Difference Equations to **D**ifferential Equations

Section 8.6

The Geometry of Solutions: The Phase Plane

As mentioned in Section 8.4, we may represent a second order differential equation

$$\ddot{x} = f(x, \dot{x}, t) \tag{8.6.1}$$

as a system of two first order equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= f(x, y, t). \end{aligned} \tag{8.6.2}$$

More generally, if g and f are functions of x, y, and t, we may consider a system of equations

$$\begin{aligned} \dot{x} &= g(x, y, t) \\ \dot{y} &= f(x, y, t), \end{aligned} \tag{8.6.3}$$

of which (8.6.2) is a particular case when $g(x, y, t) = y$. In this section we shall consider the behavior of solutions to such systems of equations, paying particular attention to those arising in the manner of (8.6.2).

Definition Suppose $x(t)$ and $y(t)$ are solutions of the system

$$\begin{aligned} \dot{x} &= g(x, y, t) \\ \dot{y} &= f(x, y, t) \end{aligned}$$

for t in an interval $[a, b]$. The curve in the plane with coordinates $(x(t), y(t))$, $a \leq t \leq b$, is called a *phase curve* of the system. The plane in which the phase curve is plotted is called the *phase plane* of the system.

Note that if the system of equations arises from a second order differential equation, then a phase curve is a plot of $\dot{x}(t)$ versus $x(t)$. In many common cases, this is a plot of velocity versus position.

Definition Suppose the constant functions $x(t) = x_0$ and $y(t) = y_0$ is a solution of the system

$$\begin{aligned} \dot{x} &= g(x, y, t) \\ \dot{y} &= f(x, y, t) \end{aligned}.$$

Then the point (x_0, y_0) is called a *stationary point* of the system.

If (x_0, y_0) is a stationary point, then the phase curve of the solution

$$\begin{aligned} x(t) &= x_0 \\ y(t) &= y_0 \end{aligned}$$

consists of only the single point (x_0, y_0). That is, if (x_0, y_0) is a stationary point and the system has initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$, then the system will remain at the point (x_0, y_0) for all time. Moreover, note that for this solution

$$\dot{x}(t) = 0$$

and

$$\dot{y}(t) = 0$$

for all t. Since we must have

$$\begin{aligned} \dot{x} &= g(x, y, t) \\ \dot{y} &= f(x, y, t), \end{aligned}$$

it follows that stationary points are precisely the points (x_0, y_0) for which

$$g(x_0, y_0, t) = 0$$

and

$$f(x_0, y_0, t) = 0$$

for all t.

Example Consider the second order linear equation

$$\ddot{x} + \frac{k}{m}x = 0,$$

where k and m are positive constants. In Section 8.5 we saw how this equation models an undamped mass-spring system consisting of an object of mass m oscillating at the end of a spring with spring constant k. This equation is equivalent to the system of equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x. \end{aligned} \tag{8.6.4}$$

Clearly, the only stationary point of this system is $(0, 0)$, corresponding to the object being at rest at the equilibrium position of the system. With initial conditions $x(0) = x_0$ and $y(0) = 0$, this system has solution

$$\begin{aligned} x &= x_0 \cos\left(\sqrt{\frac{k}{m}}t\right) \\ y &= -x_0\sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}\right). \end{aligned}$$

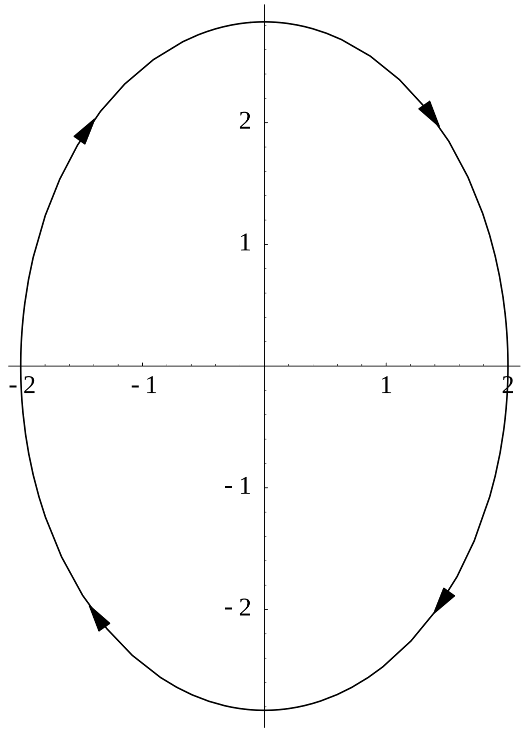

Figure 8.6.1 A phase curve for the system $\dot{x} = y$, $\dot{y} = -2x$

A plot of the phase curve for this solution is shown in Figure 8.6.1 for $k = 10$, $m = 5$, and $x_0 = 2$ for $0 \le t \le \sqrt{2}\pi$ (that is, for one full period of the motion). You should compare this plot with the graph of x in Figure 8.5.4. The arrows on the curve point in the direction of increasing t. At $t = 0$, the mass is released from a point 2 units below the equilibrium position, hence $x = 2$ and $y = 0$; as t increases from 0 to $\frac{\sqrt{2}\pi}{4}$, x decreases from 2 to 0 as the mass moves upward to the equilibrium position while y, the velocity, decreases from 0 to $-2\sqrt{2}$; as t increases from $\frac{\sqrt{2}\pi}{4}$ to $\frac{\sqrt{2}\pi}{2}$, x continues to decrease from 0 to -2 as the mass moves to its highest point, at which point its velocity is $y = 0$; at this time, the velocity becomes positive and the mass moves from -2, through the equilibrium position, back to 2, at which point the velocity is again 0 and the motion begins all over again. Notice that the phase curve is a closed curve because the motion is periodic: after a period of $\sqrt{2}\pi$ units of time, both the position and the velocity of the object have returned to their original values. Moreover, the stationary point $(0, 0)$ is at the center of this phase curve. In fact, all the phase curves for this equation are closed curves about the stationary point. Such a stationary point is called a *center*.

Note that in this example the phase curves are all ellipses. The curve in Figure 8.6.1 satisfies

$$\frac{x^2}{4} + \frac{y^2}{8} = 1. \tag{8.6.5}$$

Example Consider the second order linear equation

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0,$$

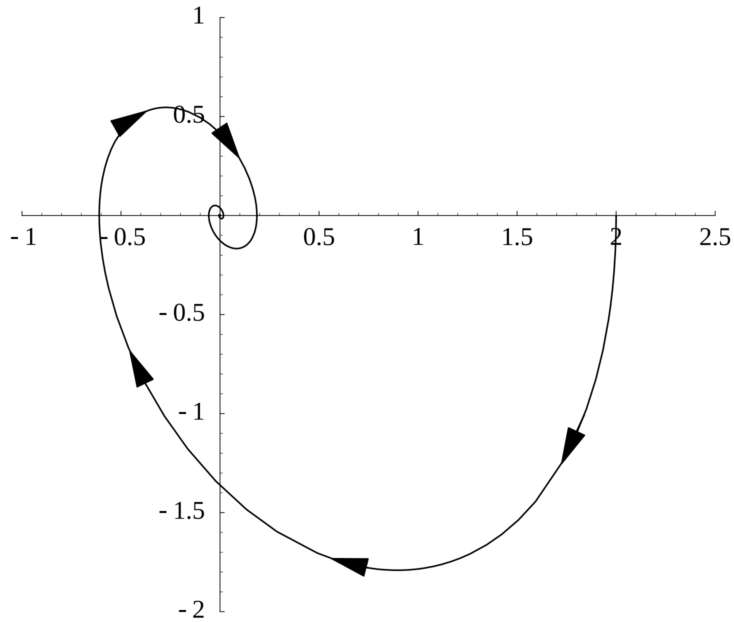

Figure 8.6.2 A phase curve for the system $\dot{x} = y$, $\dot{y} = -y - 2x$

where c, k, and m are positive constants. We saw in Section 8.5 that this equation models the motion of a damped mass-spring system consisting of an object of mass m attached to a spring with spring constant k and moving through a medium offering a resistive force proportional to $\dot{x}$. This second order equation is equivalent to the system

$$\begin{aligned} \dot{x} &= y \\ y &= -\frac{c}{m}y - \frac{k}{m}x. \end{aligned} \tag{8.6.6}$$

As in the previous example, the only stationary point of this system is $(0,0)$. We will consider an example of the underdamped case, namely, $k = 10$, $m = 5$, and $c = 5$. In that case, with initial conditions $x(0) = x_0$ and $y(0) = 0$, the solution of (8.6.6) is

$$\begin{aligned} x &= 2\sqrt{\frac{2}{7}}x_0 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{7}}{2}t - \theta\right) \\ y &= -\sqrt{2}x_0 e^{-\frac{t}{2}} \left(\sin\left(\frac{\sqrt{7}}{2}t - \theta\right) + \frac{1}{\sqrt{7}}\cos\left(\frac{\sqrt{7}}{2}t - \theta\right)\right), \end{aligned}$$

where

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{7}}\right).$$

A plot of the phase curve for this solution is shown in Figure 8.6.2 for $x_0 = 2$ with $0 \le t \le 20$. You should compare this plot with the graph of x in Figure 8.5.7. Here we see that the phase curve is not closed and the motion is not periodic; as t increases, the curve spirals in toward the stationary point $(0,0)$. This is in fact the general behavior of phase curves for this system: No matter what the initial condition, as t increases, both the position and velocity functions decay toward 0 as the mass performs smaller and smaller oscillations about the equilibrium. In this case we call the stationary point a *stable equilibrium*. In general, a stationary point (x_0, y_0) is a stable equilibrium if for any initial conditions sufficiently close to (x_0, y_0), the resulting phase curve approaches (x_0, y_0) in the limit as $t \to \infty$. In this example, every phase curve approaches $(0, 0)$ as $t \to \infty$.

A stationary point (x_0, y_0) is called an *unstable equilibrium* if there is a fixed distance d such that it is possible to find initial conditions arbitrarily close to (x_0, y_0) for which the resulting phase curve will eventually be farther than d away from (x_0, y_0).

Example Taking $k = 10$, $m = 5$, and $c = -5$ in the system (8.6.6) would lead to the solution

$$x = 2\sqrt{\frac{2}{7}}x_0 e^{\frac{t}{2}} \cos\left(\frac{\sqrt{7}}{2}t - \theta\right)$$
$$y = \sqrt{2}x_0 e^{\frac{t}{2}} \left(\frac{1}{\sqrt{7}} \cos\left(\frac{\sqrt{7}}{2}t - \theta\right) - \sin\left(\frac{\sqrt{7}}{2}t - \theta\right)\right),$$

where

$$\theta = \tan^{-1}\left(-\frac{1}{\sqrt{7}}\right).$$

In this case, the stationary point $(0, 0)$ is an unstable equilibrium because, since $e^{\frac{t}{2}}$ increases with t, the phase curves spiral away from the stationary point $(0, 0)$. A plot of the phase curve for this solution is shown in Figure 8.6.3 for $x_0 = 0.01$ with $0 \le t \le 10$.

We will see another example of an unstable equilibrium when we return to the pendulum example below.

Numerical approximations

The ideas developed above are most helpful when exact solutions are not available and we must rely upon numerical approximations to understand the behavior of our solutions. However, before we can consider such examples, we must first modify our numerical techniques from Section 8.1 to the current situation. Since the second order Runge-Kutta method is more accurate than Euler's method, we will discuss only the modification of the former.

Suppose we wish to approximate the solution of the system

$$\begin{aligned} \dot{x} &= g(x, y, t) \\ \dot{y} &= f(x, y, t), \end{aligned} \tag{8.6.7}$$

with initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$, at time $t_0 + h$. First we approximate x at $t_0 + \frac{h}{2}$ by $x_0 + m_1$, where

$$m_1 = \frac{h}{2}\dot{x}(t_0) = \frac{h}{2}g(x_0, y_0, t_0), \tag{8.6.8}$$

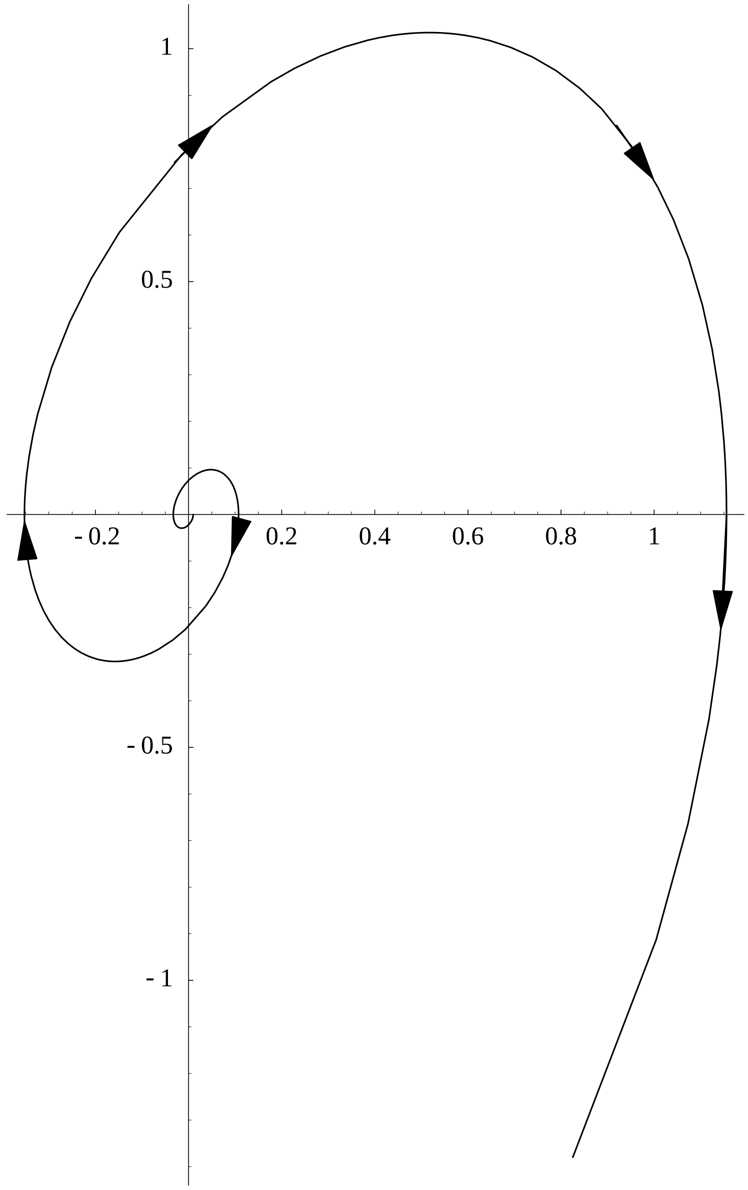

Figure 8.6.3 A phase curve for the system $\dot{x} = y$, $\dot{y} = y - 2x$

and y at $t_0 + \frac{h}{2}$ by $y_0 + m_2$, where

$$m_2 = \frac{h}{2}\dot{y}(t_0) = \frac{h}{2}f(x_0, y_0, t_0). \tag{8.6.9}$$

Then the second order Runge-Kutta approximations to $x(t_0 + h)$ and $y(t_0 + h)$ are given by

$$x_1 = x_0 + hg\Big(x_0 + m_1, y_0 + m_2, t_0 + \frac{h}{2}\Big) \tag{8.6.10}$$

and

$$y_1 = y_0 + hf\Big(x_0 + m_1, y_0 + m_2, t_0 + \frac{h}{2}\Big), \tag{8.6.11}$$

respectively. As before, to approximate the solution over an interval $[t_0, t_1]$, we iterate the above process as many times as necessary.

Second order Runge-Kutta To approximate the solution of the system of equations

$$\dot{x} = g(x, y, t)$$
$$\dot{y} = f(x, y, t)$$

with initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$ on an interval $[t_0, t_1]$, choose a small value for $h > 0$ and an integer n such that $t_0 + nh \geq t_1$. Letting $s_k = t_0 + kh$, compute

$$\begin{aligned} m_1 &= \frac{h}{2} g(x_k, y_k, s_k) \\ m_2 &= \frac{h}{2} f(x_k, y_k, s_k) \end{aligned} \tag{8.6.12}$$

and

$$\begin{aligned} x_{k+1} &= x_k + hg\left(x_k + m_1, y_k + m_2, s_k + \frac{h}{2}\right) \\ y_{k+1} &= y_k + hf\left(x_k + m_1, y_k + m_2, s_k + \frac{h}{2}\right) \end{aligned} \tag{8.6.13}$$

for $k = 0, 1, 2, \ldots, n-1$. Then x_k is an approximation for $x(t_0 + kh)$ and y_k is an approximation for $y(t_0 + kh)$.

Example In this example we consider a simple case for modeling a predator-prey environment. Suppose animals of species A prey on animals of species B. For our example, species A will be foxes and species B will be rabbits, although they could be any two species that have the predator-prey relationship we are about to describe. We assume that the food supply of the rabbits is essentially unlimited and the foxes are their only natural enemy in the given environment, while, on the other hand, we assume the foxes are dependent upon the rabbits for the bulk of their food supply. We also assume that the foxes have no natural enemies. Let $y(t)$ be the size of the fox population and let $x(t)$ be the size of the rabbit population at time t. If there were no foxes, the rabbits would enjoy uninhibited growth and we would have

$$\dot{x} = \alpha x$$

for some constant $\alpha > 0$ representing the natural growth rate of rabbits in the given environment. However, if we assume that the number of encounters between rabbits and foxes is proportional to the product of the two populations and, furthermore, that a certain proportion of these encounters results in a rabbit becoming a meal for a fox, then $\dot{x}$ will be decreased by an amount βxy for some constant $\beta > 0$. Hence we have

$$\dot{x} = \alpha x - \beta xy = x(\alpha - \beta y).$$

At the same time, if there were no rabbits, the fox population would decline for want of food, that is, we would expect

$$\dot{y} = -\gamma y$$

for some constant $\gamma > 0$, while, if there are rabbits, encounters between rabbits and foxes contributes positively to the growth of the fox population. Thus y, the size of the fox population, should make a negative contribution to $\dot{y}$ and xy should make a positive contribution to $\dot{y}$. This leads us to suppose

$$\dot{y} = -\gamma y + \delta xy = -y(\gamma - \delta x)$$

for some constants $\gamma > 0$ and $\delta > 0$. Hence we now have a system of first order equations

$$\begin{aligned} \dot{x} &= x(\alpha - \beta y) \\ \dot{y} &= -y(\gamma - \delta x), \end{aligned} \tag{8.6.14}$$

where α, β, γ, and δ are all positive constants.

The stationary points for this system are solutions of

$$\begin{aligned} 0 &= x(\alpha - \beta y) \\ 0 &= -y(\gamma - \delta x). \end{aligned}$$

Clearly, $x = 0$, $y = 0$ is one solution. If $x \neq 0$, then, from the first equation, we must have

$$0 = \alpha - \beta y,$$

and so

$$y = \frac{\alpha}{\beta}.$$

Thus $y \neq 0$, so, from the second equation, we must have

$$0 = \gamma - \delta x,$$

from which we find

$$x = \frac{\gamma}{\delta}.$$

Hence the system (8.6.14) has two stationary points: $(0, 0)$ and $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$. The first corresponds to the uninteresting situation when there are no foxes and no rabbits; the second to an equilibrium condition in which the populations are in balance.

For example, consider the case with parameters $\alpha = 0.06$, $\beta = 0.0008$, $\gamma = 0.2$, and $\delta = 0.0008$, corresponding, in part, to a natural growth rate of 6% per year for the rabbits and a decay rate, in the absence of any rabbits, of 20% per year for the foxes. The system (8.6.14) then becomes

$$\begin{aligned} \dot{x} &= x(0.06 - 0.0008y) \\ \dot{y} &= -y(0.2 - 0.0008), \end{aligned} \tag{8.6.15}$$

with nonzero stationary point

$$\left(\frac{0.2}{0.0008}, \frac{0.06}{0.0008}\right) = (250, 75).$$

Hence a population of 250 rabbits and 75 foxes would be in equilibrium and would not change over time; the natural yearly increase in the rabbit population is accounted for exactly by the appetite of the foxes. To see what happens in other cases, suppose we start with an initial population of $x_0 = 400$ rabbits and $y_0 = 50$ foxes. We will approximate the solution to (8.6.15) over the interval $[0, 150]$ using the second order Runge-Kutta method with a step size of $h = 0.05$. To start, using

$$\begin{aligned} g(x, y, t) &= x(0.06 - 0.0008y) \\ f(x, y, t) &= -y(0.2 - 0.0008x), \end{aligned}$$

we compute

$$\begin{aligned} m_1 &= \frac{h}{2} g(x_0, y_0, t_0) = (0.025)(400)(0.06 - (0.0008)(50)) = 0.2 \\ m_2 &= \frac{h}{2} f(x_0, y_0, t_0) = (0.025)(50)(0.2 - (0.0008)(400)) = 0.15 \end{aligned}$$

and

$$\begin{aligned} x_1 &= x_0 + hg\Big(x_0 + m_1, y_0 + m_2, t_0 + \frac{h}{2}\Big) \\ &= 400 + (0.05)g(400.2, 50.15, 0.025) \\ &= 400 + (0.05)(400.2)(0.06 - (0.0008)(50.15)) \\ &= 400.3978 \\ y_1 &= y_0 + hf\Big(x_0 + m_1, y_0 + m_2, t_0 + \frac{h}{2}\Big) \\ &= 50 + (0.05)f(400.2, 50.15, 0.025) \\ &= 50 - (0.05)(50.15)(0.2 - (0.0008)(400.2)) \\ &= 50.3013, \end{aligned}$$

where we have rounded x_1 and y_1 to four decimal places. Then x_1 is an approximation for $x(0.05)$ and y_1 is an approximation for $y(0.05)$. In general, x_{20t} and y_{20t} are approximations for $x(t)$ and $y(t)$ when $20t$ is an integer. Table 8.6.1 gives our results for $t = 0, 5, 10, \ldots, 150$, where we have rounded the values to the nearest integer.

Notice the cyclic nature of both x and y. In the early years, the population of foxes increases due to the plentiful supply of rabbits for food. However, eventually (sometime between 15 and 20 years) the increasing fox population causes a decrease in the rabbit population to the point where the population of foxes begins to decline. As the fox population declines, there comes a point (between 30 and 35 years) when the rabbit population begins to increase, which in turn eventually leads to an increase in the fox population, starting sometime between 55 and 60 years. At this point, the cycle begins again. This behavior is most evident in Figure 8.6.4, where the numerical solutions for x and y have been plotted over the interval $[0, 150]$. Notice how the periods of the two curves are the same, but their phases are different. This phase difference occurs because, for example, a decrease in the rabbit population does not lead to an immediate decrease in the fox population; in fact, the fox population will continue to grow until the rabbit population

t	x_{20t}	y_{20t}
0	400	50
5	408	95
10	332	158
15	225	175
20	161	137
25	138	91
30	139	58
35	156	38
40	185	28
45	226	23
50	278	23
55	339	29
60	395	47
65	411	89
70	343	152
75	235	177
80	165	142
85	139	95
90	138	61
95	153	40
100	181	28
105	221	23
110	272	23
115	333	28
120	390	44
125	413	83
130	354	145
135	245	177
140	170	147
145	141	100
150	138	64

Table 8.6.1 Predator-prey poplulations

is too small to support its growth, and it is at that point that the fox population begins to decline. The phase curve for this solution is shown in Figure 8.6.5. Here the fact that the phase curve is a closed curve reveals the periodic nature of the solution. Note that the phase curve encloses the nonzero stationary point $(250, 75)$. This point is in fact a center. Figure 8.6.6 shows several phase curves, all of which are closed curves about $(250, 75)$. We have omitted arrows on the phase curves in Figure 8.6.6, but the direction of increasing t is counter-clockwise, as it was in Figure 8.6.5.

Example For our final example in this section, we return to the pendulum problem discussed in Section 8.5. Suppose our pendulum consists of a bob of mass m at the end of a rigid rod of length b. We will assume that the upper end of the rod is attached to

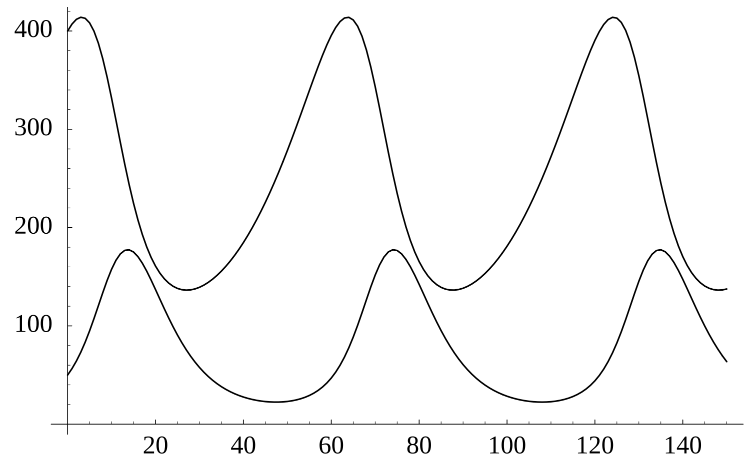

Figure 8.6.4 Predator-prey populations of Table 8.6.1

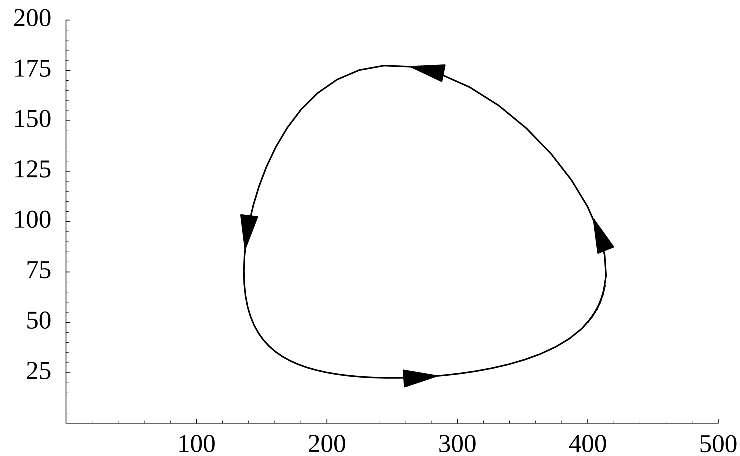

Figure 8.6.5 Phase curve for the predator-prey populations in Table 8.6.1

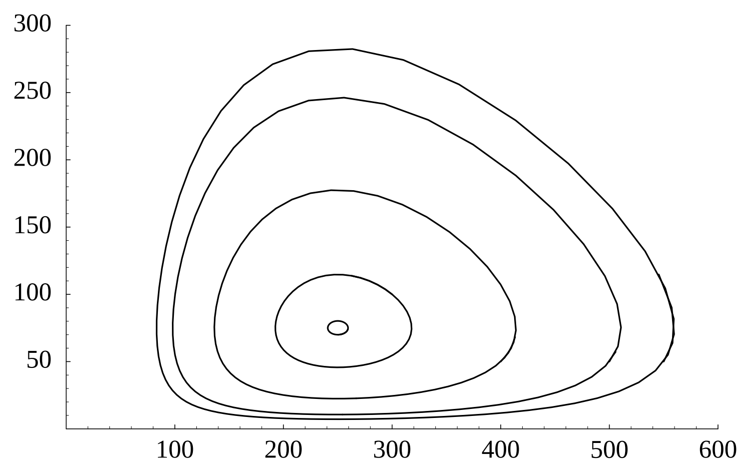

Figure 8.6.6 Phase curves for the predator-prey system (8.6.15)

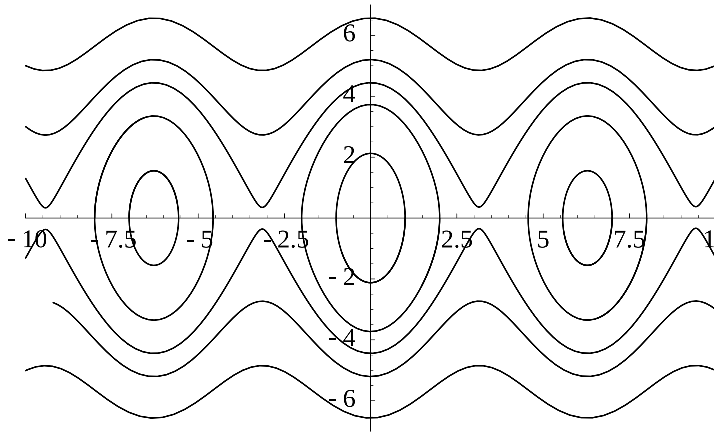

Figure 8.6.7 Phase curves for a pendulum: $\dot{x} = y$, $\dot{y} = -4.9\sin(x)$

another rod, held fixed and perpendicular to the plane of motion of the pendulum, in such a way that the pendulum is free to move through complete circles about this axis. If we let $x(t)$ be the angle between the rod and the vertical at time t, then we showed in Section 8.5 that x must satisfy the equation

$$\ddot{x} = -\frac{g}{b}\sin(x).$$

Equivalently, as a system of first order equations, we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{g}{b}\sin(x). \end{aligned} \tag{8.6.16}$$

The stationary points for this system are the points (x, y) where $y = 0$ and $\sin(x) = 0$. Hence there are an infinite number of stationary points, namely, $(n\pi, 0)$ for $n = 0, 1, 2, \ldots$. Note that for even values of n, the stationary points $(n\pi, 0)$ correspond to the pendulum hanging at rest with the bob end down. We should expect these stationary points to be centers since, without friction, any nearby initial conditions would result in the pendulum oscillating about the given stationary point. For odd values of n, the stationary points $(n\pi, 0)$ correspond to the pendulum balancing with the bob end up. We should expect that any initial condition near one of these stationary points would result in motion away from the given stationary point. That is, any slight motion away from the balanced position should cause the pendulum to fall and begin an oscillatory motion. Hence these stationary points should be unstable equilibriums. A look at the phase curves in Figure 8.6.7, shown for a pendulum of length 2 meters, supports these statements: For any integer k, $(2k\pi, 0)$ is a center and $((2k-1)\pi, 0)$ is an unstable equilibrium. We have again omitted arrows on the phase curves in Figure 8.6.7, but the direction of increasing t is from left to right above the x-axis and from right to left below the x-axis.

Problems

1. For each of the following differential equations, find the general solution and then plot the phase curves of the solutions for the given initial conditions over the given time interval I. For each equation decide whether the stationary point $(0,0)$ is a center, a stable equilibrium, or an unstable equilibrium.

   (a) $\ddot{x} + x = 0$ $\quad I = [0, 2\pi]$ $\quad$ Initial conditions:
   $x(0) = 1$, $\dot{x}(0) = 0$
   $x(0) = 2$, $\dot{x}(0) = 0$
   $x(0) = 3$, $\dot{x}(0) = 0$
   $x(0) = 4$, $\dot{x}(0) = 0$
   $x(0) = 5$, $\dot{x}(0) = 0$

   (b) $\ddot{x} + 3\dot{x} + 2x = 0$ $\quad I = [-2, 2]$ $\quad$ Initial conditions:
   $x(0) = -2$, $\dot{x}(0) = 0$
   $x(0) = -1$, $\dot{x}(0) = 0$
   $x(0) = -0.5$, $\dot{x}(0) = 0$
   $x(0) = 0.5$, $\dot{x}(0) = 0$
   $x(0) = 1$, $\dot{x}(0) = 0$
   $x(0) = 2$, $\dot{x}(0) = 0$

   (c) $\ddot{x} - x = 0$ $\quad I = [-2, 2]$ $\quad$ Initial conditions:
   $x(0) = 0$, $\dot{x}(0) = -0.2$
   $x(0) = 0$, $\dot{x}(0) = 0.2$
   $x(0) = -0.2$, $\dot{x}(0) = 0$
   $x(0) = 0.2$, $\dot{x}(0) = 0$

   (d) $\ddot{x} + 2\dot{x} + 2x = 0$ $\quad I = [-1, 5]$ $\quad$ Initial conditions:
   $x(0) = 0$, $\dot{x}(0) = 1$
   $x(0) = 0$, $\dot{x}(0) = -1$
   $x(0) = -1$, $\dot{x}(0) = 0$
   $x(0) = 1$, $\dot{x}(0) = 0$

   (e) $\ddot{x} - 2\dot{x} + 2x = 0$ $\quad I = [-5, 1]$ $\quad$ Initial conditions:
   $x(0) = 0$, $\dot{x}(0) = 1$
   $x(0) = 0$, $\dot{x}(0) = -1$
   $x(0) = -1$, $\dot{x}(0) = 0$
   $x(0) = 1$, $\dot{x}(0) = 0$

2. Plot the phase curves for the examples of overdamped and critically damped mass-spring systems given in Section 8.5.

3. Consider the equation of motion for a mass-spring system

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

   with initial conditions $x(0) = 10$ and $\dot{x}(0) = 0$.

   (a) Suppose $k = 10$ and $m = 10$. Plot the phase curves for the solutions with $c = 0$, $c = 5$, $c = 10$, $c = 20$, $c = 25$, and $c = 30$. Compare your results with your plots of $x(t)$ from Problem 3 of Section 8.5.

   (b) Suppose $m = 10$ and $c = 20$. Plot the phase curves for the solutions with $k = 2$, $k = 5$, $k = 10$, and $k = 15$. Compare your results with your plots of $x(t)$ from Problem 4 of Section 8.5.

4. For each of the following second order differential equations, write the equation as a system of first order equations and approximate the solution for the given initial conditions over the interval I using the second order Runge-Kutta method with step size h. Plot both $x(t)$ and the corresponding phase curve.

   (a) $\ddot{x} = -x^2$, $x(0) = -2$, $\dot{x}(0) = 4$, $I = [0, 3]$, $h = 0.01$

   (b) $\ddot{x} + \dot{x} = -\sin(x)$, $x(0) = -3$, $\dot{x}(0) = 2$, $I = [0, 10]$, $h = 0.02$

   (c) $\ddot{x} + x^3 = 0$, $x(0) = 2$, $\dot{x}(0) = 0$, $I = [0, 4]$, $h = 0.02$

   (d) $\ddot{x} + (x^2 - 1)\dot{x} + x = 0$, $x(0) = 0.5$, $\dot{x}(0) = 0$, $I = [0, 20]$, $h = 0.01$

   (e) $\ddot{x} - (x^2 - 1)\dot{x} + x = 0$, $x(0) = -2$, $\dot{x}(0) = 0$, $I = [0, 20]$, $h = 0.01$

   (f) $\ddot{x} + tx = 0$, $x(0) = 2$, $\dot{x}(0) = 0$, $I = [0, 10]$, $h = 0.05$

5. For each of the following systems of first order differential equations, approximate the solution for the given initial conditions over the interval I using the second order Runge-Kutta method with step size h. Plot $x(t)$, $y(t)$, and the corresponding phase curve.

   (a) $\dot{x} = 2xy$, $\dot{y} = y^2 - x^2$; $x(0) = 0.05$, $y(0) = 0.5$; $I = [0, 10]$; $h = 0.05$

   (b) $\dot{x} = x(1 - y^2)$, $\dot{y} = -y(1 - x^2)$; $x(0) = 3$, $y(0) = 2.5$; $I = [0, 10]$; $h = 0.02$

   (c) $\dot{x} = -y + x(1 - x^2 - y^2)$, $\dot{y} = -x + y(1 - x^2 - y^2)$; $x(0) = 0$, $y(0) = 4$; $I = [0, 20]$; $h = 0.04$

6. Consider the predator-prey model

$$\dot{x} = x(\alpha - \beta y)$$
$$\dot{y} = -y(\gamma - \delta x)$$

   where x is the size of the prey population, y is the size of the predator population, and α, β, γ, and δ are nonnegative constants.

   (a) Find explicit solutions for x and y if α and γ are both positive, but $\beta = \delta = 0$. Describe the behavior of the solutions in this case.

   (b) Suppose $\alpha = 0.05$, $\beta = 0.001$, $\gamma = 0.25$, and $\delta = 0.0005$. Using the initial conditions $x(0) = 700$ and $y(0) = 50$, plot x and y over an interval of time long enough to capture at least two periods (use a step size of $h = 0.05$). Plot the corresponding phase curve. What is the nonzero stationary point in this case?

   (c) For the solution found in (b), what are the maximum and minimum predator populations? What are the corresponding prey populations?

   (d) For the solution found in (b), what are the maximum and minimum prey populations? What are the corresponding predator populations?

   (e) Plot four more phase curves using the parameters specified in (b) with varying the initial conditions, plotting two inside and two outside the phase curve plotted in (b). Be sure to plot a complete cycle in each case.

7. Consider the motion of a pendulum as described by the equation

$$\ddot{x} = -\frac{g}{b}\sin(x)$$

as in the final example of the section. Use the second order Runge-Kutta method to approximate x for a pendulum of length 2 meters over the interval $[0, 10]$ using the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$ and a step size of $h = 0.05$. Graph $x(t)$ and use your results to estimate the period of $x(t)$. How does your estimate compare with the period of the linearized system

$$\ddot{x} = -\frac{g}{b}x$$

that we considered in Section 8.5?

8. For $c > 0$, consider the equation

$$\ddot{x} = -\frac{g}{b}\sin(x) - c\dot{x},$$

the equation for the motion of a pendulum of length b with a damping force proportional to its angular velocity. Suppose $b = 2$ meters and $c = 0.8$.

(a) Write this equation as a system of first order equations. What are the stationary points of this system? Which stationary points do you expect to be stable equilibriums? Which stationary points do you expect to be unstable equilibriums? Which stationary points do you expect to be centers?

(b) Plot phase curves corresponding to the initial conditions $x(0) = 0$ and, in turn, $\dot{x}(0) = -20$, $\dot{x}(0) = -15$, $\dot{x}(0) = -10$, $\dot{x}(0) = -5$, $\dot{x}(0) = 5$, $\dot{x}(0) = 10$, $\dot{x}(0) = 15$, and $\dot{x}(0) = 20$. Describe the behavior of the pendulum for each of these curves.

(c) Plot phase curves corresponding to the initial conditions $x(0) = 0$ and, in turn, $\dot{x}(0) = 6.0$, $\dot{x}(0) = 6.2$, $\dot{x}(0) = 6.4$, $\dot{x}(0) = 6.6$, $\dot{x}(0) = 6.8$, and $\dot{x}(0) = 7.0$. Describe the behavior of the pendulum for each of these curves.

(d) Do your results in (b) and (c) agree with your expectations from (a)?

9. Consider the equation

$$\ddot{x} + \alpha\dot{x} - x(1 - x^2) = 0,$$

where α is a constant.

(a) Write this equation as a system of first order equations and find all the stationary points.

(b) Let $\alpha = 1$. Plot enough phase curves to convince yourself of the proper classification of the stationary points found in (a).

(c) Let $\alpha = -1$. Plot enough phase curves to convince yourself of the proper classification of the stationary points found in (a). How do your answers compare with your results in (b)?