Direction

Consider the lines thru a particular vertex point. They must fill space by themselves since any point will be on the ray determined by the vertex and itself. Each of the rays issuing from the vertex will be distinct from all the others, having only the vertex in common with the rest. How can this spray of rays, which constitutes a sort of polar coordinate system, and their relationships be characterized?

For a vertex, if a particular one of its rays is picked out as a reference, then any other ray thru that vertex will make an angle with that reference ray which will have some derivatives. It will now be shown that, as it was previously suggested, the rays are uniquely identified and ordered by these values in, at least, the non-degenerate case. Clearly this could only be true within a plane. In more than two dimensions a planar subspace would have to be considered. Here consequences of Axiom 7 will be used for the first time.

5.1 Order Inequalities

Consider any three distinct rays, call them *R*, *S* and *T*, emanating from a common vertex, *V* where *S* is in the interior of the angle *(R,T)*. Then their mutual angle derivatives can be related by various inequalities which will now be derived.

Drop segments from arbitrary fixed points *A* on *R* and *B* on *S* to a point, *P*, moving about on *T* in the vicinity of the vertex. By the crossbar theorem there is a point *Q* where the line containing *AP* intersects the line of direction *S*. These auxiliary constructions, for both signs of $\overline{VP}$, are shown in the figure below. For small $\overline{VP}$ they create two small triangles, *BQP* and *VQP*, different from those which have been considered before. These triangles collapse as $\overline{VP}$ goes to zero and so they are good prospects for fruitful application of the triangle inequality. In this situation, the quantities $\overline{PA}$ and $\overline{PB}$ are always positive, $\overline{QB}$ is positive for sufficiently small $\overline{VP}$ and both $\overline{PQ}$ and $\overline{VQ}$ have the same sign as $\overline{VP}$.

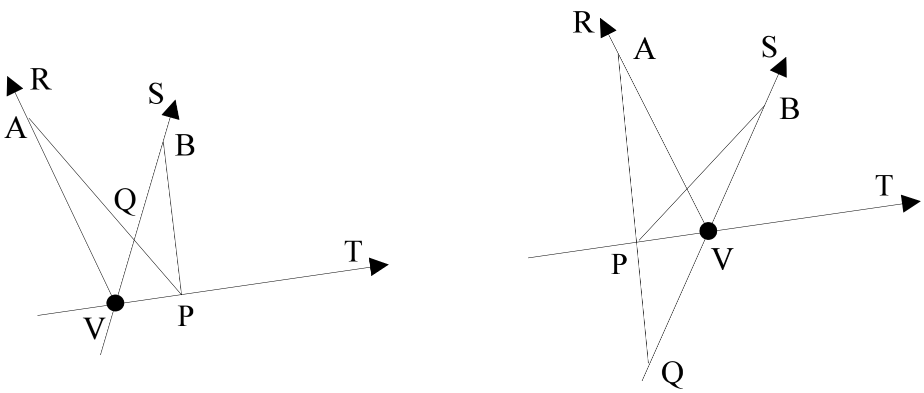

Figure 17: Order Inequalities

For the triangle BQP the appropriate triangle inequalities are $\left|\overline{PB}-\overline{QB}\right| \le \left|\overline{PQ}\right|$. (The other possible inequality is trivial for small $\overline{VP}$.) Upon division by $\left|\overline{VP}\right|$ this yields inequalities,

$$\left|\frac{\overline{PB}-\overline{VB}}{\overline{VP}}+\frac{\overline{VQ}}{\overline{VP}}\right| = \left|\frac{\overline{PB}-\overline{QB}}{\overline{VP}}\right| \le \left|\frac{\overline{PQ}}{\overline{VP}}\right| = \frac{\overline{PQ}}{\overline{VP}} = \frac{\overline{PA}-\overline{VA}}{\overline{VP}} - \frac{\overline{QA}-\overline{VA}}{\overline{VP}},$$

all of whose terms, with suitable rearrangement, converge to derivatives of functions of $\overline{VP}$ in the limit as $\overline{VP} \to 0$ because of the continuity axiom. The quantity defined by $\rho \equiv \lim \overline{VQ}/\overline{VP}$ is the derivative of the function $\overline{VQ}$ of $\overline{VP}$ at zero. By the chain rule the derivative of the function $\overline{QA}$ of $\overline{VP}$ at zero is

$$\lim \frac{\overline{QA}-\overline{VA}}{\overline{VP}} = \rho R_S$$

and so forth. The following inequalities are the result.

$$\left|S_T + \rho\right| \le R_T - \rho R_S$$

$$\Rightarrow \quad -\left(S_T + R_T\right) \le \rho\left(1 - R_S\right) \quad \text{and} \quad \rho\left(1 + R_S\right) \le R_T - S_T \qquad 5.1.1$$

Note that the quantity ρ[*] is always non-negative. (Roughly speaking, one would expect, because of simple continuity, for ρ to have a value near unity for S and T near each other.) An immediate consequence is that

$$0 \le \rho(1+R_S) \le R_T - S_T$$

since $0 \le 1+R_S$. This applies with other suitable permutations of the rays, of course. For instance, $R \to R^-$, $S \to T$ and $T \to S$ yields $T_S + R_S \le 0$. Further consequences will be elaborated in subsequent theorems.

Theorem 5.1.1. (Compound Ordering) The mutual angle derivatives of any three distinct rays *R*, *S* and *T*, with *S* in the interior of the angle *(R,T)*, satisfy

$$R_T + R_S S_T \ge 0 \ge \begin{cases} R_S + R_T T_S \\ S_R + S_T T_R \end{cases}.$$

These apply, of course, to any suitable permutations of the rays.

Proof. The inequalities (5.1.1) can be combined to give

$$-(R_T + S_T)(1+R_S) \le \rho(1-R_S)(1+R_S) \le (1-R_S)(R_T - S_T)$$

by multiplying them by factors which are non-negative. After cancellation and simplification this yields the first part of the theorem. The second part is obtained by applying the first part to the rays R^-, T and S in the two different ways possible.

There is a pattern to the pair of inequalities in the theorem which can be remembered easily. The product term starts and ends with the same rays as the other term with an intermediate ray in between. If the intermediate ray is in the interior of the others the expression is positive (or zero); if not, it is negative (or zero).

---

[*] This is clearly a function of the three rays. Altho it appears that it also is a function of the distance $\overline{VA}$ it turns out that, like the angle derivatives, it is not. This has important consequences for the theory which will not be developed in this thesis.

For the triangle *VQP* the appropriate triangle inequalities are $\left|\overline{VP}-\overline{VQ}\right| \le \left|\overline{PQ}\right|$. (The result, in the limit, of the other possible inequality is just a consequence of the magnitude of the angle derivatives being bounded by one.) In a manner similar to the above, divide by $\left|\overline{VP}\right|$, rearrange and take the limit.

$$\left|1-\frac{\overline{VQ}}{\overline{VP}}\right| \le \frac{\overline{PA}-\overline{VA}}{\overline{VP}} - \frac{\overline{QA}-\overline{VA}}{\overline{VP}}$$

$$\Rightarrow \qquad 1-R_T \le \rho\left(1-R_S\right) \text{ and } \rho\left(1+R_S\right) \le 1+R_T \qquad 5.1.2$$

A particular consequence of this is that, by combining in the manner of the previous theorem,

$$\left(1-R_T\right)\left(1+R_S\right) \le \rho\left(1-R_S\right)\left(1+R_S\right) \le \left(1+R_T\right)\left(1-R_S\right).$$

$$\Rightarrow \qquad R_S \le R_T$$

This applies to other suitable permutations of the rays, of course. For instance, $R \to S$, $S \to T$ and $T \to R^-$ yields $S_T + S_R \le 0$.

Theorem 5.1.2. (Loose Ordering) If *S* is in the interior of *(R,T)* then the angle derivatives satisfy the following inequalities.

$$S_T \le R_T \text{ and } S_R \le T_R.$$

$$T_S \le T_R \text{ and } R_S \le R_T$$

That is, the angle derivatives, of either sort, between the outer rays are never smaller than that those of the same sort between an outer ray and an interior ray.

Theorem 5.1.3. (Triangular Ordering) If *S* is in the interior of *(R,T)* then the angle derivatives satisfy the following inequality.

$$1+S_T \le \left(R_T + R_S S_T\right) + \left(R_T - R_S\right).$$

This also applies to any other suitable permutations of the rays, of course; which gives five other similar inequalities between the same rays.

Proof. Combine, in the now familiar way, the first part of (5.1.2) with the second part of (5.1.1) to obtain

$$\left(1-R_T\right)\left(1+R_S\right)\leq\rho\left(1-R_S\right)\left(1+R_S\right)\leq\left(1-R_S\right)\left(R_T-S_T\right)$$

from which, by cancellation and regrouping the conclusion follows.

The other possible combination of the above sort between (5.1.1) and (5.1.2) is a trivial redundancy. In the non-degenerate case, this last theorem will allow the inequalities to be tightened up so as to strictly order the angle derivatives and make their use for measuring directions possible.

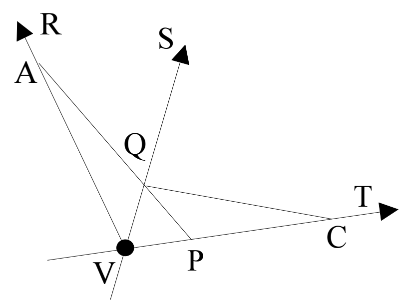

Figure 18: Other Order Inequalities

There is one other variation on the situation at hand, shown in the figure above, which needs to be mentioned. The appropriate triangle inequalities for the triangle PQC are $\left|\overline{QC}-\overline{PC}\right|\leq\left|\overline{PQ}\right|$. Applying the same process that was used before results in another set of inequalities.

$$\rho\left(R_S-T_S\right)\leq\left(1+R_T\right)\quad\text{and}\quad\left(1-R_T\right)\leq-\rho\left(R_S+T_S\right), \qquad 5.1.3$$

They relate ρ to a different angle derivative and so are worth recording. However, the first of these cannot be combined with any other inequalities in the accustomed manner because the coefficient on the left side of is of indefinite sign. The second can be so combined. However, combination with (5.1.2) produces permutations of results previously obtained. Nevertheless, the remaining possibility does give something novel.

Theorem 5.1.4. If S is in the interior of *(R,T)* then the angle derivatives satisfy the following curious inequality.

$$1 - T_S S_T \le \left(R_T + R_S S_T\right) - \left(R_S + R_T T_S\right)$$

Note that all terms are of known sign and that this is a tighter inequality than that of theorem 5.1.3. This result entails the usual suitable permutations as well, of course.

Proof. In the familiar way, the second part of (5.1.3) can be combined with the second part of (5.1.1) to obtain the following.

$$\left(1 - R_T\right)\left(1 + R_S\right) \le -\rho\left(R_S + T_S\right)\left(1 + R_S\right) \le -\left(R_S + T_S\right)\left(R_T - S_T\right).$$

After a cancellation this can be first rearranged into

$$\left(1 - R_T\right)\left(1 - T_S\right) \le -\left(R_S + T_S\right)\left(1 - S_T\right)$$

and then, after further cancellation and rearrangement, into the conclusion.

5.2 The Degenerate Case

Consider two distinct rays, say R and T, for which $R_T = -1$. (The case of +1 degeneracy is just complementary to this and need not be studied separately.) Then $S_T = -1$ and $R_S = -1$ since it has already been proven that the interior angle derivatives can't be larger and -1 is as small as they can be. Therefore all the rays in this angle are squashed together as they approach the vertex. It is appropriate to call such an angle a *degenerate angle*.

It seems reasonable to expect that all the other mutual angle derivatives between any rays in a such a degenerate angle would be similarly degenerate. In fact, if the second part of the first ordering inequality (5.1.1) is applied to the case $R \to S^-$, $S \to R^-$ and $T \to T$ the result is

$$\rho'\left(1 + S_R\right) \le R_T - S_T$$

where ρ' is another non-negative constant. In the case under consideration, this becomes

$$\rho'(1+S_R)\le 0$$

and if $\rho' \neq 0$ then $S_R = -1$ and the expectation is fulfilled. However, the possibility that $\rho' = 0$ cannot be ruled out so this result is not so easy to obtain. This is another example of how the angle derivatives in the two different directions are distinguished. The expected result may be true but no proof has been discovered yet.

Importantly, all the rays from a vertex cannot be degenerate; $R_R = -1$ and $R_R^- = +1$ and there must be a continuous transition from one value to the other. This can be seen by considering the figure below. The ray S can be identified by specifying

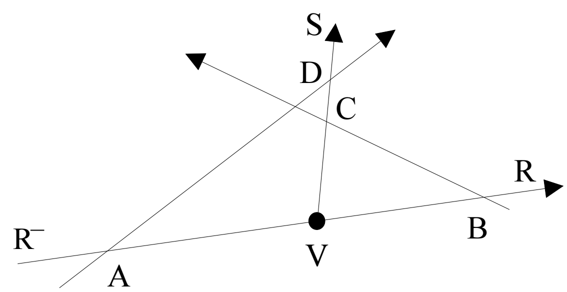

Figure 19: Continuous Transition

either $\overline{BC}$ or $\overline{AD}$ or both when there is overlap. By the continuity axiom, the angle derivatives S_R must be continuous functions of these parameters and therefore must take on all the values between the extremes.

Consequently, about any degenerate angle there must be some maximal degenerate angle outside of which the rays do not share its degeneracy. The situation is illustrated in the figure below. Perhaps there could be more than one such maximal degenerate angle if they are disjoint. But rays in different maximal degenerate angles could not be degenerate with respect to each other. And rays outside all the maximal degenerate angles cannot be degenerate with respect to any other rays.

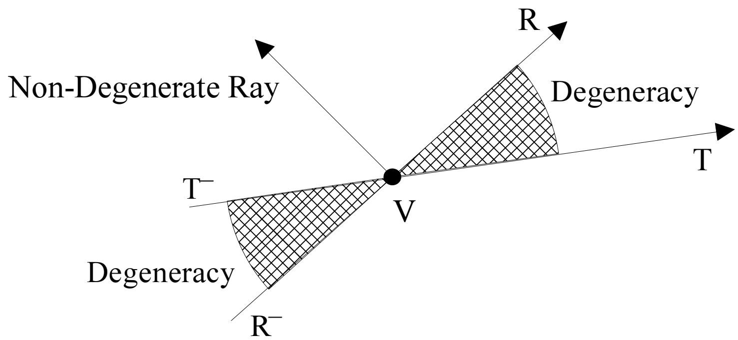

Figure 20: Degenerate Angles

Incidentally, since the angle derivatives within a degenerate angle are constant and yet all the angle derivatives at its vertex cannot be constant, this fulfills the promise made earlier to prove that an analytic geometry cannot be degenerate.

5.3 The Non-Degenerate Case

In a non-degenerate geometry or outside the degenerate angles of a degenerate geometry the angle derivatives are strictly bounded away from magnitude one. Then,

$$-2 < -1 \pm R_S < 0 < 1 \pm R_S < 2$$

for any angle derivative, R_S. Consequently the first part of inequality (5.1.2) implies that the ratio, ρ, must be strictly positive.

Theorem 5.3.1. (Strict Ordering) Excluding degeneracy, if *S* is a ray in the interior of an angle *(R,T)* then

$$\left.\begin{matrix} R_S \\ S_T \end{matrix}\right\} < R_T \quad \text{and} \quad \left.\begin{matrix} T_S \\ S_R \end{matrix}\right\} < T_R;$$

that is, all the angle derivatives with the interior ray are strictly smaller than the corresponding derivatives of the overall angle.

Proof. The expression in the proof of the Triangle Ordering Theorem (5.1.3) can be rearranged,

$$0 < \frac{(1 - R_T)(1 + R_S)}{1 - R_S} \leq R_T - S_T\,,$$

to give one of the results needed. The conclusion of that theorem can be rearranged,

$$0 < (1 - R_S)(1 + S_T) \leq 2(R_T - R_S)\,,$$

to give another. The rest follow by permutation.

5.4 Distinguishing Rays

Consequently, in a non-degenerate geometry or for non-degenerate rays in a degenerate geometry, any two distinct rays from a vertex on one side of the line of a common reference ray can be distinguished and ordered by their angle derivatives with that reference ray. First, their derivatives with respect to the reference ray must differ and the derivative of the ray closest to the head of the reference ray will be the lesser. Second, the angle derivatives of the reference ray with respect to the rays being compared must differ and the derivative with respect to the ray closest to the head of the reference ray will be the lesser.

Actually, it should be noted, that degenerate rays actually can be distinguished in a similar way. In the degenerate case, however, the arguments that lead to the above characterization apply, according to the lemma on page 25, to derivatives higher than the first instead. Specifically, inside a degenerate angle the arguments that lead to the properties of the angle derivatives devolve upon the third derivatives. [Unless they too are degenerate. But, eventually, as long as differentiation can be continued, there must be a finite odd order for which the arguments apply unless all the derivatives vanish.] Then the rays can be distinguished in a manner similar to the above using these "higher order angle derivatives". This does not mean that the angle is non-degenerate; the angle derivatives themselves are still have their degenerate values. It just means that the rays in the degenerate angle can be distinguished and ordered by other means. These remarks will not be explained any further or made precise in this paper. They are just included for completeness.

5.5 Measuring Direction

In summary, both types of angle derivatives with respect to any reference distinguish the (non-degenerate) rays and order them in the same way. As a ray "rotates" from the head towards the tail of the reference the values of both its angle derivatives must increase monotonically from -1 to $+1$, measuring off the directions. Every ray has a unique corresponding value for each of its pair of angle derivatives. However, it needs to be emphasized that there is no reason to expect the values of the two different kinds of angle derivatives that correspond to the same ray to be related to each other in any particular way.

Such an angle derivative, as previously shown, is given by

$$\frac{d}{d\overline{PB}} T_R(\overline{PA};\overline{PB}) = S_R$$

for the situation shown in the figure below. The identity of the ray, *S*, from the point

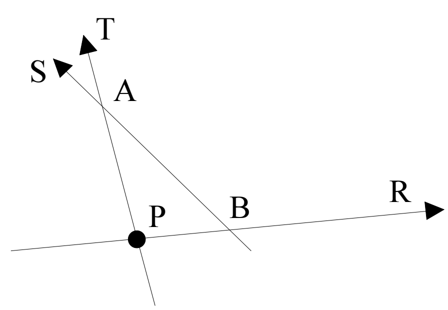

Figure 21: Continuity of Angle Derivatives #1

B is determined by the parameter $\overline{PA}$. Away from zero, because of the continuity axiom, the line-to-line functions are thrice differentiable with respect to this parameter. The angle derivative, S_R, varies continuously, therefore, with the parameter (that is as *S* changes). There are, as a result, unique rays corresponding to all values of this angle derivative in its ± 1 range (more auxiliary rays, *T*, can be used to cover the whole range of directions). The possible values of the angle derivatives of rays with respect to a reference ray are, therefore, in one-to-one correspondence with the directions themselves and can be used to measure them.

A similar argument can be made for the other angle derivative with reference to the construction below. Let the distinct points *B* and *C* on the direction *R* be fixed. Then the measure $\overline{QC}$ will be determined by $\overline{PA}$ and $\overline{BQ}$ and so, by the continuity axiom, will be a thrice differentiable function of these quantities. Consequently, the

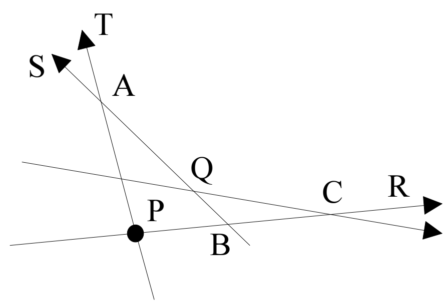

Figure 22: Continuity of Angle Derivatives #2

angle derivative R_S, which is the derivative of $\overline{QC}$ with respect to $\overline{BQ}$ (at zero) will be a continuous function of $\overline{PA}$ (which determines the direction *S*). The directions thru *B* (using more auxiliary rays, *T*, as needed to cover the full range) must, therefore, be in one-to-one and continuous correspondence with this type of angle derivative also. The possible values of the angle derivatives of a reference ray with respect a given ray are, therefore, in one-to-one correspondence with the directions themselves and can be used to measure them.

The important result is just that a numerical measure of direction exists not that a particular one exists. And, therefore, that the appropriate techniques, from algebra to calculus, can be applied to whatever measure of direction is used. But for most purposes, and all of this thesis, it will suffice to leave the measure being used arbitrary and unspecified. This is actually a rather important point. While it has been shown that either segment measures or angle derivatives could be used neither of these, as it turns out, is the most convenient measure of direction. Long experience trying to use such specific measures has taught that doing so just confounds the peculiar details of the measure with the general properties of the geometry. Leaving the measure arbitrary explicitly displays what is due to the measure and what is essential to the geometry. This will eventually make possible the discovery of a natural measure of direction (tho this will not be included in this thesis) which would probably not have been found had a specific measure been picked out in advance.

5.6 Perpendiculars

It is apparent that, unlike in classical geometry there are two distinct concepts of perpendicularity that can now be developed depending on which of the two conceptually distinct, and possibly different, angle derivatives are used to define it. This possible difference will be shown to be realized (see page 59) by working out the isotropic-scaling example.

First, thru any point, A, on a line of direction S a unique ray, R, may be erected such that the derivative of S with respect to R vanishes: $S_R = 0$. Note, in the

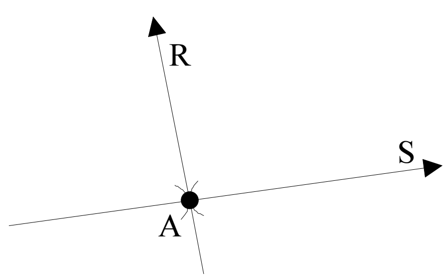

Figure 23: Perpendicular Erected to a Line at a Point

diagram, the "swing" notation with which it is convenient to denote this relationship. This can be called *erecting a perpendicular* from a point on a line. There are, of course, two possible orientations for such a perpendicular ray. The one indicated in the diagram is obtained by a “right hand rule”: with the thumb of the right hand pointing out of the page the fingers curl from the base ray, S, to the erected perpendicular ray. One could be write $R = (A,S)^{)(}$ for this right-hand erected perpendicular ray. But it will usually be more convenient to just write $R = S^{)(}$ and leave the reference to the point of erection implicit as it will usually be clear from context.

Second, given a line of direction R and a point P not on it, consider the length of the segment between P and an arbitrary point Q on the line. As argued earlier, when Q

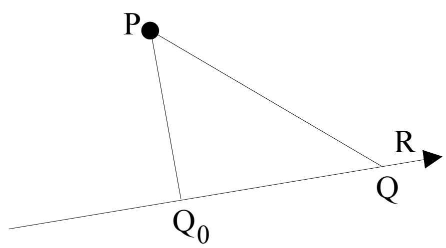

Figure 24: Point and Line

moves off toward either end of the line then $\left|\overline{PQ}\right|$ must eventually increase arbitrarily. Therefore it must attain at least one minimum somewhere in between at a some ray S; at which point $S_R = 0$. Call such a minimum segment a perpendicular dropped from P to the line. And call the special point F at which this minimum occurs the foot of that perpendicular. Again, it is convenient to indicate this

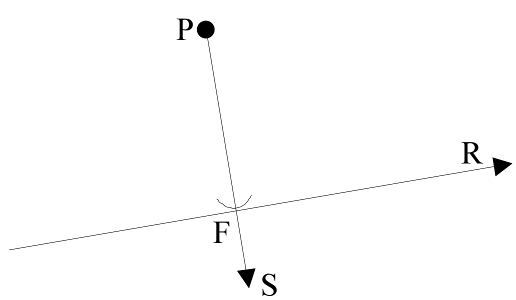

Figure 25: Perpendicular Dropped from a Point to a Line

relationship by the "swing" notation illustrated in the figure. In view of the results of the previous section, S will be the unique line thru F with this property and every point on the line will be the foot of some such perpendicular. This can be called *dropping a perpendicular* to a line.

In this situation, for a point Q in the vicinity of the foot, F, to P then since $\overline{PQ}$ has

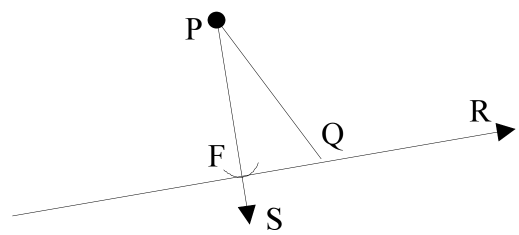

Figure 26: Near a Perpendicular Dropped from a Point

a minimum at F the first derivative of the appropriate line-to-line function $S_R(\overline{FP};\overline{FQ})$ must vanish. Then the second derivative is

$$\frac{d^2 S_R(\overline{FP};\overline{FQ})}{d\overline{FQ}^2} = \frac{1}{r}$$

where the geometric parameter r is characteristic of the perpendicular from P to the line at F. In view of the results obtained before it increases with $\overline{FP}$. In the Euclidean case $r = \overline{FP}$.

There are two possible orientations for a dropped perpendicular. The one chosen above is obtained by a left-hand rule (or a right-hand rule from perpendicular to base, if that is easier to remember) and writing $S = \breve{R}$ indicates this relationship. Strictly speaking, either the foot or the point from which the perpendicular is dropped must also be specified to determine this ray. But, as before, this will be left implicit since it will usually be clear from context.

The rationale for this convention (which may at first seem inconvenient) is that it makes the relation between R and $R^{)(}$ the same as that between $\breve{R}$ and R. With these conventions the relationships $S = \breve{R}$ and $R = S^{)(}$ are the same without involving any spurious “minus signs”. It is apparent that both types of perpendiculars are instances of the same concept viewed from a different point of view: which ray is regarded as the base and which as the perpendicular.

On the other hand, it should be emphasized that there *is* a novel possibility here. The difference is that, contrary to ordinary geometry, the point of view matters. The two types of perpendicularity are not identical in principle and may not be equivalent in a particular situation. Even tho $S = \breve{R}$ and $R = S^{)(}$ it may be that neither $R = \breve{S}^{-}$ nor

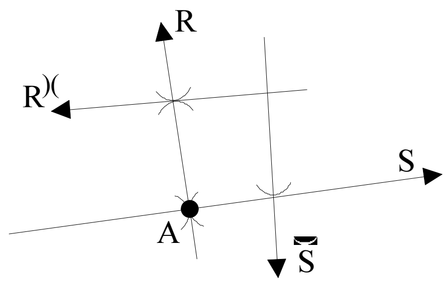

Figure 27: Perpendiculars of Perpendiculars

$S = R^{)(-}$ as in the Euclidean case. (All the rays in the diagram are intended to pass thru the point *A* in the diagram; schematically they are shown otherwise only to avoid terminal clutter.) That is, a line is not necessarily "perpendicular to its perpendiculars".

This possibility can be illustrated with the scaling, isotropic example. Let *S* be the direction of the y-axis and *T* be the direction of the x-axis. Then

$$T_S = \left[\frac{d}{dy} \sqrt{tx^2 + mxy + gy^2}\; e^{-\frac{m}{\Delta}\phi} \right]_{y=0} = \left[\frac{gy}{\sqrt{tx^2 + mxy + gy^2}}\, e^{-\frac{m}{\Delta}\phi} \right]_{y=0} = 0$$

since $\phi_y = \frac{d}{dy} \tan^{-1}\left(\frac{2g}{\Delta}\frac{y}{x} + \frac{m}{\Delta} \right) = \frac{\Delta x}{2} \frac{1}{tx^2 + mxy + gy^2}$. That is, the y-axis is the erected perpendicular to the x-axis. However,

$$S_T = \left[\frac{d}{dx} \sqrt{tx^2 + mxy + gy^2}\; e^{-\frac{m}{\Delta}\phi} \right]_{x=0} = \left[\frac{tx + my}{\sqrt{tx^2 + mxy + gy^2}}\, e^{-\frac{m}{\Delta}\phi} \right]_{x=0} = m\, e^{-\frac{m\pi}{\Delta}}$$

since $\phi_x = \frac{d}{dx}\tan^{-1}\left(\frac{2g}{\Delta}\frac{y}{x}+\frac{m}{\Delta}\right) = -\frac{\Delta y}{2}\frac{1}{tx^2+mxy+gy^2}$. This shows that the converse is not the case ($S_T \neq 0$) and the x-axis is not the erected perpendicular of the y-axis.

These non-Euclidean possibilities lead to the construction, which will be called a *t ist triangle*, illustrated in the diagram below. In general, this construction may not

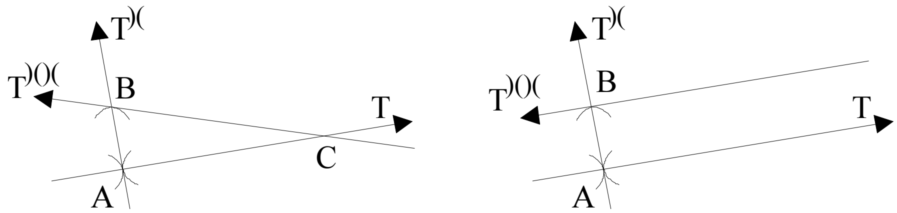

Figure 28: T ist Triangle

actually close; that is $T^{)()(} = (B,(A,T)^{)(})^{)(}$ may not intersect T; as, for instance, it does not in the Euclidean case. However, as it becomes smaller, that is $B \to A$, then because of continuity $T^{)()(} \to (A,(A,T)^{)(})^{)(}$, which does intersect T (possibly coinciding with T^-). Therefore, the construction must close in the limit, albeit possibly in a vacuous manner. If $(A,(A,T)^{)(})^{)(} \neq T^-$, however, then a sufficiently small, but finite, construction must close and form a twist triangle.

There are some other hard to conceive possibilities that have not been excluded. Altho a perpendicular can be erected from any point on a line it hasn t been guaranteed that there is a point on a line from which a perpendicular can be erected that passes thru a given point off the line. On the other hand, it may be possible to erect perpendiculars from two different points on a line which then cross. And altho a point must have at least one perpendicular to and foot on a given line there is also the possibility that it has more than one. It is possible that these are just topological possibilities and can be excluded by considering a sufficiently small but finite region. It can be shown that they do not occur in the tangent space.