Difference Equations to Differential Equations

Section 5.1

Polynomial Approximations

In Chapter 3 we discussed the problem of finding the affine function which best approximates a given function about some point. In particular, we found that the best affine approximation to a function f at a point c is given by

$$T(x) = f'(c)(x - c) + f(c), \tag{5.1.1}$$

provided that f is differentiable at c. In this section and the next, we will extend the ideas of Sections 3.1 and 3.2 to the problem of finding polynomial approximations of any given degree to a function about some specified point. We shall see that many nonlinear functions can be approximated to any desired level of accuracy over a specified interval if we use polynomials of sufficiently high degree. As an example, compare the graphs of $f(x) = \sin(x)$ and

$$P(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362,880}x^9$$

in Figure 5.1.1. They are almost indistinguishable over the interval $[-\pi, \pi]$. In practical terms, this means there is little difference in working with $P(x)$ instead of $f(x)$ for x in $[-\pi, \pi]$. Moreover, since polynomials are the simplest of functions, involving only the arithmetic operations of addition, subtraction, and multiplication, the substitution of P for f can be a very helpful step in simplifying a problem.

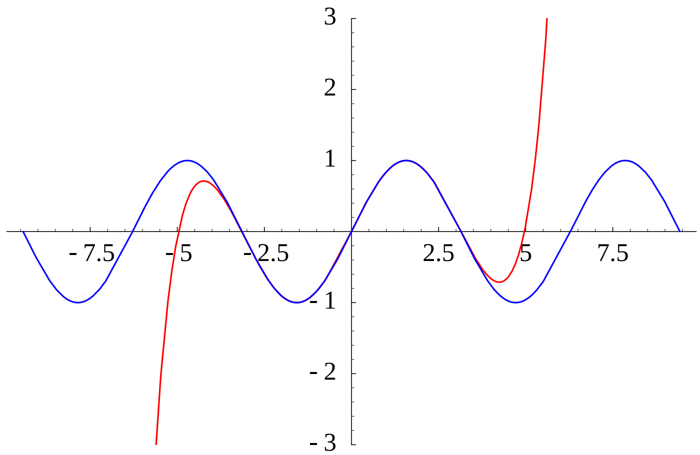

Figure 5.1.1 Graphs of $f(x) = \sin(x)$ and an approximating polynomial

To begin, we need to recall, and then generalize, some definitions and facts from Sections 3.1 and 3.2. First, recall that a function f is said to be $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0; \tag{5.1.2}$$

a function f is said to be $O(h)$ if there exist constants M and ϵ such that

$$\left| \frac{f(h)}{h} \right| \leq M \tag{5.1.3}$$

whenever $-\epsilon < h < \epsilon$. In particular, we saw that f is $O(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h}$$

exists. The following definition generalizes to other powers of h this method of characterizing the rate at which a function converges to 0.

Definition For any $n > 0$, a function f is said to be $o(h^n)$ if

$$\lim_{h \to 0} \frac{f(h)}{h^n} = 0. \tag{5.1.4}$$

For any $n > 0$, a function f is said to be $O(h^n)$ if there exist constants M and ϵ such that

$$\left| \frac{f(h)}{h^n} \right| \leq M \tag{5.1.5}$$

whenever $-\epsilon < h < \epsilon$.

Similar to our result in Section 3.1, f is $O(h^n)$ if

$$\lim_{h \to 0} \frac{f(h)}{h^n}$$

exists.

As before, we use this notation as a means of comparing the rates at which functions approach 0. As h approaches 0, a function which is $O(h^n)$ approaches 0 as least as fast as h^n does. Note that for $n > m > 0$,

$$\lim_{h \to 0} \frac{h^n}{h^m} = \lim_{h \to 0} h^{n-m} = 0 \tag{5.1.6}$$

since $n - m > 0$, and so h^n goes to 0 faster than h^m as h approaches 0. Thus if $n > m > 0$, as h goes to 0, a function which is $O(h^n)$ approaches 0 faster than does a function which

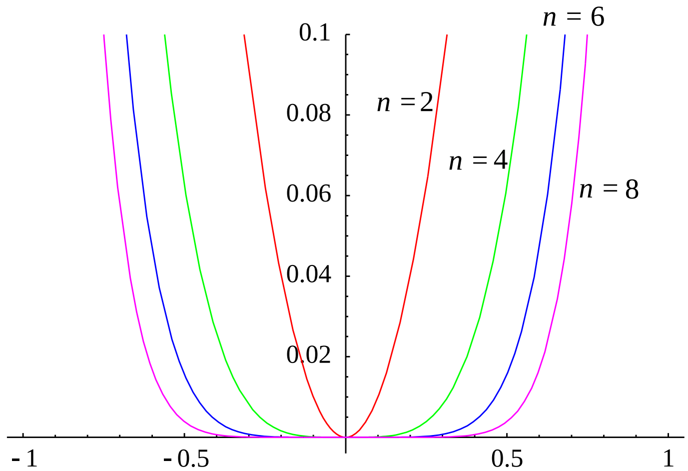

Figure 5.1.2 Graphs of $f(h) = h^n$ for $n = 2, 4, 6$, and 8

is $O(h^m)$ but not $O(h^n)$. Figure 5.1.2 illustrates this fact with the graphs of $f(h) = h^n$ for $n = 2, 4, 6$ and 8.

Example Since

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1,$$

it follows that $\sin(h)$ is $O(h)$.

Example Since

$$\lim_{h \to 0} \frac{\sin^2(h)}{h} = \lim_{h \to 0} \frac{\sin(h)}{h} \lim_{h \to 0} \sin(h) = (1)(0) = 0,$$

it follows that $\sin^2(h)$ is $o(h)$.

Example Since

$$\lim_{h \to 0} \frac{\sin^2(h)}{h^2} = \lim_{h \to 0} \frac{\sin(h)}{h} \lim_{h \to 0} \frac{\sin(h)}{h} = (1)(1) = 1,$$

it follows that $\sin^2(h)$ is $O(h^2)$.

Example Since

$$\lim_{h \to 0} \frac{\sin^3(h)}{h^3} = \lim_{h \to 0} \frac{\sin(h)}{h} \lim_{h \to 0} \frac{\sin(h)}{h} \lim_{h \to 0} \frac{\sin(h)}{h} = (1)(1)(1) = 1,$$

it follows that $\sin^3(h)$ is $O(h^3)$.

Hence, for example, we would say that as h goes to 0, $\sin^2(h)$ approaches 0 faster than h, but at about the same rate as h^2.

Now suppose that f is $O(h^n)$ for some $n > 0$. This means that as h goes to 0, f approaches 0 at least as fast as h^n does. It should follow that f goes to 0 faster than h^m, and so is $o(h^m)$, for any $0 < m < n$. To see this, let M and $\epsilon > 0$ be numbers such that

$$\left|\frac{f(h)}{h^n}\right| \leq M \tag{5.1.7}$$

for all h in the interval $(-\epsilon, \epsilon)$. Then

$$\left|\frac{f(h)}{h^m}\right| = |h^{n-m}|\left|\frac{f(h)}{h^n}\right| \leq |h|^{n-m}M \tag{5.1.8}$$

for all h in $(-\epsilon, \epsilon)$. Since

$$\lim_{h\to 0} |h|^{n-m}M = 0, \tag{5.1.9}$$

it follows that

$$\lim_{h\to 0}\left|\frac{f(h)}{h^m}\right| = 0, \tag{5.1.10}$$

and so

$$\lim_{h\to 0}\frac{f(h)}{h^m} = 0. \tag{5.1.11}$$

Thus f is $o(h^m)$.

Proposition If $n > m > 0$ and f is $O(h^n)$, then f is $o(h^m)$.

Example We saw above that $\sin^3(h)$ is $O(h^3)$, from which it now follows, for example, that $\sin^3(h)$ is $o(h^2)$.

Next, recall that if f is a function defined in an open interval about a point c and T is an affine function such that $T(c) = f(c)$ and

$$R(h) = f(c+h) - T(c+h) \tag{5.1.12}$$

is $o(h)$, then T is the best affine approximation to f at c. Moreover, as mentioned above, we saw in Chapter 3 that a function f has a best affine approximation at a point c if and only if f is differentiable at c and, in that case, the best affine approximation is given by

$$T(x) = f(c) + f'(c)(x-c). \tag{5.1.13}$$

Putting (5.1.12) and (5.1.13) together and letting $x = c + h$, or, equivalently, $h = x - c$, we have that

$$f(x) - f(c) - f'(c)(x-c)$$

is $o(x-c)$. We may express this by writing

$$f(x) - f(c) - f'(c)(x-c) = o(x-c), \tag{5.1.14}$$

or, solving for $f(x)$, simply

$$f(x) = f(c) + f'(c)(x - c) + o(x - c). \tag{5.1.15}$$

In words, (5.1.15) says that $f(x)$ is equal to $f(c) + f'(c)(x - c)$ plus some function which is $o(x - c)$, that is, some function which approaches 0 faster than $x - c$ as x approaches c.

Example Let $f(x) = \sqrt{x}$. Then

$$f'(1) = \frac{1}{2},$$

so the best affine approximation to f at 1 is

$$T(x) = 1 + \frac{1}{2}(x - 1).$$

That is,

$$\sqrt{x} = 1 + \frac{1}{2}(x - 1) + o(x - 1).$$

In words, this statement says that $\sqrt{x}$ is equal to

$$1 + \frac{1}{2}(x - 1)$$

plus a term of order higher than $x - 1$, that is, plus a term which goes to 0 faster than $x - 1$ as x approaches 1.

Example Let $f(x) = \sin(x)$. Then $f'(0) = \cos(0) = 1$, so the best affine approximation to f at 0 is

$$T(x) = x.$$

Thus

$$\sin(x) = x + o(x),$$

a fact which is often used in applications to justify replacing the function $\sin(x)$ by the function x for calculations involving only values of x close to 0.

Now suppose that f is twice continuously differentiable on an interval $(c - \delta, c + \delta)$ for some $\delta > 0$; that is, suppose both f' and f'' exist and are continuous on $(c - \delta, c + \delta)$. If T is the best affine approximation to f at c, then, as we have seen,

$$R(h) = f(c + h) - T(c + h) \tag{5.1.16}$$

is $o(h)$. We will show that R is in fact $O(h^2)$. Suppose $0 < \epsilon < \delta$ and $-\epsilon < h < \epsilon$. First, note that

$$f(c + h) - T(c + h) = f(c + h) - f(c) - f'(c)h. \tag{5.1.17}$$

By the Mean Value Theorem, there is a point u between c and $c + h$ such that

$$f(c + h) - f(c) = f'(u)h. \tag{5.1.18}$$

Hence

$$f(c+h) - T(c+h) = f'(u)h - f'(c)h = h(f'(u) - f'(c)). \tag{5.1.19}$$

Applying the Mean Value Theorem again, there exists a point v between c and u such that

$$f'(u) - f'(c) = f''(v)(u-c). \tag{5.1.20}$$

Thus

$$\frac{R(h)}{h^2} = \frac{f(c+h) - T(c+h)}{h^2} = \frac{h(u-c)f''(v)}{h^2} = \frac{(u-c)f''(v)}{h}. \tag{5.1.21}$$

If we let M be the maximum value of $|f''(x)|$ on $[c-\epsilon, c+\epsilon]$ and note that $|u-c| < |h|$, then we see that

$$\left|\frac{R(h)}{h^2}\right| = \frac{|u-c||f''(v)|}{|h|} < \frac{|h|M}{|h|} = M \tag{5.1.22}$$

for all h with $-\epsilon < h < \epsilon$. Hence $R(h)$ is $O(h^2)$.

Proposition If f is twice continuously differentiable on an open interval containing the point c and T is the best affine approximation to f at c, then

$$R(h) = f(c+h) - T(c+h) \tag{5.1.23}$$

is $O(h^2)$.

Letting $x = c + h$, we can rephrase the proposition to say that

$$r(x) = f(x) - T(x) \tag{5.1.24}$$

is $O((x-c)^2)$. Similar to our notation above, we may write

$$f(x) = f(c) + f'(c)(x-c) + O((x-c)^2). \tag{5.1.25}$$

For our previous examples, this means that

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) + O((x-1)^2)$$

and

$$\sin(x) = x + O(x^2).$$

This is the type of formulation that we wish to generalize to higher order polynomial approximations. We will introduce these polynomials, called *Taylor polynomials*, here, but save the verification that they provide the sought-for approximations until the next section.

Taylor polynomials

The best affine approximation T to a function f at a point c may be described as the only first degree polynomial satisfying both $T(c) = f(c)$ and $T'(c) = f'(c)$. This provides a clue as to where to look for higher order polynomial approximations. Namely, given a function f which is n times differentiable at a point c, we will look for a polynomial P_n of degree at most n with the property that $P_n(c) = f(c)$ and the first n derivatives of P_n at c agree with the first n derivatives of f at c. Hence we want to find constants $b_0, b_1, b_2, \ldots, b_n$ so that the polynomial

$$P_n(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + \cdots + b_n(x-c)^n \tag{5.1.26}$$

satisfies

$$P_n^{(j)}(c) = f^{(j)}(c) \tag{5.1.27}$$

for $j = 0, 1, 2, \ldots, n$, where $P_n^{(0)} = P_n$ and, for $j > 0$, $P_n^{(j)}$ is the jth derivative of P_n. Now

$$\begin{aligned} P_n(c) &= b_0 \\ P_n'(c) &= b_1 \\ P_n''(c) &= 2b_2 \\ P_n'''(c) &= (3)(2)b_3 \\ P_n^{(4)}(c) &= (4)(3)(2)b_4 \\ &\vdots \\ P_n^{(n)}(c) &= n!b_n. \end{aligned}$$

Thus, to satisfy (5.1.27), we must have

$$\begin{aligned} f(c) &= b_0 \\ f'(c) &= b_1 \\ f''(c) &= 2b_2 \\ f'''(c) &= 3!b_3 \\ f^{(4)}(c) &= 4!b_4 \\ &\vdots \\ f^{(n)}(c) &= n!b_n. \end{aligned}$$

Solving for $b_0, b_1, b_2, \ldots, b_n$, we have

$$\begin{aligned} b_0 &= f(c) \\ b_1 &= f'(c) \\ b_2 &= \frac{f''(c)}{2} \end{aligned}$$

$$b_3 = \frac{f'''(c)}{3!}$$
$$b_4 = \frac{f^{(4)}(c)}{4!}$$
$$\vdots$$
$$b_n = \frac{f^{(n)}(c)}{n!}.$$

That is,

$$b_j = \frac{f^{(j)}(c)}{j!} \tag{5.1.28}$$

for $j = 0, 1, 2, \ldots, n$. The resulting polynomial is named after Brook Taylor (1685-1731), an English mathematician who was the first to publish work on the related infinite series that we will consider later in this chapter.

Definition Suppose f is n times differentiable at a point c. Then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the *Taylor polynomial of order* n for f at c.

Example Consider $f(x) = \sin(x)$ and $c = 0$. Then

$$f'(x) = \cos(x)$$
$$f''(x) = -\sin(x)$$
$$f'''(x) = -\cos(x)$$
$$f''''(x) = \sin(x).$$

Notice that, if we were to continue finding higher derivatives, this cycle would repeat itself. Evaluating the function and its derivatives at 0, we obtain

$$f(0) = 0$$
$$f'(0) = 1$$
$$f''(0) = 0$$
$$f'''(0) = -1$$
$$f''''(0) = 0,$$

a cycle which would repeat itself if we were to continue evaluating higher-order derivatives.

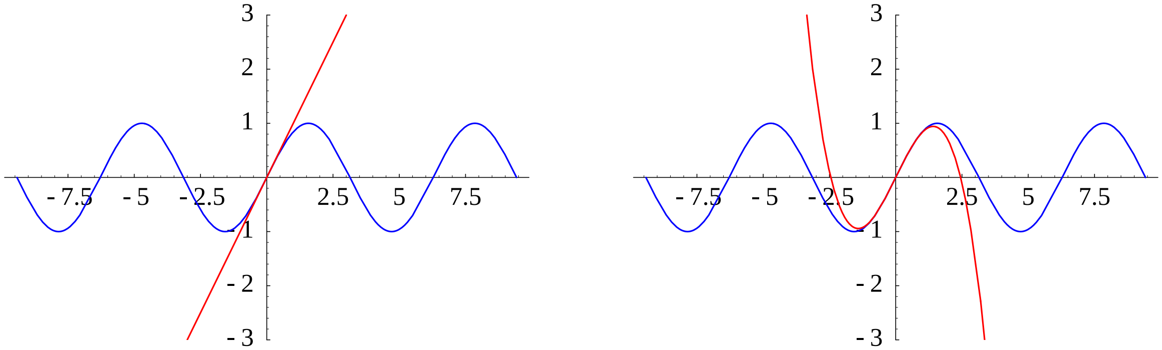

Figure 5.1.3 Graphs of $f(x) = \sin(x)$ with Taylor polynomials P_1 (left) and P_3 (right)

Thus we obtain the following Taylor polynomials for $\sin(x)$ at $x = 0$:

$$\begin{aligned}
P_1(x) &= x \\
P_2(x) &= x \\
P_3(x) &= x - \frac{x^3}{3!} \\
P_4(x) &= x - \frac{x^3}{3!} \\
P_5(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\
P_6(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\
P_7(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
P_8(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
P_9(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}.
\end{aligned}$$

The graphs of P_1, P_3, P_5, and P_7, along with the graph of f, are shown in Figures 5.1.3 and 5.1.4. We have already seen the graph of P_9 in Figure 5.1.1. Notice how the Taylor polynomials give increasingly better approximations to $\sin(x)$ as the order increases. Finally, since the values of the derivatives repeat the pattern 0, 1, 0, endlessly, in this case we can write down a simple general expression for the Taylor polynomial of any order. Namely, for any integer $n \geq 0$,

$$P_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

and $P_{2n+2}(x) = P_{2n+1}(x)$, the latter following from the fact that all the even-order derivatives are 0.

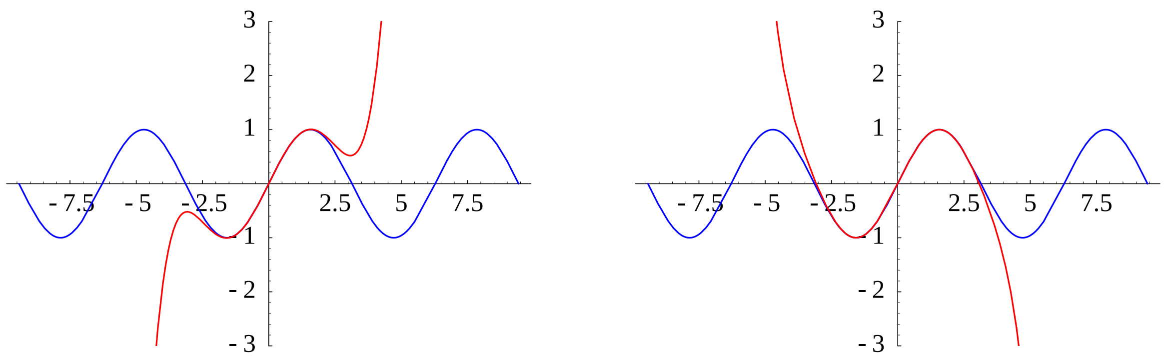

Figure 5.1.4 Graphs of $f(x) = \sin(x)$ with Taylor polynomials P_5 (left) and P_7 (right)

Example Now we will find the Taylor polynomial of order 4 for $g(x) = \sqrt{x}$ at $x = 1$. First we find that

$$\begin{aligned} g'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\ g''(x) &= -\frac{1}{4}x^{-\frac{3}{2}} \\ g'''(x) &= \frac{3}{8}x^{-\frac{5}{2}} \\ g''''(x) &= -\frac{15}{16}x^{-\frac{7}{2}}. \end{aligned}$$

Hence

$$\begin{aligned} g(1) &= 1 \\ g'(1) &= \frac{1}{2} \\ g''(1) &= -\frac{1}{4} \\ g'''(1) &= \frac{3}{8} \\ g''''(1) &= -\frac{15}{16}. \end{aligned}$$

Thus we have

$$\begin{aligned} P_4(x) &= 1 + \frac{1}{2}(x-1) - \frac{\frac{1}{4}}{2}(x-1)^2 + \frac{\frac{3}{8}}{3!}(x-1)^3 - \frac{\frac{15}{16}}{4!}(x-1)^4 \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4. \end{aligned}$$

The graphs of P_4 and g are shown in Figure 5.1.5. As we hoped, $P_4(x)$ provides a good approximation to $\sqrt{x}$ for values of x close to 1. For example, to 8 decimal places,

$$P_4(1.1) = 1.04880859,$$

while

$$\sqrt{1.1} = 1.04880884.$$

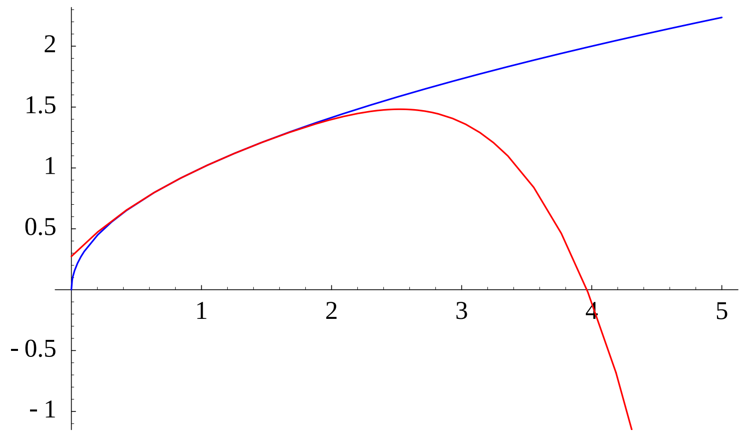

Figure 5.1.5 Graphs of $g(x) = \sqrt{x}$ with Taylor polynomial of order 4

As we should expect, the approximation worsens for x farther away from 1. For example, to 3 decimal places,

$$P_4(2) = 1.398,$$

while

$$\sqrt{2} = 1.414.$$

Although finding a Taylor polynomial of order n for a given function f involves only evaluating the derivatives of f at a specified point, nevertheless, the required computations may become unwieldy, especially if f is itself complicated or n is large. In such cases, a computer algebra system may prove useful. For example, you may find a computer algebra system helpful in working Problems 10 through 12.

In the next section we will see that the Taylor polynomials provide polynomial approximations that generalize best affine approximations. That is, we shall show that, under suitable conditions, if P_n is the Taylor polynomial of order n for f at c, then the remainder function

$$R(h) = f(c+h) - P_n(c+h) \tag{5.1.29}$$

is $O(h^{n+1})$, in agreement with our previous result that the remainder function for the best affine approximation, P_1, is $O(h^2)$.

Problems

1. Show that $f(x) = \tan^2(x)$ is $o(h)$.
2. Show that $g(x) = \tan^2(x)$ is $O(h^2)$.
3. Show that $f(z) = z^2\sin(z)$ is $o(h^2)$ and $O(h^3)$.
4. Show that $h(t) = 1 - \cos(t)$ is $O(h^2)$.
5. Show that $f(x) = \sin^2(3x)$ is $O(h^2)$.

6. Show that $f(x) = x^{\frac{4}{3}}$ is $o(h)$, but not $O(h^2)$.
7. For each of the following functions, find the Taylor polynomial of order 4 at the given point c.
   (a) $f(x) = \sin(2x)$ at $c = 0$ (b) $g(x) = \cos(x)$ at $c = 0$
   (c) $f(z) = \sqrt{z}$ at $c = 4$ (d) $f(\theta) = \tan(\theta)$ at $\theta = 0$
   (e) $h(x) = \sin(x)$ at $c = \pi$ (f) $g(t) = \cos(2t)$ at $c = \pi$
   (g) $f(x) = \dfrac{1}{x}$ at $c = 1$ (h) $f(x) = 3x^2 + 2x - 9$ at $c = 0$
   (i) $g(x) = \dfrac{1}{1 + x^2}$ at $c = 0$ (j) $h(x) = x^4 + 5x^3 + 4x^2 - 9x - 20$ at $c = 0$
   (k) $g(t) = \dfrac{1}{t^2}$ at $c = 1$ (l) $h(z) = 8z^5 - 3z^3 + 6z$ at $c = 0$
   (m) $x(t) = \sec(t)$ at $c = 0$ (n) $f(x) = 3x^4 - 4x^3 + x^2 - 3x - 2$ at $c = 1$
8. Let P_9 be the 9th order Taylor polynomial for $f(x) = \cos(x)$ at 0. Graph f and P_9 on the same axes. On what interval does $P_9(x)$ appear to give a good approximation to $\cos(x)$?
9. Let P_{13} be the 13th order Taylor polynomial for $f(x) = \sin(x)$ at 0. Graph f and P_{13} on the same axes. On what interval does $P_{13}(x)$ appear to give a good approximation to $\sin(x)$?
10. Let P_n be the nth order Taylor polynomial for $f(x) = \dfrac{1}{x^2 + 1}$ at 0.
    (a) Graph f and P_6 on the same axes. On what interval does $P_6(x)$ appear to give a good approximation to $f(x)$?
    (b) Repeat part (a) for P_{10} and P_{20}.
    (c) Do any of the polynomials in parts (a) and (b) appear to give a good approximation to f on the interval $[1, 2]$?
11. Let P_6 be the 6th order Taylor polynomial for $x(t) = \tan(t)$ at 0. Graph x and P_6 on the same axes and comment.
12. Let P_{15} be the 15th order Taylor polynomial for $g(x) = \sqrt{x}$ at 1. Graph g and P_{15} on the same axes. On what interval does $P_{15}(x)$ appear to give a good approximation to $\sqrt{x}$?