Difference Equations
to
Differential Equations

Section 8.7

Power Series Solutions

In this section we consider one more approach to finding solutions, or approximate solutions, to differential equations. Although the method may be applied to first order equations, our discussion will center on second order equations.

The idea is simple: Assuming that the equation

$$\ddot{x} = f(x, \dot{x}, t) \tag{8.7.1}$$

has a solution which is analytic on an interval about $t = t_0$, we express x as a power series

$$x(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n, \tag{8.7.2}$$

compute $\dot{x}$ and $\ddot{x}$, substitute the results into the equation, solve for the coefficients a_0, a_1, a_2, $\ldots$, and verify that the resulting series converges on an interval about t_0. As we shall see, in practice the difficult part is solving for the coefficients. This method will lead us to a closed form solution for the equation only in the rare case that we are able to recognize the resulting power series as the Taylor series of some known function. One advantage of this technique over numerical methods, such as the Runge-Kutta method, is that we are able to work with general solutions and equations involving unspecified parameters, whereas with a numerical method every quantity must be specified as a number. The disadvantage of this technique is that it is not as widely applicable, due to the difficulty of solving for the coefficients, and, when numerical results are needed, one must still approximate the infinite series which results when evaluating x at a point.

To illustrate the procedure, we will begin with an example which we know to be solvable by the techniques of Section 8.4.

Example Consider the equation

$$\ddot{x} = -x. \tag{8.7.3}$$

This is a constant coefficient homogeneous linear equation with characteristic equation $k^2 + 1 = 0$. Since the roots of the characteristic equation are $-i$ and i, we know from our work in Section 8.4 that the general solution of this equation is

$$x = c_1 \cos(t) + c_2 \sin(t),$$

where c_1 and c_2 are arbitrary constants.

We may obtain the same result using power series. If we suppose that x is analytic on an interval about $t = 0$, then we may write

$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$

for some constants a_0, a_1, a_2, ... Differentiating, we have

$$\dot{x}(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

and

$$\ddot{x}(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n.$$

Substituting into (8.7.3) gives us

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n = -\sum_{n=0}^{\infty} a_n t^n.$$

Since power series representations are unique, the coefficient of t^n in the power series on the left must equal the coefficient of t^n in the power series on the right for all values of n. That is, we must have

$$(n+2)(n+1) a_{n+2} = -a_n$$

for $n = 0, 1, 2, \ldots$. Hence the coefficients of the power series representation of x satisfy the difference equation

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \tag{8.7.4}$$

for $n = 0, 1, 2, \ldots$. Note that (8.7.4) does not restrict either a_0 or a_1, but determines all of the other coefficients once these values are specified. Thus, given any values for a_0 and a_1,

$$\begin{aligned}
a_2 &= -\frac{a_0}{(2)(1)} = -\frac{a_0}{2}, \\
a_3 &= -\frac{a_1}{(3)(2)} = -\frac{a_1}{3!}, \\
a_4 &= -\frac{a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)} = \frac{a_0}{4!}, \\
a_5 &= -\frac{a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)} = \frac{a_1}{5!}, \\
a_6 &= -\frac{a_4}{(6)(5)} = -\frac{a_0}{(6)(5)(4)(3)(2)} = -\frac{a_0}{6!}, \\
a_7 &= -\frac{a_5}{(7)(6)} = -\frac{a_1}{(7)(6)(5)(4)(3)(2)} = -\frac{a_1}{7!},
\end{aligned}$$

and so on. In fact, we see that for $k = 0, 1, 2, \ldots$,

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}.$$

In most cases, this is as far as we can go; we would now check for the interval of convergence of the resulting power series and conclude that x is a solution of (8.7.3) on that interval. However, in this case we see that

$$\begin{aligned} x &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_0 + a_1 t - \frac{a_0}{2}t^2 - \frac{a_1}{3!}t^3 + \frac{a_0}{4!}t^4 + \frac{a_1}{5!}t^5 - \frac{a_0}{6!}t^6 - \frac{a_1}{7!}t^7 + \cdots \\ &= a_0\left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right) + a_1\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) \\ &= a_0\cos(t) + a_1\sin(t), \end{aligned}$$

the general solution that we noted above. Hence there is no need to check for the interval of convergence since we recognize our power series representation of x as the Taylor series of a familiar function.

In general, if

$$x = \sum_{n=0}^{\infty} a_n (t - t_0)^n, \tag{8.7.5}$$

then $x(t_0) = a_0$ and $\dot{x}(t_0) = a_1$. Hence if we are seeking the solution of a differential equation in this form, then the values of a_0 and a_1 are determined by any initial conditions which specify $x(t_0)$ and $\dot{x}(t_0)$. Thus we shall see that all of our examples will be of the general form of the previous example. Namely, after substituting x, $\dot{x}$, and $\ddot{x}$ into the equation, we will find a difference equation which determines the coefficients, a_2, a_3, a_4, $\ldots$, in terms of a_0 and a_1. However, unlike the first example, our remaining examples will not result in closed form expressions for our solutions. Nevertheless, we will find power series representations for the solutions which may be used to approximate a specific solution to any desired order on some interval of convergence.

Example Consider the equation

$$\ddot{x} - tx = 0. \tag{8.7.6}$$

Suppose x is analytic on an interval about $t = 0$ and write

$$x = \sum_{n=0}^{\infty} a_n t^n$$

for some constants a_0, a_1, a_2, ... Then, as in the previous example,

$$\dot{x} = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

and

$$\ddot{x} = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n.$$

Substituting into (8.7.6) gives us

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - t \sum_{n=0}^{\infty} a_n t^n = 0,$$

from which it follows that

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n = \sum_{n=0}^{\infty} a_n t^{n+1}.$$

Since the powers of t in the series on the left begin with 0 while that the powers of t in the series on the right begin with 1, we will move the constant term of the series on the left out of the summation and adjust the index of the sum on the right so that it agrees with the index of the sum on the left. We then have

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} t^n = \sum_{n=1}^{\infty} a_{n-1} t^n.$$

We can now use the uniqueness of power series representations to equate the coefficients on the two sides of this equation, giving us

$$2a_2 = 0$$

and, for $n = 1, 2, 3, \ldots$,

$$(n+2)(n+1) a_{n+2} = a_{n-1}.$$

Hence the coefficients of the power series for x are specified by

$$a_2 = 0$$

and the difference equation

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)} \tag{8.7.7}$$

for $n = 1, 2, 3, \ldots$. As in the previous example, these equations do not restrict the values of a_0 and a_1. However, after specifying a_0 and a_1 by the initial conditions $x(0) = a_0$ and $\dot{x}(0) = a_1$, we may compute

$$\begin{aligned} a_2 &= 0, \\ a_3 &= \frac{a_0}{(3)(2)} = \frac{a_0}{6}, \\ a_4 &= \frac{a_1}{(4)(3)} = \frac{a_1}{12}, \\ a_5 &= \frac{a_2}{(5)(4)} = 0, \\ a_6 &= \frac{a_3}{(6)(5)} = \frac{a_0}{180}, \\ a_7 &= \frac{a_4}{(7)(6)} = \frac{a_1}{504}, \end{aligned}$$

and so on for as many terms as are desired. We then have

$$\begin{aligned} x &= a_0 + a_1 t + \frac{a_0}{6}t^3 + \frac{a_1}{12}t^4 + \frac{a_0}{180}t^6 + \frac{a_1}{504}t^7 + \cdots \\ &= a_0\left(1 + \frac{t^3}{6} + \frac{t^6}{180} + \cdots\right) + a_1\left(t + \frac{t^4}{12} + \frac{t^7}{504} + \cdots\right). \end{aligned}$$

To find the interval of convergence for x, we look at the two series on the right individually. Applying the ratio test to the first series, and making use of the difference equation (8.7.7) to find a_{3n+3} in terms of a_{3n}, we have, for any value of t,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{3n+3}t^{3n+3}}{a_{3n}t^{3n}}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{a_{3n}}{(3n+3)(3n+2)}}{a_{3n}}\right||t|^3 = \lim_{n\to\infty}\frac{|t|^3}{(3n+3)(3n+2)} = 0.$$

Hence $\rho < 1$ for all t and the series converges on $(-\infty, \infty)$. Similarly, for the second series we have, for any value of t,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{3n+4}t^{3n+4}}{a_{3n+1}t^{3n+1}}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{a_{3n+1}}{(3n+4)(3n+3)}}{a_{3n+1}}\right||t|^3 = \lim_{n\to\infty}\frac{|t|^3}{(3n+4)(3n+3)} = 0.$$

Again, $\rho < 1$ for all t and this series also converges on $(-\infty, \infty)$. Thus we have found a solution for (8.7.6) which is analytic on $(-\infty, \infty)$.

The computation of the interval of convergence of a solution found in the manner of the last example can be very involved. Although the justification of the following proposition is itself too involved for us to go into at this point, we will make use of it in our final two examples.

Proposition Suppose $p(t)$ and $q(t)$ are analytic on the interval (t_0-R, t_0+R). Then for any two constants a_0 and a_1, there is a unique function $x(t)$, analytic on (t_0-R, t_0+R), which satisfies the differential equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{8.7.8}$$

with initial conditions $x(t_0) = a_0$ and $\dot{x}(t_0) = a_1$.

In our previous example, we have, in the notation of the proposition, $p(t) = 0$ and $q(t) = -t$, both of which are analytic on $(-\infty, \infty)$. Hence it follows from the proposition, as we saw by direct computation, that our power series solution converges on $(-\infty, \infty)$.

Note that this proposition also tells us the we analytic solutions to an equation of the form (8.7.8) will exist provided p and q are both analytic. Equation (8.7.8) is similar to the equations we studied in Section 8.4, the difference being that (8.7.8) does not require the coefficients of $\dot{x}$ and x to be constants.

Example Consider the equation

$$(1-t)\ddot{x} + x = 0. \tag{8.7.9}$$

Suppose x is analytic on an interval about $t = 0$ and write

$$x = \sum_{n=0}^{\infty} a_n t^n$$

for some constants a_0, a_1, a_2, ... Then, as before,

$$\dot{x} = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

and

$$\ddot{x} = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n.$$

Substituting into (8.7.9) gives us

$$(1-t)\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n + \sum_{n=0}^{\infty} a_n t^n = 0.$$

Expanding the first term, we have

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^{n+1} + \sum_{n=0}^{\infty} a_n t^n = 0.$$

To adjust for the fact that the powers of t begin with 1 in the middle series, but with 0 for the other series, we move the constant terms of the latter series out of the summation and adjust the index of the middle series to obtain

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=1}^{\infty}(n+1)na_{n+1}t^n + a_0 + \sum_{n=1}^{\infty} a_n t^n = 0,$$

from which we obtain

$$a_0 + 2a_2 + \sum_{n=1}^{\infty}((n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + a_n)t^n = 0.$$

Using the uniqueness of power series representations, we conclude that all the coefficients on the left-hand side of this equation must be 0. Hence

$$a_0 + 2a_2 = 0$$

and, for $n = 1, 2, 3, \ldots$,

$$(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + a_n = 0.$$

Thus

$$a_2 = -\frac{a_0}{2} \tag{8.7.10}$$

and

$$a_{n+2} = \frac{(n+1)na_{n+1} - a_n}{(n+2)(n+1)} \tag{8.7.11}$$

for $n = 1, 2, 3, \ldots$. Since (8.7.11) becomes (8.7.10) when $n = 0$, we may combine them into a single difference equation,

$$a_{n+2} = \frac{(n+1)na_{n+1} - a_n}{(n+2)(n+1)} \tag{8.7.12}$$

for $n = 0, 1, 2, \ldots$. As always, a_0 and a_1 are determined by the initial conditions and a_2, a_3, a_4, $\ldots$ may be computed from (8.7.12). For example,

$$\begin{aligned}
a_2 &= -\frac{a_0}{2}, \\
a_3 &= \frac{2a_2 - a_1}{(3)(2)} = -\frac{a_0 + a_1}{6}, \\
a_4 &= \frac{(3)(2)a_3 - a_2}{(4)(3)} = \frac{-(a_0 + a_1) + \frac{a_0}{2}}{12} = -\frac{a_0 + 2a_1}{24},
\end{aligned}$$

and

$$a_5 = \frac{(4)(3)a_4 - a_3}{(5)(4)} = \frac{-\frac{1}{2}(a_0 + 2a_1) + \frac{1}{6}(a_0 + a_1)}{20} = -\frac{2a_0 + 5a_1}{120}.$$

Hence

$$x = a_0 + a_1 t - \frac{a_0}{2}t^2 - \frac{(a_0 + a_1)}{6}t^3 - \frac{(a_0 + 2a_1)}{24}t^4 - \frac{(2a_0 + 5a_1)}{120}t^5 + \cdots$$
$$= a_0\left(1 - \frac{t^2}{2} - \frac{t^3}{6} - \frac{t^4}{24} - \frac{t^5}{60} - \cdots\right) + a_1\left(t - \frac{t^3}{6} - \frac{t^4}{12} - \frac{t^5}{24} - \cdots\right).$$

Finally, if we rewrite (8.7.9) as

$$\ddot{x} + \frac{1}{1-t}\,x = 0,$$

then, in the notation of the previous proposition,

$$p(t) = 0$$

and

$$q(t) = \frac{1}{1-t}.$$

Now p is analytic on $(-\infty, \infty)$, but, considering intervals about 0, q is analytic on only $(-1, 1)$. Thus the proposition guarantees only that our solution will be analytic on $(-1, 1)$. That is, we know that the two power series in the expression for x converge at least on $(-1, 1)$.

Example For an example involving an unspecified parameter, consider the equation

$$\ddot{x} - 2t\dot{x} + 2rx = 0, \tag{8.7.13}$$

where r is a constant. Known as *Hermite's equation*, the solutions to this equation are important in certain areas of mathematics and quantum mechanics. As usual, we suppose x is analytic on an interval about $t = 0$, write

$$x = \sum_{n=0}^{\infty} a_n t^n$$

for some constants a_0, a_1, a_2, ..., and compute

$$\dot{x} = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

and

$$\ddot{x} = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n.$$

Substituting into (8.7.13), we have

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - 2t \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n + 2r \sum_{n=0}^{\infty} a_n t^n = 0.$$

Thus

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=0}^{\infty}2(n+1)a_{n+1}t^{n+1} + \sum_{n=0}^{\infty}2ra_nt^n = 0.$$

Adjusting all these series to start with t raised to the first power gives us

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=1}^{\infty}2na_nt^n + 2ra_0 + \sum_{n=1}^{\infty}2ra_nt^n = 0.$$

Hence

$$2ra_0 + 2a_2 + \sum_{n=1}^{\infty}((n+2)(n+1)a_{n+2} + 2(r-n)a_n)t^n = 0.$$

Therefore, by the uniqueness of power series representations, we must have

$$2ra_0 + 2a_2 = 0$$

and, for $n = 1, 2, 3, \ldots$,

$$(n+2)(n+1)a_{n+2} + 2(r-n)a_n = 0.$$

Thus

$$a_2 = -ra_0 \tag{8.7.14}$$

and

$$a_{n+2} = -\frac{2(r-n)a_n}{(n+2)(n+1)} \tag{8.7.15}$$

for $n = 1, 2, 3, \ldots$. Since (8.7.15) becomes (8.7.14) when $n = 0$, we see that, after a_0 and a_1, the coefficients of the solution are determined by the difference equation

$$a_{n+2} = -\frac{2(r-n)a_n}{(n+2)(n+1)}, \tag{8.7.16}$$

$n = 0, 1, 2, \ldots$. For example, we have

$$\begin{aligned}
a_2 &= -ra_0, \\
a_3 &= -\frac{2(r-1)a_1}{(3)(2)} = -\frac{2(r-1)a_1}{3!}, \\
a_4 &= -\frac{2(r-2)a_2}{(4)(3)} = \frac{2^2r(r-2)a_0}{4!}, \\
a_5 &= -\frac{2(r-3)a_3}{(5)(4)} = \frac{2^2(r-1)(r-3)a_1}{5!}, \\
a_6 &= -\frac{2(r-4)a_4}{(6)(5)} = -\frac{2^3r(r-2)(r-4)a_0}{6!},
\end{aligned}$$

and

$$a_7 = -\frac{2(r-5)a_5}{(7)(6)} = -\frac{2^3(r-1)(r-3)(r-5)a_1}{7!}.$$

Thus

$$\begin{aligned} x &= a_0 + a_1 t - r a_0 t^2 - \frac{2(r-1)a_1}{3!}t^3 + \frac{2^2 r(r-2)a_0}{4!}t^4 + \frac{2^2(r-1)(r-3)a_1}{5!}t^5 \\ &\quad - \frac{2^3 r(r-2)(r-4)a_0}{6!}t^6 - \frac{2^3(r-1)(r-3)(r-5)a_1}{7!}t^7 + \cdots \\ &= a_0\left(1 - rt^2 + \frac{2^2 r(r-2)}{4!}t^4 - \frac{2^3 r(r-2)(r-4)}{6!}t^6 + \cdots\right) \\ &\quad + a_1\left(t - \frac{2(r-1)}{3!}t^3 + \frac{2^2(r-1)(r-3)}{5!}t^5 - \frac{2^3(r-1)(r-3)(r-5)}{7!}t^7 + \cdots\right). \end{aligned}$$

In the notation of the previous proposition, we have $p(t) = 2t$ and $q(t) = 2r$, both of which are analytic on $(-\infty, \infty)$. Hence it follows that the two series in our solution converge for all values of t.

Moreover, note that if we let

$$x_1(t) = 1 - rt^2 + \frac{2^2 r(r-2)}{4!}t^4 - \frac{2^3 r(r-2)(r-4)}{6!}t^6 + \cdots$$

and

$$x_2(t) = t - \frac{2(r-1)}{3!}t^3 + \frac{2^2(r-1)(r-3)}{5!}t^5 - \frac{2^3(r-1)(r-3)(r-5)}{7!}t^7 + \cdots,$$

so that

$$x(t) = a_0 x_1(t) + a_1 x_2(t),$$

then x_1 is a polynomial when r is an nonnegative even integer and x_2 is a polynomial when r is a positive odd integer. That is, when r is a nonnegative integer, Hermite's equation will have a polynomial solution. When suitably normalized, as described in Problem 6 below, these polynomials are called *Hermite polynomials*.

Our final example shows the strength of the power series method of solving differential equations. Through one computation we have found analytic solutions to an entire family of equations parametrized by the real number r. As an added consequence, we have discovered that the equation has polynomial solutions for certain values of the parameter r. If we were only interested in numerical values of a solution of Hermite's equation for one value of r and one set of initial conditions, then using a numerical method, such as the Runge-Kutta method of Section 8.6, would be the proper approach; however, we can see that the power series approach leads to a much richer understanding of the solutions to the general form of the equation.

Problems

1. Solve the following first order differential equations using power series with the initial condition $x(0) = a_0$. Verify your answer by finding a closed form solution for the equation using the techniques of Sections 8.2 and 8.3

   (a) $\dot{x} = 3x$ (b) $\dot{x} = 2tx$

   (c) $\dot{x} = x - 1$ (d) $\dot{x} = -x$

2. Solve the following second order differential equations using power series with the initial conditions $x(0) = a_0$ and $\dot{x}(0) = a_1$. Write the solution out through the first six nonzero terms and give an interval of convergence for each solution.

   (a) $\ddot{x} + tx = 0$ (b) $\ddot{x} + \dot{x} - tx = 0$

   (c) $\ddot{x} + t\dot{x} + x = 0$ (d) $\ddot{x} - (1 + t^2)x = 0$

   (e) $(1 - t^2)\ddot{x} - 2t\dot{x} - x = 0$ (f) $(1 + t)\ddot{x} - x = 0$

3. (a) Use power series to show that the solution of

   $$\ddot{x} = x$$

   satisfying $x(0) = a_0$ and $\dot{x}(0) = a_1$ is given by $x = a_0 \cosh(t) + a_1 \sinh(t)$.

   (b) Solve the equation in (a) using the techniques of Section 8.4 and show that your answer agrees with the answer in (a).

4. Use the ratio test to verify that the solutions x_1 and x_2 of Hermite's equation found in the last example of this section converge for all t in $(-\infty, \infty)$.

5. Find polynomial solutions of Hermite's equation for $r = 0$, $r = 1$, $r = 2$, $r = 3$, $r = 4$, and $r = 5$.

6. A polynomial solution of Hermite's equation with highest degree term of the form $2^n t^n$ is called a *Hermite polynomial* and is denoted $H_n(t)$.

   (a) Show that $H_0(t) = 1$, $H_1(t) = 2t$, $H_2(t) = 4t^2 - 2$, and $H_3(t) = 8t^3 - 12t$.

   (b) Find $H_4(t)$ and $H_5(t)$.

7. The equation

   $$(1 - t^2)\ddot{x} - 2t\dot{x} + r(r + 1)x = 0,$$

   where r is a constant, is known as *Legendre's equation.*

   (a) Show that the general solution to Legendre's equation may be written as

   $$x(t) = a_0 x_1(t) + a_1 x_2(t),$$

   where

   $$x_1(t) = 1 - \frac{r(r+1)}{2!}t^2 + \frac{r(r-2)(r+1)(r+3)}{4!}t^4 - \frac{r(r-2)(r-4)(r+1)(r+3)(r+5)}{6!}t^6 + \cdots,$$

$$x_2(t) = t - \frac{(r-1)(r+2)}{3!}t^3 + \frac{(r-1)(r-3)(r+2)(r+4)}{5!}t^5 - \frac{(r-1)(r-3)(r-5)(r+2)(r+4)(r+6)}{7!}t^7 + \cdots,$$

and a_0 and a_1 are constants.

(b) Explain why the radius of convergence of each of these series is at least 1.

(c) Note that if r is a nonnegative even integer, then x_1 is a polynomial, and if r is a positive odd integer, then x_2 is a polynomial. If r is an even nonnegative integer, let

$$P_r(t) = \frac{x_1(t)}{x_1(1)}$$

and if r is a positive odd integer let

$$P_r(t) = \frac{x_2(t)}{x_2(1)}.$$

Then $P_r(t)$, $r = 0, 1, 2, \ldots$, is a polynomial solution of Legendre's equation, known as a *Legendre polynomial*, normalized so that $P_r(1) = 1$. Find $P_0(t)$, $P_1(t)$, $P_2(t)$, $P_3(t)$, $P_4(t)$, and $P_5(t)$ and plot their graphs on the interval $[-1, 1]$.

8. Discuss all the interconnections we have seen between difference equations and differential equations.