Difference Equations to **D**ifferential Equations

Section 7.2

The Calculus of Complex Functions

In this section we will discuss limits, continuity, differentiation, and Taylor series in the context of functions which take on complex values. Moreover, we will introduce complex extensions of a number of familiar functions. Since complex numbers behave algebraically like real numbers, most of our results and definitions will look like the analogous results for real-valued functions. We will avoid going into much detail; the complete story of the calculus of complex-valued functions is best left to a course in complex analysis. However, we will see enough of the story to enable us to make effective use of complex numbers in elementary calculations.

We begin with a definition of the limit of a sequence of complex numbers.

Definition We say that the *limit* of a sequence of complex numbers $\{z_n\}$ is L, and write

$$\lim_{n\to\infty} z_n = L,$$

if for every $\epsilon > 0$ there exists an integer N such that

$$|z_n - L| < \epsilon$$

whenever $n > N$.

Notice that the only difference between this definition and the definition of the limit of a sequence given in Section 1.2 is the use of the magnitude of a complex number in place of the absolute value of a real number. Even here, the notation is the same. The point is the same as it was in Chapter 1: the limit of the sequence $\{z_n\}$ is L if we can always ensure that the values of the sequence are within a desired distance of L by going far enough out in the sequence.

Now if $z_n = x_n + y_n i$ and $L = a + bi$, then $\lim_{n\to\infty} z_n = L$ if and only if

$$\lim_{n\to\infty} |z_n - L| = \lim_{n\to\infty} \sqrt{(x_n - a)^2 + (y_n - b)^2} = 0,$$

the latter of which occurs if and only if $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$. Hence we have the following useful result.

Proposition Let $z_n = x_n + y_n i$ and $L = a + bi$. Then

$$\lim_{n\to\infty} z_n = L$$

if and only if

$$\lim_{n\to\infty} x_n = a \text{ and } \lim_{n\to\infty} y_n = b.$$

Thus to determine the limiting behavior of a sequence $\{z_n\}$ of complex numbers, we need only consider the behavior of the two sequences of real numbers, $\{\Re(z_n)\}$ and $\{\Im(z_n)\}$.

Example Suppose

$$z_n = \frac{3n-1}{2n+2} + \frac{n+1}{n-1}i$$

for $n = 1, 2, 3, \ldots$. Then

$$\lim_{n\to\infty} \Re(z_n) = \lim_{n\to\infty} \frac{3n-1}{2n+2} = \lim_{n\to\infty} \frac{3-\frac{1}{n}}{2+\frac{2}{n}} = \frac{3}{2}$$

and

$$\lim_{n\to\infty} \Im(z_n) = \lim_{n\to\infty} \frac{n+1}{n-1} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1-\frac{1}{n}} = 1,$$

so

$$\lim_{n\to\infty} z_n = \frac{3}{2} + i.$$

Example Suppose

$$z_n = \frac{1}{n}\left(\cos\left(\frac{n\pi}{3}\right) + \sin\left(\frac{n\pi}{3}\right)i\right)$$

for $n = 1, 2, 3, \ldots$. Then

$$\lim_{n\to\infty} \Re(z_n) = \lim_{n\to\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n} = 0$$

and

$$\lim_{n\to\infty} \Im(z_n) = \lim_{n\to\infty} \frac{\sin\left(\frac{n\pi}{3}\right)}{n} = 0,$$

so

$$\lim_{n\to\infty} z_n = 0.$$

Geometrically, since $|z_n| = \frac{1}{n}$ and $\arg(z_n) = \frac{n\pi}{3}$, the points in this sequence are converging to 0 along a spiral path, as seen in Figure 7.2.1.

Having defined the limit of a sequence of complex numbers, we may define the limit of a complex-valued function, as in Section 2.3, and then define continuity, as in Section 2.4.

Definition Suppose $f : \mathbb{C} \to \mathbb{C}$, that is, f is a complex-valued function of a complex variable. We say the *limit* of $f(z)$ as z approaches a is L, written

$$\lim_{z\to a} f(z) = L,$$

if whenever $\{z_n\}$ is a sequence of points with $z_n \neq a$ for all n and $\lim_{n\to\infty} z_n = a$, then

$$\lim_{n\to\infty} f(z_n) = L.$$

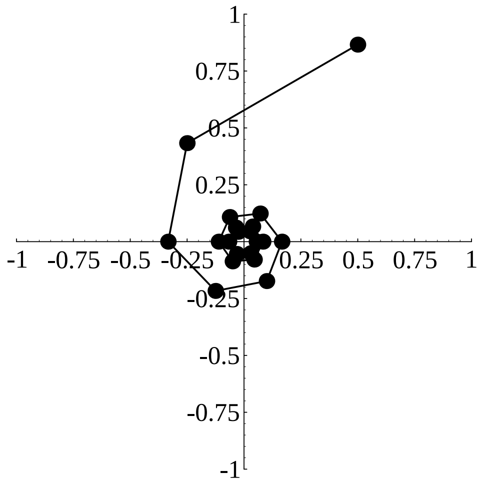

Figure 7.2.1 Plot of the points $z_n = \frac{1}{n}\left(\cos\left(\frac{n\pi}{3}\right) + \sin\left(\frac{n\pi}{3}\right)i\right)$, $n = 1, 2, 3, \ldots 20$

Definition We say the function $f : \mathbb{C} \to \mathbb{C}$ is *continuous* at a if $\lim_{z \to a} f(z) = f(a)$.

As with real-valued functions of a real variable, it is easy to show that algebraic functions of a complex variable are continuous wherever they are defined. In particular, complex polynomials, that is, functions P of the form

$$P(z) = a_n z_n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

where n is a nonnegative integer and the coefficients $a_0, a_1, \ldots, a_n$ are complex numbers, are continuous at all points in the complex plane. Complex rational functions, that is, functions R of the form

$$R(z) = \frac{P(z)}{Q(z)},$$

where both P and Q are polynomials, are continuous at all points where they are defined.

Example Since $f(z) = 3z^2 - iz + 4 - 5i$ is a polynomial, it is continuous at all points in the complex plane. In particular,

$$\lim_{z \to i} f(z) = \lim_{z \to i}(3z^2 - iz + 4 - 5i) = 3i^2 - (i)(i) + 4 - 5i = 2 - 5i.$$

Example Algebraic simplification may be useful in evaluating limits here as it was in Section 2.3. For example,

$$\lim_{z \to i} \frac{z - i}{z^2 + 1} = \lim_{z \to i} \frac{z - i}{(z - i)(z + i)} = \lim_{z \to i} \frac{1}{z + i} = \frac{1}{2i} = \frac{1}{2i}\frac{i}{i} = -\frac{1}{2}i.$$

Although this is not the time to go into any detail about the geometric meaning of the derivative of a function $f : \mathbb{C} \to \mathbb{C}$, the algebraic definition and manipulation of derivatives follows the pattern of the results for real-valued functions in Chapter 3.

Definition If $f : \mathbb{C} \to \mathbb{C}$, then the *derivative* of f at a, denoted $f'(a)$, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \tag{7.2.1}$$

provided the limit exists.

Note that h in this definition is, in general, a complex number, not just a real number. Since the algebraic properties of the complex numbers are very similar to the algebraic properties of the real numbers, much of what we learned about differentiation in Chapter 3 still holds true in our new situation. For example, if n is a nonzero rational number, then

$$\frac{d}{dz} z^n = nz^{n-1}. \tag{7.2.2}$$

Moreover, all the techniques we learned for computing derivatives in Sections 3.3 and 3.4, including the quotient, product, and chain rules, still hold.

Example If $f(z) = 3z^5 + iz^3 - (3 + 2i)z$, then

$$f'(z) = 15z^4 + 3iz^2 - 3 - 2i.$$

Example If

$$g(w) = \frac{(3+i)w^2}{2w-1},$$

then, using the quotient rule,

$$g'(w) = \frac{(2w-1)(6+2i)w - (3+i)w^2(2)}{(2w-1)^2} = \frac{(6+2i)(w^2-w)}{(2w-1)^2}.$$

From this point it is possible to follow the pattern of Chapter 5 and develop the theory of polynomial approximations using Taylor polynomials, defined in a manner analogous to the definition in Section 5.1, as well as the theory of power series and Taylor series. In particular, a power series

$$\sum_{n=0}^{\infty} a_n (z-a)^n, \tag{7.2.3}$$

where $a_0, a_1, a_2, \ldots$ and a are complex numbers, is said to *converge absolutely* at those points z for which the series

$$\sum_{n=0}^{\infty} |a_n||z-a|^n \tag{7.2.4}$$

converges. Since the latter series involves only real numbers, its convergence may be determined using the tests developed in Chapter 5. As before, absolute convergence implies convergence. Moreover, if the series (7.2.3) converges at points other than a, then there exists an R, either a positive real number or ∞, such that the series converges absolutely for all z such that $|z-a|<R$ and diverges for all z such that $|z-a|>R$. However, note that in this case the set of all points in the complex plane such that $|z-a|<R$ is a disk of radius R centered at a, not an interval as it was in the real number case.

Example Consider the power series

$$\sum_{n=0}^{\infty} z^n. \tag{7.2.5}$$

Since the series

$$\sum_{n=0}^{\infty} |z|^n$$

is a geometric series, it converges for all values of z for which $|z|<1$. Hence

$$\sum_{n=0}^{\infty} z^n$$

converges for all z for which $|z|<1$, that is, for all z inside the unit circle centered at the origin of the complex plane. Thus the radius of convergence of (7.2.5) is $R=1$. Using the same argument as we used in Section 1.3, we can show that

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

for all z with $|z|<1$. For example,

$$\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n = \frac{1}{1-\frac{i}{2}} = \frac{2}{2-i} = \frac{2(2+i)}{(2-i)(2+i)} = \frac{4}{5} + \frac{2}{5}i.$$

Example Consider the power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{7.2.6}$$

To determine its radius of convergence, we apply the ratio test to the series

$$\sum_{n=0}^{\infty} \left|\frac{z^n}{n!}\right| = \sum_{n=0}^{\infty} \frac{|z|^n}{n!}, \tag{7.2.7}$$

obtaining

$$\rho = \lim_{n\to\infty} \frac{\frac{|z|^{n+1}}{(n+1)!}}{\frac{|z|^n}{n!}} = \lim_{n\to\infty} \frac{|z|}{n+1} = 0$$

for all values of z. Since $\rho = 0$ for any value of z, (7.2.7) converges for all z in the complex plane. That is, the radius of convergence of (7.2.6) is $R = \infty$. Of course, we also know that (7.2.7) converges for all z because, from our work in Section 6.1, it is equal to $e^{|z|}$.

The power series in the last example is the extension to complex numbers of the series we used to define the exponential function in Section 6.1. With it, we can define the complex exponential function.

Definition The *complex exponential function*, with value at z denoted by $\exp(z)$, is defined for all points in the complex plane by

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{7.2.8}$$

Of course, this definition agrees with our old definition when z is real.

In Chapter 6 we used the exponential function to give meaning to exponents which were not rational numbers. Similarly, the complex exponential function may be used to define complex exponents. However, we will only consider the case of raising e to a complex power.

Definition If z is a complex number with $\Im(z) \neq 0$, then we define $e^z = \exp(z)$.

With this definition we now have $e^z = \exp(z)$ for all z in the complex plane, the case when $\Im(z) = 0$, that is, when z is real, having been treated in Section 6.1. Although we will not repeat them here, the arguments from Section 6.1 come over to establish the following proposition.

Proposition For any complex numbers w and z,

$$e^{w+z} = e^w e^z \tag{7.2.9}$$

and

$$e^{w-z} = \frac{e^w}{e^z}. \tag{7.2.10}$$

Also, as in Section 6.1, direct differentiation of (7.2.8) yields the following result.

Proposition

$$\frac{d}{dz} e^z = e^z. \tag{7.2.11}$$

Example Using the product and chain rules,

$$\frac{d}{dz}\left(z^2 e^{-z^2}\right) = z^2(-2z)e^{-z^2} + 2ze^{-z^2} = 2z(1-z^2)e^{-z^2}.$$

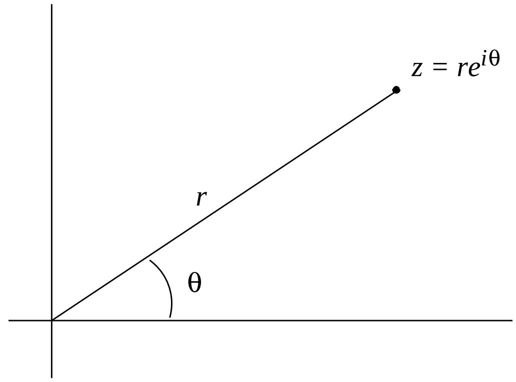

Figure 7.2.2 Plot of the point $re^{i\theta}$ in the complex plane

The exponential of a pure imaginary number is particularly interesting. To see why, let θ be a real number and consider

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

Proposition For any real number θ,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{7.2.12}$$

As a consequence, if θ is a real number, then $|e^{i\theta}| = 1$ and $\arg(e^{i\theta}) = \theta$. That is, $e^{i\theta}$ is a point in the complex plane on the unit circle centered at the origin, a distance of θ radians away, in a counterclockwise direction, along the circle from $(1, 0)$. Moreover, if z is a nonzero complex number with $|z| = r$ and $\arg(z) = \theta$, then

$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}. \tag{7.2.13}$$

This exponential notation provides a compact way to display any nonzero complex number in polar form. See Figure 7.2.2.

Example If $z = 1 - i$, then $|z| = \sqrt{2}$ and $\text{Arg}(z) = -\frac{\pi}{4}$, so

$$z = \sqrt{2}e^{-i\frac{\pi}{4}}.$$

Moreover,

$$\bar{z} = \sqrt{2}e^{i\frac{\pi}{4}}$$

and

$$z^2 = 2e^{-2i\frac{\pi}{4}} = 2e^{-i\frac{\pi}{2}} = -2i.$$

Example If $w = 3e^{i\frac{\pi}{3}}$ and $z = 5e^{i\frac{\pi}{8}}$, then

$$wz = \left(3e^{i\frac{\pi}{3}}\right)\left(5e^{i\frac{\pi}{8}}\right) = 15e^{i\left(\frac{\pi}{3}+\frac{\pi}{8}\right)} = 15e^{i\frac{11\pi}{24}}$$

and

$$\frac{w}{z} = \frac{3e^{i\frac{\pi}{3}}}{5e^{i\frac{\pi}{8}}} = \frac{3}{5}e^{i\left(\frac{\pi}{3}-\frac{\pi}{8}\right)} = \frac{3}{5}e^{i\frac{5\pi}{24}}.$$

Since for any real number θ,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

it follows that

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta).$$

Hence

$$e^{i\theta} - e^{-i\theta} = \cos(\theta) + i\sin(\theta) - (\cos(\theta) - i\sin(\theta)) = 2i\sin(\theta). \tag{7.2.14}$$

Solving (7.2.14) for $\sin(\theta)$, we have

$$\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

Similarly,

$$e^{i\theta} + e^{-i\theta} = \cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta) = 2\cos(\theta), \tag{7.2.15}$$

from which we obtain

$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}).$$

Proposition For any real number θ,

$$\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \tag{7.2.16}$$

and

$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}). \tag{7.2.17}$$

These formulas are very similar to the formulas we used to define the hyperbolic sine and cosine functions in Section 6.7. We will now use these formulas to define the complex sine and cosine functions; at the same time, we will extend the definitions of the hyperbolic sine and cosine functions. In doing so, we will see just how closely related the circular and hyperbolic trigonometric functions really are.

Definition The *complex sine function*, with value at z denoted by $\sin(z)$, and the *complex cosine function*, with value at z denoted by $\cos(z)$, are defined for all z in the complex plane by

$$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}) \tag{7.2.18}$$

and

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz}). \tag{7.2.19}$$

The *complex hyperbolic sine function*, with value at z denoted by $\sinh(z)$, and the *complex hyperbolic cosine function*, with value at z denoted by $\cosh(z)$, are defined for all z in the complex plane by

$$\sinh(z) = \frac{1}{2}(e^{z} - e^{-z}) \tag{7.2.20}$$

and

$$\cosh(z) = \frac{1}{2}(e^{z} + e^{-z}). \tag{7.2.21}$$

Note that these functions are defined so that they agree with their original versions when evaluated at real numbers.

With these definitions it is a simple matter to prove that

$$\frac{d}{dz}\sin(z) = \cos(z), \tag{7.2.22}$$

$$\frac{d}{dz}\cos(z) = -\sin(z), \tag{7.2.23}$$

$$\frac{d}{dz}\sinh(z) = \cosh(z), \tag{7.2.24}$$

and

$$\frac{d}{dz}\cosh(z) = \sinh(z). \tag{7.2.25}$$

For example,

$$\begin{aligned}
\frac{d}{dz}\cos(z) &= \frac{d}{dz}\left(\frac{1}{2}(e^{iz} + e^{-iz})\right) \\
&= \frac{1}{2}(ie^{iz} - ie^{-iz}) \\
&= \frac{i}{2}(e^{iz} - e^{-iz}) \\
&= \frac{i^2}{2i}(e^{iz} - e^{-iz}) \\
&= -\frac{1}{2}(e^{iz} - e^{-iz}) \\
&= -\sin(z).
\end{aligned}$$

Example Note that

$$\begin{aligned}\sin(i) &= \frac{1}{2i}(e^{i^2} - e^{-i^2})\\ &= \frac{1}{2i}(e^{-1} - e^{1})\\ &= -\frac{1}{i}\sinh(1)\\ &= -\frac{i}{i^2}\sinh(1)\\ &= i\sinh(1).\end{aligned}$$

Example Using the product and chain rules, we have

$$\begin{aligned}\frac{d}{dz}\sin(2z)\cos(3z) &= \sin(2z)(-\sin(3z))(3) + \cos(3z)\cos(2z)(2)\\ &= 3\sin(2z)\sin(3z) + 2\cos(2z)\cos(3z).\end{aligned}$$

The final complex-valued function we will define is the complex logarithm function. Analogous to our other definitions in this section, we would like this function to share the basic characteristic properties of the ordinary logarithm function and to agree with that function when evaluated at a positive real number. In particular, if we let $\mathrm{Log}(z)$ denote the complex logarithm of a complex number z and $\log(r)$ denote the real logarithm of a positive real number r, then for a nonzero complex number z with $|z| = r$ and $\mathrm{Arg}(z) = \theta$ we would like to have

$$\mathrm{Log}(z) = \mathrm{Log}(re^{i\theta}) = \mathrm{Log}(r) + \mathrm{Log}(e^{i\theta}) = \log(r) + i\theta. \tag{7.2.26}$$

Moreover, using (7.2.26) to define the complex logarithm function will guarantee that our new function agrees with the ordinary logarithm function when evaluated at positive real numbers, for if z is a positive real number, then $|z| = z$ and $\mathrm{Arg}(z) = 0$, giving us $\mathrm{Log}(z) = log(z)$.

Definition The *complex logarithm function*, with value at z denoted by $\mathrm{Log}(z)$, is defined for all nonzero complex numbers z with $|z| = r$ and $\mathrm{Arg}(z) = \theta$ by

$$\mathrm{Log}(z) = \log(r) + i\theta, \tag{7.2.27}$$

where $\log(r)$ is the ordinary real-valued logarithm of r.

Note that we have used the principal value of $\arg(z)$, that is, $\mathrm{Arg}(z)$, in the definition of $\mathrm{Log}(z)$ in order to give $\mathrm{Log}(z)$ a unique value. Moreover, note that this definition gives meaning to the logarithm of a negative real number, although it still does not define the logarithm of 0.

Example Since $|2 - 2i| = \sqrt{8}$ and $\mathrm{Arg}(2 - 2i) = -\frac{\pi}{4}$, we have

$$\mathrm{Log}(2 - 2i) = \log(\sqrt{8}) - \frac{\pi}{4}i = \frac{1}{2}\log(8) - \frac{\pi}{4}i = \frac{3}{2}\log(2) - \frac{\pi}{4}i.$$

Example Since $|-4| = 4$ and $\text{Arg}(-4) = \pi$, we have

$$\text{Log}(-4) = \log(4) + \pi i = 2\log(2) + \pi i.$$

Problems

1. For each of the following, find $\lim_{n\to\infty} z_n$. Also, plot $z_1, z_2, z_3, \ldots, z_{15}$ in the complex plane.

   (a) $z_n = \frac{3-n}{n} + \frac{n+1}{2n+3}i$ (b) $\frac{2-n}{n^2} - \left(4 + \frac{6}{n}\right) i$

   (c) $z_n = 3e^{i\frac{\pi}{n}}$ (d) $z_n = e^{i\frac{\pi(n-1)}{n}}$

2. Evaluate each of the following limits.

   (a) $\lim_{z\to i}(4z^3 - 6z + 3)$ (b) $\lim_{z\to 1-i}(z^2 - 3z)$

   (c) $\lim_{w\to 3i} \frac{z^2+9}{z-3i}$ (d) $\lim_{z\to i} \frac{z^4-1}{z^2+1}$

3. Find the derivative of each of the following functions.

   (a) $f(z) = 3z^2 - 6z^5 + 18i$ (b) $g(w) = \frac{13w - 6i + 3}{w+i}$

   (c) $f(z) = (z-4i)e^{-z^2}$ (d) $h(s) = (s^2+1)\exp(3s^2 - si)$

4. (a) Show that

   $$\frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-1)^n z^{2n}$$

   for all z with $|z| < 1$.

   (b) How does (a) help explain why, for real values of x, the Taylor series

   $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

   converges only on the interval $(-1, 1)$?

5. (a) If $z = x + yi$, show that

   $$\Re(e^z) = e^x \cos(y)$$

   and

   $$\Im(e^z) = e^x \sin(y).$$

   (b) If $z = x + yi$, find $|e^z|$ and $\arg(e^z)$.

6. Show that $e^{i\pi} + 1 = 0$.

7. Verify the differentiation formulas for $\sin(z)$, $\sinh(z)$, and $\cosh(z)$.
8. (a) Show that
$$\int_{-2}^{-1} \frac{1}{x}\, dx = -\log(2).$$
   (b) Some computer algebra systems evaluate the integral in (a) as
$$\int_{-2}^{-1} \frac{1}{x}\, dx = \mathrm{Log}(-1) - \mathrm{Log}(-2).$$
   Reconcile this answer with the answer in (a).
9. Let z and w be complex numbers. Verify the following two properties of the complex logarithm.
   (a) $\mathrm{Log}(wz) = \mathrm{Log}(w) + \mathrm{Log}(z)$
   (b) $\mathrm{Log}\left(\frac{w}{z}\right) = \mathrm{Log}(w) - \mathrm{Log}(z)$
10. For a positive integer n, an *nth root of unity* is a complex number z with the property that $z^n = 1$. Show that for $m = 0, 1, \ldots, n-1$,
$$z_m = e^{i\frac{2m\pi}{n}}$$
   is an nth root of unity. Plot these points in the complex plane for $n = 10$.
11. (a) Use the fact that
$$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$$
   to find the complex power series representation for $\sin(z)$.
   (b) Use the fact that
$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$$
   to find the complex power series representation for $\cos(z)$.
12. Define a complex version of the tangent function and show that
$$\tan(z) = \frac{1}{i}\left(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right).$$
13. (a) Show that $\sin(ix) = i\sinh(x)$ for every real number x.
   (b) Show that $\cos(ix) = \cosh(x)$ for every real number x.
14. Let $z = x + yi$.
   (a) Show that
$$\Re(\sin(z)) = \sin(x)\cosh(y)$$

and

$$\Im(\sin(z)) = \cos(x)\sinh(y).$$

(b) Show that

$$\Re(\cos(z)) = \cos(x)\cosh(y)$$

and

$$\Im(\sin(z)) = -\sin(x)\sinh(y).$$

15. (a) Show that for any nonzero complex number z, $e^{\mathrm{Log}(z)} = z$.

(b) If z is a nonzero complex number, does it necessarily follow that $\mathrm{Log}(e^z) = z$?