

Two Dimensions

The initial results which have been obtained so far are independent of dimension and it is dimension 4 which is of physical interest. However, most the rest of this paper will be specialized to the case of two dimensions by taking Pasch's Postulate as a new axiom. When doing research it is always methodologically prudent to investigate simple cases because what is learned in solving them is often the key to the harder ones; this is the stage of the present work.

4.1 Pasch's Postulate

To state this axiom first requires the formal introduction of the concept of a triangle which has hitherto been used implicitly. For any three noncollinear points A , B and C define the *triangle* $ABC = AB \cup BC \cup CA$. The points are called the *vertices* and the segments are called the *sides* of the triangle. Note that it is evident from this definition that the order in which the defining vertices are specified is irrelevant.

Axiom 7. If a line intersects a triangle then it intersects two sides of the triangle, possibly at a common vertex.

Corollary. If A , B and C are three noncollinear points not on a line m then m cannot intersect all three sides of the triangle ABC .

Proof. Suppose m intersects AB at D , BC at E and CA at F then, without loss of generality, take E to be between D and F . The line containing BC then intersects the segment DF of the triangle AFD . By Pasch's Postulate, that line must also intersect AF or AD . But it already intersects the line containing AF elsewhere at C and the line containing AD elsewhere at B and so must coincide with one of those lines by Axiom 3. This is a contradiction.

This is the same proof you are likely to find in most any textbook on Euclidean geometry. In a similar way much of the same two dimensional terminology can be introduced and theorems can be proven as in ordinary synthetic geometry. This is because these kind of results only depend on the properties of metric geometry (that is, Axioms 1-4 of physical geometry). These necessities will be completely familiar to the reader and will, accordingly, be presented as briefly as possible by deferring to standard textbooks for the proofs.

4.2 Plane Geometry

For any two points A and B that do not lie on a line, m , if the segment AB does not intersect m then A and B are said to lie *on the same side* of m . Otherwise they are *on opposite sides* of m .

Theorem 4.2.1. (Plane Separation). Consider any line m and any three points A , B and C not on m . (i) If A and B are on the same side of m and B and C are on the same side of m then A and C are also on the same side of the line. (ii) If A and B are on the opposite side of m and B and C are on the opposite side of m then A and C are on the same side of the line.*

Plane separation and Pasch's Postulate are equivalent (i.e. the proof goes the other way as well). Pasch's Postulate was used because of its physical appeal and for its potential for generalization to characterizing higher dimensions.

Accordingly all the points on the same side (or opposite side) of a line as a test point are on the same side as each other. These two *sides* are separated by the line. The line and its sides clearly exhaust the point set. A set is *convex* if it contains the

* Millman and Parker, p77, Theorem 4.3.3.

segment between two points of the set. The sides of a line are, virtually by definition, convex sets.

Consider a nontrivial angle, (S,T) , with vertex V ($S \neq T, T^-$ etc.). The only point the rays have in common is V and it is an *extreme point* which is not between any of the points on either ray. Therefore all the points of either ray, except V , are on the same side of the (line containing the) other ray. That side of the ray is said to be the side on which the other ray lies. The *interior* of the angle is the intersection of the side of S on which T lies with the side of T on which S lies. An *interior ray* of the angle is any other ray from V all of whose points, except V , lie in the interior of the angle. A *crossbar* of an angle is any “bar which lies across” the angle; that is, a line which intersects both rays of an angle at points other than the vertex. Note that it would be a mistake to assume that any line in the interior of an angle must be a crossbar.

Theorem 4.2.2. (Crossbar). If (S,T) is a nontrivial angle with vertex V then for any points $A \neq V$ on S and $B \neq V$ on T an interior ray of that angle intersects the interior of the segment AB .*

Obviously, the parts into which a crossbar is divided by an interior ray measure smaller in magnitude than the whole.

Corollary. In the situation of the crossbar theorem, if there are several crossbars divided by two interior rays then the respective parts into which the crossbars are divided by the interior rays are monotonic with respect to each other.

For example, referring to the diagram in the figure below, if the segment from P to the ray Y measure less than the segment from P to the ray X then so does the segment

* Millman and Parker, p84, Theorem 4.4.7.

from A to the ray Y measure less than the segment from A to the ray X . After all, in that situation, the ray Y is in the interior of the angle (X, T) and the crossbar theorem applies to both crossbars in that “smaller” angle.

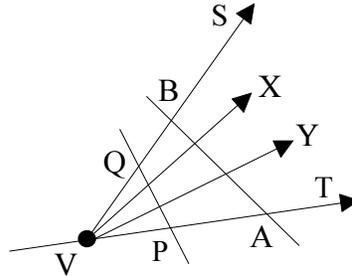


Figure 16: Mutually Monotonic Crossbars

This can be thought of as if a ray were moving about in the interior of (S, T) : then the distance from P to the interior ray on one crossbar is a monotonic function of the distance from A to the interior ray on the other crossbar. The same is true of the segments from Q and B on the other side, of course.

4.3 Expectations

Working out the character of physical geometry in 2 dimensions, the simplest non-trivial geometry, will be very instructive. It indicates that the physical geometry is fruitful. It can be hoped and expected that the ideas developed and the experience gained in this way will reveal how to proceed in more complex situations; in particular, for the physically more relevant four-dimensional case.