curved, simply connected Riemannian manifolds*) and the examples in this appendix provide such demonstrations.

Apparently interesting possibilities in physical geometry might, in fact, not exist. It could be that an inconsistency or disproof has just not been found yet. Or a proof which excludes some possibility might be erroneous, in whole or in part. In such cases an example which exhibits the characteristic in question can settle the issue conclusively. Thus, such models and examples can provide guidance, correction and confirmation for the investigation.

Furthermore, such concrete examples are an invaluable aid to human understanding and the development of insight sorely needed in unfamiliar territory.

A.1 The Continuity Axiom

It needs to be proven that examples do, in fact, satisfy the axiom system of which they are purported to be models. While the continuity axiom specifies exactly the sort of physical properties one would expect of space it could be objected that it poses formidable difficulties for proving that anything could satisfy it. How could it ever be shown that "in any geometric construction, those segment measures which are mutually dependent are mutually thrice differentiable"?

This turns out not to be as difficult as it may seem; at least, in certain cases. We know that the continuity axiom is satisfied in the Euclidean case. And for models which can be suitably related to the Euclidean case it can be used to establish the axiom for those models. If the points (and lines) in the model are in a one-to-one correspondence with those of Euclidean geometry then so are the geometrical constructions. For this, for instance, it is sufficient that the model be coordinatized

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* Ballmann.

and that the Cartesian lines of its coordinate system be the lines of the model. If the segment measure in the model can be shown to be a sufficiently differentiable function of the Euclidean segment measure then, by functional composition and inversion, the continuity axiom can be established in the model. Since the segment measures of concern are differentiable functions of each other in Euclidean space, by going thru Euclidean space they can be shown to be so in the model. This process is illustrated in the figure below. Thus, in such cases, all that need be proven is the

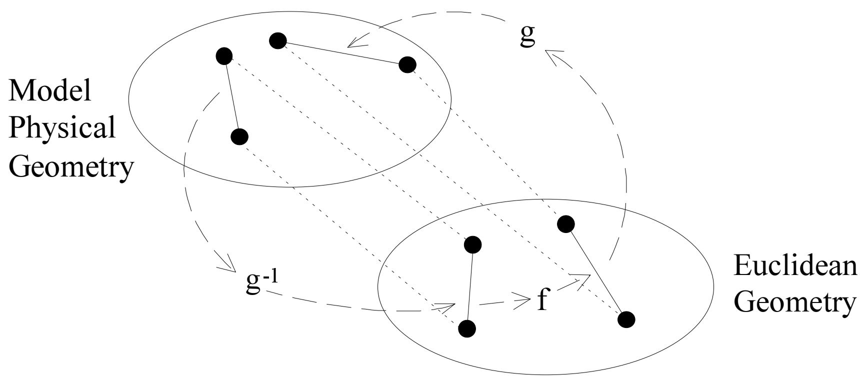

Figure 29: Using a Euclidean Correspondence

differentiability of the model's segment measure as a function of Euclidean segment measure. No detailed consideration of particular geometrical figures is required.

Two examples of this sort are presented in this appendix.

A.2 G-Space

In 1955 Herbert Busemann published a book, *The Geometry of Geodesics*, in which he introduced a mathematical concept intended as a basis for the synthetic construction of differential geometry. He defined a *G-space* by the following five axioms: (1) a metric space which is (2) finitely compact (that is, any bounded infinite set has an accumulation point), (3) M-convex and (4) locally prolongable (5) uniquely. *M-convex* means that given any two points, *A* and *C*, there is a third point, *B*, such that

$$d(A,B) + d(B,C) = d(A,C).$$

Prolongable means that given any two points A and B there is a third point C such that the same relation holds. A *straight* G-space is one which is prolongable in the large.

It has already been pointed out that physical geometry is a metric space. Furthermore, there are points between its points, its lines can be extended globally and the extensions are unique. The remaining property of finite compactness plays a role in G-space similar to that of the continuity axiom. So it is reasonable to suspect that physical geometry and straight G-spaces have similarities.

A.3 Busemann's Examples

Later[*], Busemann gave a class of examples of G-spaces which may also be used as examples of physical geometries. They are spaces in which a distortion of the Euclidean distance function is imposed on the Cartesian plane (that is, the points are ordered pairs of numbers and the lines are the usual lines). As such, Axioms 1-4 and 7 are immediately satisfied and only Axioms 5 and 6 need be established.

Busemann's distance function between an arbitrary point and the origin is fairly simple to specify. Distort the Euclidean projection onto a line of arbitrary direction of the segment (between the point and the origin). Then integrate the result over all such lines. That is,

$$d(x,0) = \int_{-\pi/2}^{\pi/2} g(|x|\cos\alpha)\,d\alpha$$

where $|x|$ is the Cartesian distance from the origin to the Cartesian point x and g is a continuous, strictly increasing function on $[0,\infty)$ with $g(0) = 0$ and

[*] *Recent Synthetic Differential Geometry*, p.27-29.

$g(t) \to \infty$ for $t \to \infty$. The distance between two arbitrary points in this scheme is obtained by taking these distorted projections with respect to the origin for each and combining them algebraically (that is, adding or subtracting them depending on whether the projections are contiguous or superimposed) before integrating over all directions as above. It should be noted that this procedure picks out the origin as a special, very peculiar point and so all of these spaces are metrically inhomogeneous.

One way to express this procedure is in terms of the constructions illustrated in the figure below. Let the ray from the origin to *x* be *S* and the ray from the origin to *y* be

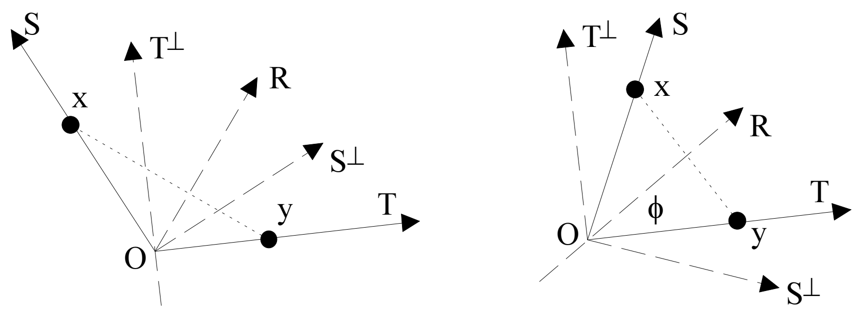

Figure 30: Constructing Busemann's Metric

T. Without loss of generality assume that the interior angle, θ, measured in a c.c. direction from *T* to *S* is between *0* and π; otherwise exchange *x* and *y*. The ray, *R*, from the origin which is perpendicular to the line between *x* and *y* can range anywhere between the two perpendiculars ($T^{\perp}$ and $S^{\perp}$) depending on the relative magnitudes of *x* and *y*. The interior angle, ϕ, from *T* to *R* (measured in a c.c. direction) is given,

$$\tan\phi = \frac{|y|/|x| - \cos\theta}{\sin\theta},$$

in terms of θ and the lengths of the segments from the origin to *x* and *y*. Let a particular line onto which to project be specified by the interior angle, α, it makes with *T* (measured in the c.c. direction). The projections add for lines outside of the perpendiculars. They subtract for lines inside the perpendiculars with, upon passing thru *R*, a reversal of order. There are a variety of analytic ways to express this procedure. In the terms established above one of them is the following.

$$\begin{aligned} d(x,y) = & \int_{-\pi/2}^{-\pi/2+\theta} \Big(g(|y|\cos(\alpha)) + g(|x|\cos(\pi+\alpha-\theta)) \Big) d\alpha \\ & + \int_{-\pi/2+\theta}^{\phi} \Big(g(|y|\cos(\alpha)) - g(|x|\cos(\alpha-\theta)) \Big) d\alpha \qquad \text{A.3.1} \\ & + \int_{\phi}^{\pi/2} \Big(g(|x|\cos(\alpha-\theta)) - g(|y|\cos(\alpha)) \Big) d\alpha \end{aligned}$$

This just treats each of the three different kinds of situations in a separate but well defined way.

A.4 Busemann Continuity

The only virtue of this unwieldy expression is to give Busemann's metric in terms of Euclidean quantities in a form which is manifestly as differentiable as the *g*-function.

From the above integral, this might appear not to be the case when one of the points passes thru the origin (i.e. $|x|$ or $|y|$ passes thru zero) where things are peculiar. Consideration of the functional, geometric description of the construction (page 66), however, does not support this suspicion. The appearance is just an artifact of using an analytic description which alters form in this circumstance so as to maintain differentiability thru the origin.

This process will be illustrated by explicitly calculating the first derivatives. In order to apply the formula for the distance when *x* (as in Figure 28) passes thru the origin the substitutions: $S \to T$, $x \to y$, $T \to S^-$, $y \to x$ and $\theta \to \pi - \theta$ must be made in order to conform to the specification given above of how to construct this function.

$$
\begin{aligned}
d(y,x) = & \int_{-\pi/2}^{\pi/2-\theta} \Big(g(|x|\cos(\beta)) + g(|y|\cos(\beta+\theta)) \Big) d\beta \\
& + \int_{\pi/2-\theta}^{\varphi} \Big(g(|x|\cos(\beta)) - g(|y|\cos(\pi+\beta+\theta)) \Big) d\beta \\
& + \int_{\varphi}^{\pi/2} \Big(g(|y|\cos(\pi+\beta+\theta)) - g(|x|\cos(\beta)) \Big) d\beta
\end{aligned}
$$

where $\tan\varphi = \dfrac{|x|/|y| + \cos\theta}{\sin\theta}$. The new variable of integration, β, can be expressed in terms of the old as $\beta = \pi - \theta + \alpha$.

$$
\begin{aligned}
d(y,x) = & \int_{\theta-\pi/2}^{\pi/2} \Big(g(|x|\cos(\alpha-\theta)) + g(|y|\cos(\alpha)) \Big) d\alpha \\
& + \int_{-\pi/2}^{\varphi-\pi+\theta} \Big(g(|x|\cos(\alpha-\theta+\pi)) - g(|y|\cos(\alpha)) \Big) d\alpha \\
& + \int_{\varphi-\pi+\theta}^{\theta-\pi/2} \Big(g(|y|\cos(\alpha)) - g(|x|\cos(\alpha-\theta+\pi)) \Big) d\alpha
\end{aligned}
$$

This expression for x on the S^- side can be differentiated to give

$$
\begin{aligned}
\frac{d}{d|x|} d(y,x) = & \int_{\theta-\pi/2}^{\pi/2} \cos(\alpha-\theta) g'(|x|\cos(\alpha-\theta)) d\alpha + \int_{\pi/2}^{\varphi+\theta} \cos(\alpha-\theta) g'(|x|\cos(\alpha-\theta)) d\alpha \\
& + 2\begin{pmatrix} g(|x|\cos(\varphi)) \\ -g(|y|\cos(\varphi+\theta-\pi)) \end{pmatrix} \frac{d\varphi}{d|x|} - \int_{\varphi+\theta}^{\theta+\pi/2} \cos(\alpha-\theta) g'(|x|\cos(\alpha-\theta)) d\alpha
\end{aligned}
$$

with a simplifying change of variable in the second and last terms. Likewise the original (*S* side) expression becomes, upon differentiation,

$$
\begin{aligned}
\frac{d}{d|x|} d(x,y) = & \int_{\pi/2}^{\pi/2+\theta} \cos(\alpha-\theta) g'(|x|\cos(\alpha-\theta)) d\alpha - \int_{-\pi/2+\theta}^{\phi} \cos(\alpha-\theta) g'(|x|\cos(\alpha-\theta)) d\alpha \\
& + 2\begin{pmatrix} g(|y|\cos(\phi)) \\ -g(|x|\cos(\phi-\theta)) \end{pmatrix} \frac{d\phi}{d|x|} + \int_{\phi}^{\pi/2} \cos(\alpha-\theta) g'(|x|\cos(\alpha-\theta)) d\alpha
\end{aligned}
$$

with a simplifying change of variable in the first term.

Taking the limit of the first of these expressions as $|x| \to 0$ yields

$$\left[\frac{d}{d|x|}d(y,x)\right]_{|x|\to 0} = g'(0)\left(\int_{\theta-\pi/2}^{\pi/2}\cos(\alpha-\theta)d\alpha - \int_{\pi/2}^{\pi/2+\theta}\cos(\alpha-\theta)d\alpha\right)$$

since $\varphi_{|x|=0} = \pi/2 - \theta$ and $\frac{d\varphi}{d|x|} = \frac{\cos^2\varphi}{|y|\sin\theta} \to \frac{\sin\theta}{|y|}$ in that limit. Similarly

$$\left[\frac{d}{d|x|}d(x,y)\right]_{|x|\to 0} = g'(0)\left(\int_{\pi/2}^{\pi/2+\theta}\cos(\alpha-\theta)d\alpha - \int_{\theta-\pi/2}^{\pi/2}\cos(\alpha-\theta)d\alpha\right)$$

since $\phi_{|x|=0} = \pi/2$ and $\frac{d\phi}{d|x|} = -\frac{|y|}{|x|^2}\frac{\cos^2\phi}{\sin\theta} = -\frac{|y|\sin\theta}{|y|^2 - 2|x||y|\cos\theta + |x|^2} \to -\frac{\sin\theta}{|y|}$ in that same limit. These are the same except for sign which is only an artifact of the fact that the variables of differentiation in the two calculations differ by a sign. The derivatives have the same limit at the origin from both sides and therefore the derivative exists there; this provides the illustration sought.

If the g function is thrice differentiable then any Busemann distance in any figure will also be so as a function the smooth Euclidean expressions that specify it. Consequently, according to the discussion of section (A.1), the continuity axiom is satisfied for this distance function.

A.5 The Busemann Triangle Inequality

If three (distinct) points *x*, *y* and *z* are projected onto an arbitrary line then there are only two possibilities. Either their projections are in the same order as they are listed above or they are not. Let $I(x,y)$ denote the integrand (the combination of the distortions of the Euclidean distance of the projection from the origin) corresponding to a line in Busemann's metric integral. Then in the "same order" case

$$I(z,x) = I(z,y) + I(y,x)$$

and the “different order” case

$$I(z,x) < I(z,y) + I(y,x).$$

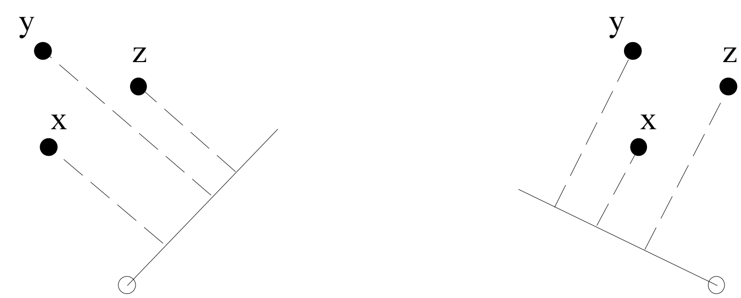

Figure 31: Examples of Projecting Three Points

If the three points are collinear then the projections onto all lines are in the same order so the integral gives

$$d(z,x) = d(z,y) + d(y,x).$$

This same result can, of course, be obtained by application of the explicit expression for the metric (A.1.1) above after combining terms, simplifying and canceling.

If the three points are not collinear then there is some range of lines in the integration for which the order is not the same. Consequently, in that case

$$d(z,x) < d(z,y) + d(y,x).$$

That is, these examples of Busemann satisfy the geodesic hypothesis also.

A.6 Busemann’s Specific Cases

Busemann’s examples are, therefore, physical geometries. Busemann gives two particular cases which reveal some kinds of asymptotic behavior that are possible for parallel lines. Both cases are produced by analytic g-functions; so they clearly are sufficiently differentiable to yield physical geometries. These cases bear on the “distance between parallel lines at infinity”. More precisely: what is the minimum distance between one line and a point on another line parallel to it as the distance from that point to a fixed point grows arbitrarily large. According to Busemann when $g(t) = \ln(1+t)$ this always vanishes and when $g(t) = e^t - 1$ it always diverges to infinity. That is, in the first case all parallel lines converge at infinity and in the second case all parallel lines diverge at infinity. This represents a wide range of possible singular behavior!

That physical geometries can have such behavior is not, in itself, physically objectionable. Singular behavior, at least as a limit, can not be ruled out and on the basis of current theories is expected.

A.7 Isotropic-Scaling Geometry

The isotropic-scaling geometry (given as an example in Chapter 2) is coordinatized and clearly the segment measures are sufficiently differentiable functions of the corresponding Cartesian coordinates. Then the only axiom which needs to be verified for it is the geodesic axiom.

This is a coordinatized model in which the distance between two points (x_1, y_1) and (x_2, y_2) is of the form

$$|x| f(\frac{y}{x})$$

where $x = x_1 - x_2$, $y = y_1 - y_2$ and *f* is the function given. Consider a curve whose points have the coordinates $(X, Y(X))$ for an arbitrary function *Y(X)*. With the above distance, the arc length of this curve is

$$\int_{X_1}^{X_2} f(Y'(X))dX$$

between two fixed points $(X_1, Y(X_1))$ and $(X_2, Y(X_2))$. This will be stationary provided Lagrange's equations are satisfied.

$$\frac{d}{dX}\left(\frac{\partial f(Y')}{\partial Y'}\right) = 0$$

That is*, $Y'(X) = const.$ or $Y(X) = aX + b$. These Cartesian lines are then the geodesics of any model of this form provided only that $f'' > 0$ so that they are stationary minimums.

To verify this for the particular case under consideration it is necessary to calculate this derivative. Now in this case

$$f(u) = \sqrt{t + mu + gu^2}\; e^{-\frac{m}{\Delta}\phi}$$

where $\phi = \tan^{-1}\left(\frac{2g}{\Delta}u + \frac{m}{\Delta}\right)$, $\Delta = \sqrt{4tg - m^2}$, $\sqrt{t} = e^{\frac{m}{\Delta}\tan^{-1}\frac{m}{\Delta}}$ and $\sqrt{g} = e^{\frac{m\pi}{2\Delta}}$. Now

$$f'(u) = \left[\frac{m/2 + gu}{t + mu + gu^2} - \frac{m}{\Delta}\phi'\right] f(u) = \frac{gu}{t + mu + gu^2} f(u)$$

since $\phi' = \frac{\Delta}{2}\frac{1}{t + mu + gu^2}$. Differentiating again gives

$$\begin{aligned} f''(u) &= \left[\frac{g}{t + mu + gu^2} - \frac{gu(m + g2u)}{\left(t + mu + gu^2\right)^2}\right] f(u) + \frac{gu}{t + mu + gu^2} f'(u) \\ &= \left[\frac{g\left(t - gu^2\right)}{\left(t + mu + gu^2\right)^2} + \left(\frac{gu}{t + mu + gu^2}\right)^2\right] f(u) \\ &= \frac{gt}{\left(t + mu + gu^2\right)^2} f(u) \end{aligned}$$

which is always a positive quantity. Therefore, the Cartesian straight lines are the geodesics of the isotropic-scaling example.

The isotropic-scaling example satisfies all the axioms of a physical geometry. And so does any other model of the general form considered for which $f'' > 0$. It is worth

* Essentially this same calculation is done on p16 of Forsyth.

pointing out that this happens to be just the general form which the arbitrary general tangent spaces of physical geometry must take.