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Commonality in Liquidity:
Transmission of Liquidity Shocks across Investors and Securities*

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Abstract

Recent findings of common factors in liquidity raise many issues pertaining to the determinants of commonality and its impact on asset prices. We explore some of these issues using a model of liquidity trading in which liquidity shocks are decomposed into common (systematic) and idiosyncratic components. We show that common liquidity shocks do not give rise to commonality in trading volume, raising questions about the sources of commonality that is detected in the literature. Indeed, trading volume is independent of systematic liquidity risk, which is always priced independently of the liquidity in the secondary market. In contrast, idiosyncratic liquidity shocks create liquidity demand and volume, and investors can diversify their risk by trading. Hence, the pricing of the risk of idiosyncratic liquidity shocks depends on the market's liquidity, with idiosyncratic liquidity risk being fully priced only in perfectly illiquid markets. While trading volume is increasing in the variance of idiosyncratic liquidity shocks, price volatility is increasing in the variance of both systematic liquidity shocks and idiosyncratic liquidity shocks. Surprisingly, our results are largely independent of the number of different securities traded in the market. When asset returns are uncorrelated, there is no transmission of liquidity across assets even when investors experience common (systematic) liquidity shocks, suggesting that such liquidity shocks may not be the source of commonality in liquidity across assets detected in the literature. However, under limited conditions, more liquid securities can act as substitutes for less liquid securities. Overall, our findings suggest that common factors in liquidity may be the outcome of covariation in investor heterogeneity (e.g. as measured by co-movements in the volatility of idiosyncratic liquidity shocks) rather than of common liquidity shocks. Moreover, we find that different liquidity proxies measure different things, which has implications for future empirical analysis.

1. INTRODUCTION

With the proliferation of financial securities and the markets in which they trade, considerable attention has been focused on the role of liquidity in financial markets. While the traditional focus of research in this area has been on the liquidity of individual securities, recent studies have detected common factors in prices, trading volume, and transactions-cost measures such as bid-ask spreads.[1] These findings highlight the importance of understanding the mechanics by which liquidity demand and supply is transmitted across investors and securities. Chordia, Roll and Subrahmanyam (2000) note that drivers of common factors in liquidity may be related to market crashes and other market incidents, pointing to recent incidents such as the Summer 1998 collapse of the global bond market and the October 1987 stock market collapse which did not seem to be accompanied by any significant news. They also identify as an important area of future research the question of whether and to what extent common factors in liquidity affect asset prices.

This paper develops a model aimed at exploring some of the issues pertaining to the determinants of commonality and its impact on asset prices. Our model follows the basic intuition provided by Karpoff (1986), who characterizes non-informational trading as the outcome of differences in personal valuation of assets by investors, due to their differential liquidity needs. In our model, liquidity shocks which cause investors to revise their personal valuations can have both systematic (i.e. common across all investors) and idiosyncratic components. This formulation permits us to examine the transmission of liquidity shocks across

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[1] See, for example, Tkac (1999), Chordia, Roll and Subrahmanyam (2000), Gibson and Mougeot (2000), Lo and Wang (2000), Huberman and Halka (2001), and Hasbrouck and Seppi (2001).

assets and across the investor base of individual assets. Indeed, our analysis highlights the importance of variations in liquidity demand across investors as a crucial determinant of the liquidity of assets they hold.

Common factors in liquidity seem to imply that liquidity shocks apply systematically across investors, and are transmitted across investors and/or securities causing market-wide effects. We show that systematic and idiosyncratic liquidity shocks have significantly different effects on asset prices, trading volume and volatility. The demand for liquidity arises from investor heterogeneity caused by idiosyncratic liquidity shocks, and is manifested in trading volume. Contingent upon the state of liquidity in the market, trading volume increases with the intensity of idiosyncratic liquidity shocks (measured by their variance). In contrast, systematic liquidity shocks do not give rise to a demand for liquidity or affect trading volume, although they have a significant impact on price volatility. The risk of systematic liquidity shocks is always priced and is independent of the state of liquidity in the secondary market, since investors are unable to diversify this risk by trading.[2] The price volatility associated with systematic liquidity shocks is also not contingent upon the state of liquidity in the market. Indeed, as in Milgrom and Stokey (1982), systematic liquidity shocks will not induce trading even if the market is liquid. In contrast, the state of liquidity in the market is very important in the case of idiosyncratic liquidity shocks. Since investors are differentially impacted by the shocks, they can be transmitted across the investor base by trading, to the benefit of all investors. Hence, investors will seek to exploit the benefits of trading if the market is liquid and the state of liquidity in the market will

[2] Gibson and Mougeot (2000) confirm that systematic market liquidity is priced in the US stock market.

determine the extent to which the risk of idiosyncratic liquidity shocks is incorporated in the price.

These results suggest the importance of carefully differentiating between systematic and idiosyncratic liquidity drivers when using standard liquidity measures as proxies for liquidity. They also raise questions about the sources of commonality in liquidity detected in the literature. It is especially interesting to observe that systematic liquidity shocks do not cause co-movement in volume. Idiosyncratic liquidity shocks are the principal determinant of volume, which expands as the intensity of these shocks increases. Commonality in the context of recent findings in the literature of covariation in volume suggests the existence of covariation in investor heterogeneity, as measured, for example, by co-movements in the volatility of idiosyncratic liquidity shocks experienced by investors. The tax cycle is one potential source of such covariation although as conjectured by Chordia, Roll and Subrahmanyam (2000), behavioral factors may also be at work. Huberman and Halka (2001) conjecture that commonality emerges due to noise traders, which is consistent with our model if the volatility of idiosyncratic liquidity shocks is considered as a proxy for the level of noise in the market.

We provide new insights into the pricing of illiquidity. Amihud and Mendelson (1986) empirically demonstrate that asset returns are increasing in the cost of transacting (bid-ask spread) and hypothesize that in equilibrium, assets with higher bid-ask spreads will be held by investors with longer investment time horizons. Brennan and Subrahmanyam (1996) also find a significant relationship between required rates of return and measures of illiquidity, after adjusting for the Fama and French risk factors and the stock price level. However, Eleswarapu and Reinganum (1993) find a significant liquidity premium only in January. As noted by

Brennan and Subrahmanyam (1996), these differences may be due in part to the noisiness of transactions cost measures. However, as our analysis suggests, different liquidity variables measure different things, which may also be a confounding factor in empirical analysis. Moreover, whereas the traditional focus has been on factors related to the supply of liquidity, we show that liquidity is the outcome of both demand and supply factors, with the demand side having a much more significant and varied impact than previously thought to be the case in the literature. When investors have differences in liquidity demand due to differences in their exposure to liquidity shocks, we show that investors with lower exposure to liquidity shocks will supply liquidity to investors with higher exposure, and benefit from a higher risk-adjusted return for doing so. Thus, in addition to receiving higher returns by holding less liquid assets (as in Amihud and Mendelson (1986)) low-exposure investors will also receive a higher risk-adjusted return than high-exposure investors from the assets that they hold in common.

Surprisingly, our results are largely independent of the number of different securities traded in the market. With multiple securities, systematic liquidity shocks continue to be fully priced, since they are, by definition, perfectly correlated across investors, making them impossible to diversify by trading. This would be the case even if these shocks were not common across assets. In contrast, idiosyncratic liquidity shocks are priced only if they cannot be mutualized by trading. Even if idiosyncratic liquidity shocks were common across assets while being idiosyncratic across investors, there will be no transmission across assets as long as all assets can be freely traded. The only case in which one asset can be a "liquidity substitute" for another asset is if liquidity shocks on one asset can be better mitigated by trading another asset, which would arise if there were significant liquidity differences between the assets, all else equal. In such cases, the market price of liquid substitutes can be used to benchmark the value of illiquid securities. Indeed, in the extreme case when perfectly liquid but otherwise identical substitutes exist for illiquid securities, the price discount due to illiquidity should be zero in the absence of short-sale constraints. The magnitude of the discounts observed empirically suggests that the unavailability of liquid substitutes and/or short sale restrictions may be significant impediments to hedging the liquidity risk of illiquid securities in this way.

The rest of the paper is organized as follows. In the next section, we develop the benchmark model of our paper. In Section 3, we examine the transmission of liquidity across investors, and study the differential effects of systematic and idiosyncratic liquidity shocks on asset prices, trading volume and price volatility. In Section 4, we extend the analysis to the case of multiple securities to examine liquidity transmission across securities, and study cases in which liquid securities can act as substitutes for their illiquid counterparts. Section 5 concludes.

2. THE MODEL

We consider a two-period, three-date economy with a group of *M* risk-averse investors. We assume that each agent is endowed at time 0 with 1 unit of a single risky asset and 1 unit of the riskless asset. The risky asset pays off a random quantity of the numeraire riskless asset, $\tilde{v}$, at time 2, where $E(\tilde{v}) > 1$. The return, $\tilde{v}$, is common knowledge, and is distributed normally with mean $\bar{v}$ and variance σ_v^2. The risk-free return is assumed to be zero. Investors maximize negative exponential utility functions of their wealth at time 2, W_2: $U(W_2) = -\exp(-aW_2)$, where $a \geq 0$ is the coefficient of risk aversion.

All investors experience liquidity shocks at time 1, with the distribution of these shocks being known *ex ante* at time 0. These liquidity shocks can arise due to a broad range of events that give rise to a change in the investor's marginal valuation of the risky asset without new information about the fundamental value of the security. Following Karpoff (1986), Michaely and Vila (1995), and Michaely, Vila and Wang (1996), we characterize this shock as a random additive change, $\tilde{\theta}_i$, to investor *i*'s valuation of the payoff $\tilde{v}$ from the risky asset.[3] $\tilde{\theta}_i$ is also distributed normally with mean 0 and variance σ_θ^2, and is independent of $\tilde{v}$.

In our model, liquidity shocks can change each investor's demand for the risky asset, and induce trading when it is rational and feasible for an investor to do so. Unlike in Grossman and Stiglitz (1980), where the magnitude of liquidity trades is specified exogenously, liquidity trading is discretionary in our model since investors have the ability to rationally determine the size of their trades after taking account of all the costs and benefits of rebalancing their portfolios.

We assume that in general, liquidity shocks can be decomposed into normally distributed systematic and idiosyncratic components:

$$\tilde{\theta}_i = \gamma_i \tilde{\delta} + \tilde{\varepsilon}_i \tag{1}$$

[3] In general, liquidity shocks can be caused by changes in preferences (Tobin (1965), and Diamond and Dybvig (1983)), changes in endowments (Glosten (1989), Madhavan (1992), Bhattacharya and Spiegel (1991), and Spiegel and Subrahmanyam (1992)), or changes in personal valuations due to taxes and other non-informational reasons (Karpoff (1986), Michaely and Vila (1995), and Michaely, Vila and Wang (1996)), that change each investor's marginal valuation of the security without affecting its fundamental return. We use the latter formulation to preserve tractability.

where $\tilde{\delta}$, the systematic component, is perfectly correlated across all investors, whereas $\tilde{\varepsilon}_i$, the idiosyncratic component, is assumed to be identically and independently distributed (i.i.d.) across investors. $\tilde{\delta}$ is normally distributed with a mean of 0 and a variance of σ_δ^2 while $\tilde{\varepsilon}_i$ is normally distributed with a mean of 0 and a variance of σ_ε^2. $\gamma_i \geq 0$ measures investor *i*'s exposure to the systematic liquidity shock.[4]

Liquidity shocks affect investors' marginal valuation of the risky asset and lead them to optimally rebalance their portfolios by trading shares in the risky asset when this is possible. There are no restrictions on short holdings of the risky asset.

We assume that trading in the secondary market at time 1 occurs in a simple batch market where all trades clear at the same price subject to transactions costs. For tractability, we assume a transactions cost formulation that is commonly used in the literature:[5]

$$P_{1i} = P_1 + \lambda \Delta X_{1i} \tag{2}$$

$\lambda \geq 0$ is the transactions cost parameter, P_1 is the market-clearing price in the absence of transactions costs, ΔX_{1i} is the trade size of individual *i*, and P_{1i} is the actual price paid or received by individual *i*.

In general, the portfolio selection problem of individual *i* may be expressed as:

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[4] We are grateful to a referee for suggesting this formulation.

[5] See, for example, Kyle (1985), and Brennan and Subrahmanyam (1996). The market microstructure that gives rise to transactions costs is assumed to be exogenous to the model.

$$\max E\left[U(W_{2i})\right]; \qquad i \in M$$

$$\begin{aligned} s.t. \qquad & W_{2i} = W_{1i} + X_{1i}(\tilde{v} + \tilde{\theta}_i - \tilde{P}_1); \\ & W_{1i} = W_{0i} + X_{0i}(\tilde{P}_1 - \tilde{P}_0) - TC_i. \end{aligned} \tag{3}$$

where W_{ti} = wealth of individual i at the end of time t;

P_t = price of risky asset at time t (in units of the riskless asset);

X_{ti} = amount of risky asset held by individual i at the end of time t; and

TC_i = transactions cost ($\lambda \Delta X_{1i}^2$) incurred by individual i in rebalancing time 1 portfolio.

Given our assumption of negative exponential utility, (3) can be stated as:

$$\max_{X_{0i}} E_0\left[\max_{X_{1i}} E_1\left[-\exp\left\{-a\left[W_{0i} + X_{0i}(\tilde{P}_1 - P_0) + \tilde{X}_{1i}(\tilde{v} + \tilde{\theta}_i - \tilde{P}_1) - \lambda(\tilde{X}_{1i} - X_{0i})^2\right]\right\}\right]\right] \tag{4}$$

Individuals solve this portfolio problem recursively. In the rest of the paper, we use this model to examine how liquidity shocks affect an investor's portfolio selection decision, and study the implications for liquidity transmission across investors and securities in order to better understand the causes and consequences of commonality in liquidity.

3. TRANSMISSION OF LIQUIDITY ACROSS INVESTORS

In this section we examine how systematic and idiosyncratic liquidity shocks affect the transmission of liquidity across investors, and the impact they have on overall market liquidity and asset prices. We also analyze the implications for trading volume and price volatility in order to link our results to the existing literature on commonality in liquidity. We begin by presenting the general case in which investors are affected by liquidity shocks consisting of heterogeneous systematic and idiosyncratic components. Thereafter, we examine special cases to derive closed-form solutions and to strengthen the insights provided by our model.

3.1 Asset Pricing and Liquidity Transmission across Investors

The general case where investor *i* experiences a liquidity shock of $\tilde{\theta}_i$ as specified by (1) gives rise to both *ex ante* and *ex post* differences across investors due to liquidity shocks. The *ex ante* differences arise because investors have different exposures (γ_i) to systematic liquidity shocks. The *ex post* differences arise because of the differences across investors in the realization of idiosyncratic liquidity shocks. Thus, as in Amihud and Mendelson (1986), investors will make their time 0 portfolio decisions not only by rationally anticipating their time 1 liquidity needs but also by taking account of the currently known differences across the investor base. Since the effect of differences in γ_i across the investor base is to create differences in the incidence of systematic liquidity shocks, this will cause investors who are less impacted by systematic shocks (possibly because of portfolio composition or hedging strategies exogenous to the model) to benefit by providing liquidity to those investors who are more impacted by systematic shocks. Lemma 1 summarizes the key results for the time 1 equilibrium.

Lemma 1. *At time 1, the market clearing price,* P_1, *and the equilibrium holding of the risky asset by investor i,* X_{1i} *are respectively:*

$$
\begin{aligned}
P_1 &= \overline{v} + \gamma_A \hat{\delta} + \hat{\varepsilon}_A - a\sigma_v^2 \\
X_{1i} &= \frac{(\gamma_i - \gamma_A)\hat{\delta} + (\hat{\varepsilon}_i - \hat{\varepsilon}_A) + a\sigma_v^2 + 2\lambda X_{0i}}{a\sigma_v^2 + 2\lambda}
\end{aligned}
\tag{5}
$$

where $\hat{\delta}$ *and* $\hat{\varepsilon}_i$ *denote the realizations of* $\tilde{\delta}$ *and* $\tilde{\varepsilon}_i$, *respectively, and* $\gamma_A = \frac{\sum_{i=1}^{M} \gamma_i}{M}$, $\hat{\varepsilon}_A = \frac{\sum_{j=1}^{M} \hat{\varepsilon}_j}{M}$ *are the average exposure to systematic liquidity shocks and the average incidence of idiosyncratic liquidity shocks, respectively, across the investor base.*

Proof. See Appendix.

While all investors experience systematic liquidity shocks, only $\gamma_A \hat{\delta}$, the average systematic shock (which represents the undiversifiable component) is reflected in the price. The "idiosyncratic" component of the systematic liquidity shock experienced by investor $i, (\gamma_i - \gamma_A)\hat{\delta}$, is mutualized by trading at time 1, as reflected in the expression for X_{1i}. It would be noted that in this sense, $(\gamma_i - \gamma_A)\hat{\delta}$ manifests itself identically to the idiosyncratic shock experienced by investor $i, \hat{\varepsilon}_i$.[6] Thus, differences in exposure to systematic liquidity shocks alleviate the impact of these shocks and lead to partial risk sharing through trading between high and low-exposure investors. We explore this risk-sharing in more detail later.

Although transactions costs do not affect the equilibrium price at time 1, they have an impact on trading volume. We examine the impact of liquidity shocks on price volatility and trading volume in Subsection 3.2 under different assumptions about transactions costs.

In order to solve for the equilibrium at time 0, we need to make a specific assumption about the distribution of γ_i across the investor base. As we noted previously, systematic liquidity shocks in the general case can be divided into uniform (undiversifiable) and idiosyncratic components. Since the latter component is already captured in our formulation through $\tilde{\varepsilon}_i$, we lose little generality by assuming that systematic shocks consist only of the average component in the previous formulation, $\gamma_A \hat{\delta}$. Specifically, we assume that $\gamma_i = \gamma_A = 1$.[7] This assumption makes all investors *ex ante* identical at t = 0.[8]

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[6] For finite *M*, idiosyncratic shocks also contain an undiversifiable component, $\hat{\varepsilon}_A$. Unlike $\gamma_A \hat{\delta}$, however, $\hat{\varepsilon}_A \to 0$ as $M \to \infty$.

[7] Alternatively, we could assume a specific distribution for γ_i that is non-uniform across investors but this also reduces to the current case of a uniform systematic component and an idiosyncratic component. *Ex ante* heterogeneity across investors, the feature in the general model that we lose by this assumption, is explored more fully in Subsection 3.3 in the context of liquidity risk-sharing across investors.

[8] It will be observed that in the case where γ_i is equal across all investors, the systematic component, $\tilde{\delta}$, manifests itself identically to a shock to the fundamental payoff of the risky asset. The same effect will result if investors experience a systematic shock to their endowments or preferences. A systematic liquidity shock that impacts all investors with equal intensity, regardless of how it originates, causes all investors to revise their valuation of the risky asset identically. In this sense, a systematic liquidity shock is "fundamental," making it more difficult to empirically differentiate it from a shock to the asset's fundamental returns. This difficulty, which persists with all formulations of systematic liquidity shocks, does not detract from the importance of understanding the consequences of such shocks, and differentiating their effect from that of idiosyncratic liquidity shocks.

Noting that $X_{0i} = 1$, Lemma 2 states the result for the time 0 equilibrium price.

Lemma 2. *The equilibrium price at time 0, P_0, is:*

$$P_0 = \overline{v} - a\sigma_v^2 - a\sigma_\delta^2 - \frac{\sigma_\varepsilon^2}{M} - \frac{2\lambda a\sigma_\varepsilon^2\left(\frac{M-1}{M}\right)}{a\sigma_v^2 + 2\lambda + a\sigma_\varepsilon^2\left(\frac{M-1}{M}\right)} \tag{6}$$

Proof. See Appendix.

In the case considered here where all investors are *ex ante* identical, they will hold their initial endowments in equilibrium at time 0, in contrast to time 1 when idiosyncratic liquidity shocks are realized. By trading with each other, the idiosyncratic liquidity shocks are transmitted across investors as the rational response to the valuation changes caused by the shocks.

The price at time 0 incorporates a discount for the liquidity risk that investors face at time 1, given by

$$\Phi = a\sigma_\delta^2 + \frac{\sigma_\varepsilon^2}{M} + \frac{2\lambda a\sigma_\varepsilon^2\left(\frac{M-1}{M}\right)}{a\sigma_v^2 + 2\lambda + a\sigma_\varepsilon^2\left(\frac{M-1}{M}\right)} \tag{7}$$

The transactions cost parameter, λ, is a proxy for the external factors that determine market liquidity at time 1, and parameterizes the liquidity continuum between the case in which the time 1 market is frictionless, when $\lambda = 0$, and the case in which it is *perfectly illiquid* (*de facto* closed), when $\lambda \to \infty$. Conditional on a given distribution of liquidity shocks, the size of the market as measured by the number of investors, M, also determines its liquidity. This can be seen by examining the limiting case of $M = 1$ in which the market will, by definition, be perfectly illiquid. A frictionless market in which $M \to \infty$ can be thought of as a *perfectly liquid* market.

For a given value of $M > 1$, the time 0 equilibrium price P_0 decreases monotonically with the transactions cost parameter λ. This price decline reflects the corresponding increase in the discount for illiquidity, Φ, as the cost of trading in the secondary market rises.

Our principal conclusions on the pricing of liquidity risk follow directly from Lemmas 1 and 2, and are stated in Proposition 1.

Proposition 1 *(Pricing of Liquidity Risk). The pricing of idiosyncratic liquidity risk is contingent upon the state of liquidity in the market, whereas systematic liquidity risk is always priced and is independent of the state of liquidity in the secondary market. The systematic liquidity risk premium* $a\sigma_\delta^2$ *is increasing in the variance of systematic liquidity shocks. Idiosyncratic liquidity risk is fully priced only if the secondary market is perfectly illiquid, and unpriced if the secondary market is perfectly liquid. The idiosyncratic liquidity risk premium* $a\sigma_\varepsilon^2$ *is increasing in the variance of idiosyncratic liquidity shocks.*

The result for systematic liquidity risk parallels the no-trade equilibrium in Milgrom and Stokey (1982). If liquidity shocks are common to all investors, they cannot be diversified away by trading, nor will they induce trading even if the market is liquid. At time 1, the price will simply adjust without trading to reflect the systematic liquidity shock and at time 0, the risk of the systematic liquidity shock will be fully discounted in the price.

In contrast, the state of liquidity in the market is critical in the case of idiosyncratic liquidity shocks. Since investors are differentially impacted by the shocks, they can potentially be transmitted across the investor base by trading, to the benefit of all investors. Hence, investors will seek to exploit the benefits of trading if the market is liquid. The extent to which the risk of idiosyncratic liquidity shocks is incorporated in the price depends on the state of liquidity in the market, which in turn is determined by λ and M. When $\lambda = 0$ and $M \to \infty$, idiosyncratic liquidity shocks will no longer be priced since investors are able to perfectly offset the effect of the shocks in the market. On the other hand, idiosyncratic liquidity shocks will be fully priced when the market is perfectly illiquid, when $\lambda \to \infty$ or $M = 1$.

In the following subsections, we further investigate issues pertaining to volume and volatility in the context of systematic and idiosyncratic liquidity shocks, as well as the impact of investor heterogeneity on a security's liquidity and pricing.

3.2 Volume and Volatility

Trading arises at time 1 in the secondary market when individual valuations of the security differ from the market price because of idiosyncratic liquidity shocks. This leads to the trade of marginal quantities until the price in equilibrium equals each investor's marginal valuation. Noting that when $\gamma_i = 1$ all investors, being ex ante identical, will hold their initial endowments in the time 0 equilibrium, i.e. $X_{0i} = 1$, the equilibrium time 1 trade size for individual i, $\Delta X_{1i} = X_{1i} - X_{0i}$, becomes

$$\Delta X_{1i} = \frac{(\hat{\varepsilon}_i - \hat{\varepsilon}_A)}{a\sigma_v^2 + 2\lambda} \tag{8}$$

ΔX_{1i} is perfectly positively correlated with the differential between his individual liquidity shock and the average shock to the aggregate base of investors. If his personalized valuation at time 1 due to the shock exceeds the price in the market, he will exercise his choice to buy the risky asset. Likewise, if his personalized valuation is less than the market price, he will sell the risky asset. ΔX_{1i} is normally distributed with mean 0 and variance $\sigma_X^2 = \dfrac{\sigma_\varepsilon^2\left(\dfrac{M-1}{M}\right)}{(a\sigma_v^2 + 2\lambda)^2}$.

The expected individual and market-wide trading volume follow directly from (8) and are presented in Lemma 3.

Lemma 3. *The expected size of each individual's trade is given by:*

$$E_0\left[|\Delta X_{1i}|\right] = \frac{\sigma_\varepsilon}{a\sigma_v^2 + 2\lambda}\sqrt{\frac{2}{\pi}\left(\frac{M-1}{M}\right)} \tag{9}$$

and the expected total volume of trade in the market, Q_1*, is:*

$$Q_1 = \frac{M}{2} E_0\left[|\Delta X_{1i}|\right] = \frac{\sigma_\varepsilon}{a\sigma_v^2 + 2\lambda}\sqrt{\frac{M(M-1)}{2\pi}} \tag{10}$$

Proof. See Appendix.

The result for price volatility is presented in Lemma 4.

Lemma 4. *If* σ_P^2 *denotes price volatility at time 1, then* $\sigma_P^2 = \sigma_\delta^2 + \frac{\sigma_\varepsilon^2}{M}$.

Proof. Follows directly from (5) for the case of $\gamma_i = 1$.

These results establish the relationship between liquidity shocks, and volume of trade and price volatility in the secondary market. Proposition 2 summarizes the key result of this subsection.

Proposition 2 *(Volume and Volatility). Common (systematic) liquidity shocks do not affect trading volume. Trading volume increases with the variance of idiosyncratic liquidity shocks and decreases with transactions costs. Both common (systematic) and idiosyncratic liquidity shocks affect price volatility. The price volatility associated with systematic liquidity shocks is not contingent upon the state of liquidity in the market, and is increasing in the variance of systematic liquidity shocks. Contingent upon a liquid market at time 1, the price volatility associated with idiosyncratic liquidity shocks is increasing in the variance of idiosyncratic liquidity shocks and decreasing in M.*

These results suggest the importance of carefully differentiating between systematic and idiosyncratic liquidity shocks when using standard liquidity measures as proxies for liquidity. In particular, systematic liquidity shocks exacerbate price volatility but have no effect on trading volume. The state of liquidity in the market is another important determinant of these liquidity measures. While the state of liquidity in the market depends on whether it is open for trading and if so, the cost of undertaking transactions, it will also depend on the degree to which investors are exposed to liquidity shocks, and thus, the level of liquidity that they demand. In the next subsection, we further examine the sharing of liquidity risk across investors arising from their differential exposure to systematic liquidity shocks.

3.3 Sharing of Liquidity Risk across Heterogeneous Investors

In Subsection 3.1, we noted that when investors have non-uniform exposure to systematic liquidity shocks, they would in general be heterogeneous at time 0. In that case we observed partial risk-sharing across investors through trading at time 1. We also noted that *ex ante* differences could also affect portfolio decisions at time 0. In this subsection, we examine this specific question, using simplifying assumptions to preserve tractability. We first study the general case of liquidity shocks under special assumptions about transactions costs. Specifically, we examine in turn the two polar cases of a *perfectly liquid market* ($\lambda = 0$) and a *perfectly illiquid market* ($\lambda \to \infty$) at time 1. Next, we assume a specific non-uniform distribution of systematic liquidity shocks to examine risk sharing through differences in portfolio holding at time 0. Since we have previously considered the comparative statics associated with *M*, we simplify the analysis by assuming that $M \to \infty$, causing $\hat{\varepsilon}_A \to 0$.

Lemma 5 states the result for the time 0 equilibrium price and holding of the risky asset for the general case of liquidity shocks when the time 1 market is *perfectly liquid*.

Lemma 5. *When the secondary market at time 1 is perfectly liquid, the time 0 market clearing price, P_0^* and the equilibrium holding of the risky asset by investor i, X_{0i}^* respectively, are:*

$$P_0^* = \overline{v} - a\sigma_v^2 - a\gamma_A^2\sigma_\delta^2; \quad X_{0i}^* = 1 \tag{11}$$

Proof. See Appendix.

Interestingly, despite their *ex ante* differences, all investors hold the same portfolio at time 0 since the perfectly liquid market enables them to respond to their liquidity shocks by trading costlessly at time 1. Therefore, no prior hedging by rebalancing portfolios at time 0 takes place. As suggested by Proposition 1, only the average systematic liquidity shock (the undiversifiable component) is priced. Investors who have a low exposure to the systematic liquidity shock will reap the benefit of a lower price at time 0 than would be justified by the liquidity risk that they bear. In contrast, investors who have a high exposure to systematic liquidity shocks will pay a higher price than would be justified by their liquidity risk. This is the cost of being able to transfer their liquidity risk to low-exposure investors by trading with them in the liquid market at time 1.

The situation changes when the time 1 market is *perfectly illiquid*. In this case, the option to rebalance portfolios in the secondary market is no longer available, and investors need to take account of this knowledge when making their portfolio decisions at time 0. We state the first order condition for this case in Lemma 6 below.

Lemma 6. *When the secondary market at time 1 is perfectly illiquid, the investor's portfolio problem collapses to a single-period problem in which the time 0 first order condition becomes:*

$$\bar{v} - P_0 - a(\sigma_v^2 + \gamma_i^2\sigma_\delta^2 + \sigma_\varepsilon^2)X_{0i} = 0 \tag{12}$$

Proof. The two-period problem collapses to a single-period problem. The proof parallels the proof of Lemma 1.

Noting that in (12) γ_i is the only term that differs across investors, we can observe that in equilibrium, investors with a high exposure to systematic liquidity shocks will hold a lower amount X_{0i} of the risky asset at time 0 than investors with a low exposure.

To explore this point further, and to examine market clearing, equilibrium prices and holdings at time 0, we need to impose a distribution for the γ_i across the investor base. Unlike previously when we assumed a uniform distribution in which case there would be no potential for risk sharing at time 0, we assume that the investor base consists of two investor clienteles. A fraction ϕ is of type 1, for whom $\gamma_i = 1$. The rest of the investor base is of type 2, for whom $\gamma_i = 0$.

When the market is perfectly liquid, the equilibrium price and holding at time 0 become:

$$P_0^* = \bar{v} - a\sigma_v^2 - a\phi^2\sigma_\delta^2; \quad X_{0i}^* = 1 \tag{13}$$

The result here is consistent with (11) for the general case. Type 2 investors obtain an additional return of $a\phi^2\sigma_\delta^2$ for meeting the liquidity needs of type 1 investors at time 1, while type 1 investors forego a return of $a(1-\phi^2)\sigma_\delta^2$ for the benefit of the option to alleviate the systematic component of their liquidity risk at time 1.

As we have noted, when the market is perfectly illiquid at time 1, due to the differences in liquidity demand across the two clienteles, equilibrium holdings of the risky asset at time 0 need not in general be identical across the two clienteles. We assume that in equilibrium, type 1 investors hold a fraction μ of the risky asset, with type 2 investors holding the remainder. From the optimal choice of type 1 investors, we obtain the following expression from (12) for the time 0 equilibrium price of the risky asset, P_0:

$$P_0 = \bar{v} - \frac{a\mu}{\phi}(\sigma_v^2 + \sigma_\delta^2 + \sigma_\varepsilon^2) \tag{14}$$

and similarly, from the optimal choice of type 2 investors:

$$P_0 = \bar{v} - \frac{a(1-\mu)}{(1-\phi)}(\sigma_v^2 + \sigma_\varepsilon^2) \tag{15}$$

The equilibrium values of μ and P_0 are stated in Lemma 7 below.

Lemma 7. *When the secondary market is perfectly illiquid, the fraction of the risky asset held in equilibrium by type 1 investors,* μ*, and the market clearing price at time 0,* P_0 *are as follows:*

$$\mu = \phi \left[\frac{1}{\phi + (1-\phi)\left(\frac{\sigma_v^2 + \sigma_\delta^2 + \sigma_\varepsilon^2}{\sigma_v^2 + \sigma_\varepsilon^2} \right)} \right] \leq \phi$$

$$P_0^* = \bar{v} - a(\sigma_v^2 + \sigma_\delta^2 + \sigma_\varepsilon^2) \left[\frac{\sigma_v^2 + \sigma_\varepsilon^2}{\phi(\sigma_v^2 + \sigma_\varepsilon^2) + (1-\phi)(\sigma_v^2 + \sigma_\delta^2 + \sigma_\varepsilon^2)} \right] \tag{16}$$

Proof. See Appendix.

It is observed that in the time 0 equilibrium, type 2 investors hold more of the risky asset than type 1 investors do. Type 1 investors are exposed to a higher risk from their holdings of the risky asset due to their higher exposure. Thus, since they are not able trade this asset after observing their liquidity shocks, type 1 investors hedge the effect of the anticipated shock by selling a part of their endowment to the type 2 investors who are not exposed to the systematic component of liquidity risk that type 1 investors face.

The pricing result in the case of an illiquid time 1 market is also interesting. We denote the valuations attached by type 1 and type 2 investors at time 0 to their initial endowments (i.e. prior to any portfolio rebalancing at time 0) as P_0^1 and P_0^2, which are given by $P_0^1 = \bar{v} - a(\sigma_v^2 + \sigma_\delta^2 + \sigma_\varepsilon^2)$ and $P_0^2 = \bar{v} - a(\sigma_v^2 + \sigma_\varepsilon^2)$. We observe the following relationship relative to the equilibrium price at time 0:

$$P_0^1 \leq P_0^* \leq P_0^2 \tag{17}$$

If only type 2 investors populated the investor base, the price they would pay for the risky asset is higher since holding the asset does not expose them to systematic liquidity risk. The equilibrium price is lowered due to the presence of the type 1 investors, who transfer a part of their liquidity risk to type 2 investors by selling them part of their endowment in the primary (time 0) market. Thus, while type 2 investors hold more of the risky illiquid asset, they are compensated for doing so by the price differential between the two cases, permitting them to earn a higher return. This result is consistent with and extends the intuition in Amihud and Mendelson (1986) where the liquidity needs of investor clienteles are measured by holding period, and investors who have a longer holding period (lower liquidity need) earn a higher return.

Proposition 3 summarizes the key result of this subsection.

Proposition 3 *(Sharing of liquidity risk across investors). When the secondary market is perfectly liquid, high exposure investors will seek to manage their risk by trading with low exposure investors after observing their liquidity shocks. When the secondary market is perfectly illiquid, high exposure investors will seek to manage their risk by reducing their holdings of the risky asset in the primary market in anticipation of future shocks. In either case, low exposure investors will earn a higher risk-adjusted return, as a reward for the transmission to them of the liquidity risk of high exposure investors.*

This section has discussed commonality of liquidity shocks across investors for the case of a single risky asset, both when investors are *ex ante* identical and when they differ due to their exposure to systematic liquidity shocks. Next, we turn to the case of multiple assets and the transmission of liquidity across assets as well as investors.

4. TRANSMISSION OF LIQUIDITY ACROSS SECURITIES

In this section, we examine the case in which investors hold multiple risky assets, to study liquidity transmission and commonality across securities. We develop our basic results for the general multi-security case and examine specific examples in the context of two risky assets. As in Section 3, we begin our analysis with our general formulation of liquidity shocks (adapted for the case with multiple securities) and transactions costs, before moving to specific cases.

4.1 Systematic and Idiosyncratic Liquidity Shocks with Multiple Assets

We assume that individuals have the same preferences as before and are endowed with one unit of the riskless asset and one unit each of K risky assets, indexed by k, k = 1,..., K. As before, risky asset k pays off a random quantity of the numeraire riskless asset, $\tilde{v}^k$, at time 2, where $E(\tilde{v}^k) > 1$. The $\tilde{v}^k$, k = 1,..., K, are identically and independently (i.i.) normally distributed with mean $\bar{v}$ and variance σ_v^2. The risk-free return is assumed to be zero as before. Since we have previously studied the comparative statics associated with variations in the size of the investor base, M, we simplify the analysis in this section by assuming that $M \to \infty$. All investors experience liquidity shocks at time 1, which are characterized as random additive increments,

$\tilde{\theta}_i^k$, to the payoff $\tilde{v}^k$ of the risky asset to investor *i*. As in the single-asset case, $\tilde{\theta}_i^k$ can be decomposed into normally distributed systematic and idiosyncratic components:

$$\tilde{\theta}_i^k = \gamma_i \tilde{\delta} + \tilde{\varepsilon}_i^k \tag{18}$$

where $\tilde{\delta}$, the systematic component, impacts investor *i* with an exposure of $\gamma_i \geq 0$. Thus, the systematic component is, by design, common across the asset base. In contrast, $\tilde{\varepsilon}_i^k$, the idiosyncratic component, is assumed to be i.i.d. across both investors and assets. $\tilde{\delta}$ has a mean of 0 and a variance of σ_δ^2 while $\tilde{\varepsilon}_i^k$ has a mean of 0 and a variance of σ_ε^2. We assume that the transactions cost parameter, λ, is uniform across assets.

The individual's problem in this case is:

$$\max_{\{X_{0i}^k\}_{k=1}^K} E_0\left[\max_{\{X_{1i}^k\}_{k=1}^K} E_1\left[-\exp\left\{-a\left(W_{0i} + \sum_{k=1}^K X_{0i}^k(\tilde{P}_1^k - P_0^k) + \sum_{k=1}^K \tilde{X}_{1i}^k(\tilde{v}^k + \tilde{\theta}_i^k - \tilde{P}_1^k) - \sum_{k=1}^K \lambda(\tilde{X}_{1i}^k - X_{0i}^k)^2\right)\right\}\right]\right] \tag{19}$$

where X_{0i}^k and X_{1i}^k are the time 0 and time 1 holdings of asset *k* by individual *i*. Paralleling the single-asset case, we state the key results for the time 1 equilibrium in Lemmas 8.

Lemma 8. *The market clearing price at time 1,* P_1^k*, and the equilibrium holding of asset k by investor i,* X_{1i}^k *are respectively:*

$$P_1^k = \bar{v} + \gamma_A \hat{\delta} - a\sigma_v^2$$

$$X_{1i}^k = \frac{(\gamma_i - \gamma_A)\hat{\delta} + \hat{\varepsilon}_i^k + a\sigma_v^2 + 2\lambda X_{0i}^k}{a\sigma_v^2 + 2\lambda} \tag{20}$$

where $\hat{\delta}$ *and* $\hat{\varepsilon}_i^k$ *are the realizations of* $\tilde{\delta}$ *and* $\tilde{\varepsilon}_i^k$*, respectively.*

Proof. Parallels proof of Lemma 1 for the single-asset case.

Surprisingly, it can be seen from Lemma 8 that the results for each individual asset in the multi-asset case are identical to those obtained in Lemma 1 for the single asset case. In particular, we observe no spillover of liquidity effects across assets. The systematic component of liquidity shocks is reflected only in the equilibrium price at time 1, whereas the idiosyncratic component of liquidity shocks is reflected only in the equilibrium asset holding. Although it would appear as though there's a common component in the time 1 asset holdings of $\frac{(\gamma_i - \gamma_A)\hat{\delta}}{a\sigma_v^2 + 2\lambda}$, this component pertains only to the investor-specific "idiosyncratic" element of the common liquidity shock that applies to each specific asset. This can be easily seen by setting $\gamma_i = \gamma_A$ which makes the systematic shock common across both assets and investors, causing it to completely disappear from holdings. Hence, as in the single-asset case, liquidity matters only if liquidity shocks are heterogeneous across investors. If liquidity shocks are systematic, investors cannot trade and the availability of liquidity will not be a factor. Thus, prices will simply adjust without trade. If liquidity shocks are idiosyncratic across investors and assets, liquidity will be transmitted across investors by trading, and since $M \to \infty$, the idiosyncratic liquidity shocks will be perfectly mutualized. However, there's no transmission of liquidity across assets even in this case.

We next turn to the market equilibrium at time 0. As in the single-asset case, in order to solve completely for the equilibrium at time 0 we need to make specific assumptions about the distribution of γ_i across the investor base, and about transactions costs. As before, we assume that $\gamma_i = \gamma_A = 1$ and restrict our analysis to the two polar cases of perfectly liquid and perfectly illiquid markets.

Lemma 9. *If there is trading at time 1, the equilibrium price of asset k at time 0, P_0^k, is:*

$$P_0^k = \overline{v} - a\sigma_v^2 - Ka\sigma_\delta^2 \tag{21}$$

If the market is not open for trading at time 1, P_0^k is:

$$P_0^k = \overline{v} - a\sigma_v^2 - Ka_\delta^2 - a\sigma_\varepsilon^2 \tag{22}$$

Proof. Parallels proofs of Lemma 2 for the case $\lambda = 0$, and Lemma 6, respectively.

We state the main result of this subsection in Proposition 4.

Proposition 4 *(Liquidity transmission across securities). When asset returns are uncorrelated, there is no transmission of liquidity across assets even when investors experience common (systematic) liquidity shocks. Systematic liquidity shocks are always fully priced, whereas idiosyncratic liquidity shocks are priced only if they cannot be mutualized by trading.*

Thus, it is seen that the results for the multi-security case exactly parallel our benchmark case with a single risky asset. It is especially important to note that despite a multiplicity of risky assets, there is no transmission of liquidity across assets in the above case. Since systematic shocks are, by definition, perfectly correlated across investors, there is no possibility to eliminate their effect by trading. As shown in (20), systematic shocks are simply reflected in the equilibrium price, without trading. This would be the case even if these shocks were not common across assets. Idiosyncratic liquidity shocks are perfectly mutualized by trading in each asset, without spillover effects across assets. Thus, they are reflected in volume but not in price, since the mutualization is perfect when $M \to \infty$. Even if idiosyncratic liquidity shocks were common across assets while being idiosyncratic across investors, there will be no transmission across assets as long as all assets can be freely traded. Thus, the only case in which liquidity transmission across assets will occur is if liquidity shocks on one group of assets can be better mitigated by trading another group of assets, which would arise if liquidity shocks are correlated across the two groups of assets while there being significant liquidity differences across the two groups. We turn to this case next.

4.2 Liquidity Transmission from Liquid to Illiquid Assets

Due to the increased complexity introduced by the assumption of correlated asset returns in this subsection, we examine the special case of $K = 2$ in which liquidity shocks are common (i.e. perfectly correlated) across the two risky assets but idiosyncratic across investors. Thus, in this case, $\tilde{\theta}_i^k = \tilde{\varepsilon}_i^k = \tilde{\varepsilon}_i$. We compare three subcases:

a. Both assets perfectly liquid;

b. Both assets perfectly illiquid; and

c. Asset 1 is perfectly liquid while Asset 2 is perfectly illiquid.

In “a)”, since $M \to \infty$, investors will be able to completely eliminate the risk of liquidity shocks by trading with each other. Hence, this risk will not be priced and the time 0 equilibrium price will be $P_0^k = \bar{v} - a\sigma_v^2;\ k = 1,2$. Since both assets are liquid, they will trade independently, without any transmission of liquidity between them. The optimal holdings at time 1 are $X_{1i}^1 = X_{1i}^2 = 1 + \frac{\hat{\varepsilon}_i}{a\sigma_v^2}$, indicating a trade size of $\frac{\hat{\varepsilon}_i}{a\sigma_v^2}$ in each case.

From the standpoint of liquidity risk, “b)” is the polar opposite of “a)”. The inability to trade the assets means that investors will be fully exposed to the risk of liquidity shocks. Hence, this risk will be fully priced, yielding a time 0 equilibrium price of

$P_0^k = \bar{v} - a\sigma_v^2 - 2a\sigma_\varepsilon^2;\ \ k = 1,2.$

In “c)”, the individual's problem becomes:

$$\max_{X_{0i}^1, X_{0i}^2} E_0\left[\max_{X_{1i}^1} E_1\left[-\exp\left\{-a\left(W_{0i} + X_{0i}^1(P_1^1 - P_0^1) + X_{1i}^1(\tilde{v}^1 + \tilde{\varepsilon}_i - P_1^1) + X_{0i}^2(\tilde{v}^2 + \tilde{\varepsilon}_i - P_0^2)\right)\right\}\right]\right] \quad (23)$$

where the notation follows (19). Lemma 10 states the relevant results for the time 1 and time 0 equilibria.

Lemma 10. *When Asset 1 is perfectly liquid and Asset 2 is perfectly illiquid, the equilibrium time 1 holding of asset 1, and equilibrium prices at time 0 for both assets are as follows:*

$$X_{1i}^1 = 1 + \frac{\hat{\varepsilon}_i}{a\sigma_v^2}$$

$$P_0^1 = \bar{v} - a\sigma_v^2 \quad (24)$$

$$P_0^2 = \bar{v} - a\sigma_v^2 - a\sigma_\varepsilon^2$$

Proof. Parallels proofs of Lemma 1, and Lemma 2 for the case $\lambda = 0$, respectively.

The comparison, especially across cases "a)" and "c)" is interesting. Intuitively, since the liquidity shock is common across the two assets, one would expect transmission of liquidity in "c)" from Asset 1 to Asset 2. This is not what we find. The optimal holding of Asset 1, and therefore the optimal trade size in response to the liquidity shock, is the same in "a)" and "c)". On the other hand, there is a differential price impact across the three cases associated with the pricing of liquidity risk. As we would expect, liquidity risk is not priced for Asset 1 in either "a)" or "c)", whereas for Asset 2, the price in "c)" includes a liquidity risk premium of $a\sigma_\varepsilon^2$. However, in "b)", where both assets are illiquid, the liquidity risk premium is $2a\sigma_\varepsilon^2$. In "c)", the investor's "total exposure" to liquidity risk is lowered, since the liquidity risk associated with Asset 1 can be completely mitigated by trading at time 1. However, this does not affect trading volume, and from the standpoint of Asset 1, "a)" and "c)" are identical. Hence, despite the commonality in liquidity shocks, Asset 1 remains independent of Asset 2.

These results are somewhat surprising. In order to gain further insight into the issue of liquidity transmission across assets, we examine how the results in "a)" and "c)" above change when Asset 1 is a perfect substitute for Asset 2, i.e. their payoffs $\tilde{v}^1$ and $\tilde{v}^2$ are also identical. When both assets are perfectly liquid as in "a)", the relevant results from the time 1 and time 0 equilibria are stated in Lemma 11.

Lemma 11. *When the assets are perfectly liquid and their payoffs are perfectly correlated, the time 1 holdings and time 0 equilibrium prices are as follows:*

$$X_{1i}^1 = X_{1i}^2 = 1 + \frac{\hat{\varepsilon}_i}{2a\sigma_v^2}$$
$$P_0^1 = P_0^2 = \overline{v} - 2a\sigma_v^2 \tag{25}$$

Proof. Parallels proofs of Lemma 1, and Lemma 2 for the case $\lambda = 0$, respectively.

When Asset 1 is perfectly liquid while Asset 2 is perfectly illiquid, as in "c)", the relevant time 1 and time 0 results are stated in Lemma 12.

Lemma 12. *When Asset 1 is perfectly liquid while Asset 2 is perfectly illiquid, and their payoffs are perfectly correlated, the time 1 holding of Asset 1 and time 0 equilibrium prices are as follows:*

$$X_{1i}^1 = 1 + \frac{\hat{\varepsilon}_i}{a\sigma_v^2}$$
$$P_0^1 = P_0^2 = \overline{v} - 2a\sigma_v^2 \tag{26}$$

Proof. Parallels proofs of Lemma 1, and Lemma 2 for the case $\lambda = 0$, respectively.

The comparison of these results is interesting. The illiquidity of Asset 2 in the second case is not a factor in pricing. The ability to trade Asset 1 permits investors to transfer the entire liquidity risk of Asset 2 to Asset 1. This is seen in the doubling of the trade size of Asset 1, from $\frac{\hat{\varepsilon}_i}{2a\sigma_v^2}$ in the case where both assets are liquid, to $\frac{\hat{\varepsilon}_i}{a\sigma_v^2}$ in the case where only Asset 1 is liquid. Thus, the two assets are perfect substitutes, and transmission of liquidity occurs from Asset 1 to Asset 2, allowing investors to perfectly offset the illiquidity of Asset 2 by trading in Asset 1.

Thus, as suggested by Huberman and Halka (2001), the availability of substitutes will give rise to liquidity transmission across assets and commonality in liquidity. The spillover effects in volume are as in Caballe and Krishnan (1994) and seem to occur only when fundamentals are also correlated, regardless of whether or not liquidity shocks are correlated as well. Our model captures these spillover effects only imperfectly since Asset 2 is assumed to be perfectly illiquid. However, it is not difficult to visualize the more general case in which both assets are partially but differentially liquid, where a liquidity shock to one asset causes co-movement in volume across the two assets.

These results have useful implications for the valuation of illiquid securities. Since these securities are, by definition, traded only infrequently if at all, they cannot be accurately valued based on market prices. Our analysis suggests that at least in some situations, the market prices of liquid substitutes can be used to benchmark the value of illiquid securities. Our framework may be especially relevant for valuing restricted stocks and other securities which are privately placed issues of public corporations.[9] Restricted stocks are frequently issued by public companies and are identical to their publicly placed counterparts in all respects except that they are subject to trading restrictions. These restrictions are formidable and are designed to effectively force the purchaser to withhold the securities from public trading for the duration of the restricted period, except when exceptional circumstances justify the cost of overcoming these restrictions.

These restrictions appear to induce a significant discount in the price of restricted securities compared to their publicly traded counterparts. Wruck (1989) reports an average discount of 14% in a sample of 83 sales of NYSE and AMEX firms making private sales of restricted shares between July 1979 and December 1985. In contrast, a sample of 45 sales of registered securities during the same period reveals a premium of 4%. Silber (1991) reports an average discount of 33.75% in his study of the impact of illiquidity on restricted stock.

Our analysis suggests that when liquid but otherwise identical substitutes exist for illiquid securities, the price discount due to illiquidity should be zero. As noted previously, the intuition for this result is trivial -- if an investor wishes to reduce (increase) his holding in an illiquid security, he can short sell (buy) the liquid substitute. The magnitude of the discounts observed empirically suggests that the unavailability of liquid substitutes and/or short sale restrictions may be significant impediments to hedging the liquidity risk of illiquid securities in

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[9] The Securities and Exchange Commission (SEC, 1971) classifies restricted securities as securities acquired from an issuer in a transaction (private placement) exempt from registration as stipulated in the Securities Act of 1933. The basis for this exemption is that the transaction is not a public offering and the securities are privately placed with a group of investors who do not need the usual public stock issue protections offered by the Securities Act. Investors who buy these securities are required to provide a letter of undertaking that they are being bought for long-term investment purposes.

this way. However, it is not possible to rule out significant levels of mispricing, especially since restricted securities are typically placed on a negotiated basis that relies on long-standing industry norms about the "appropriate" discount for illiquidity.

5. CONCLUDING REMARKS

This paper has explored some of the issues raised by recent findings of common factors in liquidity. We have examined how such common factors could arise and what impact they might have on asset prices. Differentiating between systematic and idiosyncratic components of liquidity shocks has permitted us to more thoroughly analyze these questions. Surprisingly, common liquidity shocks do not give rise to commonality in trading volume. Covariation in standard liquidity measures such as volume requires covariation in the levels of investor heterogeneity. Idiosyncratic liquidity shocks, which give rise to investor heterogeneity, create liquidity demand and volume. Investors can potentially diversify their risk by trading, and therefore, the pricing of idiosyncratic liquidity risk depends on the market's liquidity. Our results are largely independent of the number of different securities traded in the market, lending support to our conjecture about commonality being the outcome of covariation in investor heterogeneity rather than of systematic liquidity shocks.

We have advanced the notion that understanding the structure of liquidity demand is key to understanding the economic effects of liquidity. Since investors are likely to fall into a wide variety of clienteles with regard to their liquidity demand, this issue is important from an empirical standpoint. Furthermore, this is also likely to be an important consideration for companies that have the choice of targeting their securities to different investor clienteles and/or markets.

The theoretical framework presented here provides a useful basis for further empirical investigations on the role of liquidity in asset pricing, and differentiating between systematic and idiosyncratic drivers of liquidity. Subject to the prevailing market structure, volume is the manifestation of investor heterogeneity arising from idiosyncratic liquidity shocks, while price volatility is largely the manifestation of systematic liquidity shocks. Differentiating between these effects is key to determining the economic effects of liquidity.

Many questions remain. Our analysis implies that the supply of liquidity will arise endogenously from liquidity demand, in addition to other factors such as market structure and the presence of informed trading. In the past, this issue has been approached largely from the standpoint of asymmetric information models, with uninformed traders being assumed to have limited discretion in trading, if at all. In contrast, we provide a microscopic examination of liquidity-driven trading to the exclusion of information-driven trading. A combination of the two will provide a richer framework to analyze the origins and consequences of liquidity variations. Finally, our analysis has shown that different liquidity variables measure different things due to the differential impact of systematic and idiosyncratic liquidity shocks. Moreover, whereas the traditional focus has been on factors related to the supply of liquidity, we show that liquidity is the outcome of both demand and supply factors, with the demand side having a much more significant and varied impact than previously thought to be the case in the literature. The implications of these new insights will need to be considered in future empirical analysis.

Appendix

Proof of Lemma 1

The investor's time 1 problem can be expressed as:

$$\max_{X_{1i}} \; E_1\left[-\exp\left\{-a\left[W_{1i} + X_{1i}(\tilde{v} + \hat{\theta}_i - P_1) - \lambda(X_{1i} - X_{0i})^2\right]\right\}\right] \tag{A.1}$$

which yields the first order condition:

$$(\bar{v} + \hat{\theta}_i - P_1) - a\sigma_v^2 X_{1i} - 2\lambda(X_{1i} - X_{0i}) = 0 \tag{A.2}$$

Equating aggregate demand and supply for the risky asset across all investors yields:

$$\sum_{i=1}^{M} X_{1i} = M \tag{A.3}$$

Noting that $\hat{\theta}_i = \gamma_i \hat{\delta} + \hat{\varepsilon}_i$, and aggregating across all investors, we obtain the market clearing price at time 1, P_1, and the equilibrium holding of the risky asset by investor i, X_{1i}, as:

$$\begin{aligned} P_1 &= \bar{v} + \gamma_A \hat{\delta} + \hat{\varepsilon}_A - a\sigma_v^2 \\ X_{1i} &= \frac{(\gamma_i - \gamma_A)\hat{\delta} + (\hat{\varepsilon}_i - \hat{\varepsilon}_A) + a\sigma_v^2 + 2\lambda X_{0i}}{a\sigma_v^2 + 2\lambda} \end{aligned} \tag{A.4}$$

Q.E.D.

Proof of Lemma 2

At time 0, the individual's problem becomes:

$$\max_{X_{0i}} E_0\left[\max_{X_{1i}} E_1\left[-\exp\left\{-a\left[W_{0i} + X_{0i}(\tilde{P}_1 - P_0) + \tilde{X}_{1i}(\tilde{v} + \tilde{\theta}_i - \tilde{P}_1) - \lambda(\tilde{X}_{1i} - X_{0i})^2\right]\right\}\right]\right] \tag{A.5}$$

which is equivalent to:

$$\max_{X_{0i}} E_0\left[\text{-}\exp\left\{\text{-}a\left[W_{0i} + X_{0i}(\tilde{P}_1^* - P_0) + \tilde{X}_{1i}^*(\bar{v} + \tilde{\theta}_i - \tilde{P}_1^*) - \frac{1}{2}a\sigma_v^2 \tilde{X}_{1i}^{*2} - \lambda(\tilde{X}_{1i}^* - X_{0i})^2\right]\right\}\right] \tag{A.6}$$

where $\tilde{X}_{1i}^*$, $\tilde{P}_1^*$ are the optimal time 1 holding and equilibrium price, respectively, both normally distributed variables from the perspective of time 0. This is of the form:

$$\max_{X_{0i}} E_0\left[-\exp\left\{-a\left[W_{0i}+\tilde{Z}(X_{0i})\right]\right\}\right] \tag{A.7}$$

Substituting from the time 1 first order condition, $\tilde{Z}$ reduces to:

$$\tilde{Z}=X_{0i}(\tilde{P}_1^*-P_0)+\lambda(\tilde{X}_{1i}^{*2}-X_{0i}^2)+\frac{1}{2}a\sigma_v^2\tilde{X}_{1i}^{*2} \tag{A.8}$$

and substituting for $\tilde{X}_{1i}^*$, $\tilde{P}_1^*$ from Lemma 1:

$$\begin{aligned}\tilde{Z}=X_{0i}(\overline{v}-a\sigma_v^2-P_0)-\lambda X_{0i}^2+\frac{\left(a\sigma_v^2+2\lambda X_{0i}\right)^2}{2\left(a\sigma_v^2+2\lambda\right)}\\+\frac{1}{2\left(a\sigma_v^2+2\lambda\right)}\Delta\tilde{\theta}_i^2+\frac{\left(a\sigma_v^2+2\lambda X_{0i}\right)}{\left(a\sigma_v^2+2\lambda\right)}\Delta\tilde{\theta}_i+X_{0i}\tilde{\theta}_A\end{aligned} \tag{A.9}$$

where $\tilde{\theta}_A=\frac{\sum_{j=1}^{M}\theta_j}{M}$ and $\Delta\tilde{\theta}_i=\tilde{\theta}_i-\tilde{\theta}_A$. This expression is of the form:

$$\tilde{Z}=A+B(\Delta\tilde{\theta}_i^2)+E(\Delta\tilde{\theta}_i)+F(\tilde{\theta}_A) \tag{A.10}$$

where A, B, E and F are non-random. From the moment generating function of $\tilde{Z}$, $E[-\exp(-\tilde{Z})]$, disregarding terms uncorrelated with X_{0i}, (A.7) reduces to:

$$\underset{X_{0i}}{Max}\,E_0\left[-\exp\left\{-a\tilde{Z}\right\}\right]\equiv\underset{X_{0i}}{Max}\left[aA-\frac{1}{2L_B}\left[M_1^2+\frac{(L_BM_2)^2}{L_B}\right]\right] \tag{A.11}$$

where:

$$
\begin{aligned}
L_B &= 1 + 2aB\sigma_{\Delta\theta}^2 \\
M_1 &= (-a)\frac{a\sigma_v^2 + 2\lambda X_{0i}}{a\sigma_v^2 + 2\lambda}\sigma_{\Delta\theta} \\
M_2 &= (-a)X_{0i}\sigma_{\theta_A}
\end{aligned}
\tag{A.12}
$$

where $\sigma_{\Delta\theta}^2$ and $\sigma_{\theta_A}^2$ are the variances of $\Delta\tilde{\theta}_i$ and $\tilde{\theta}_A$, respectively. Taking the first order condition with respect to X_{0i} and noting that $\sum_{i=1}^{M} X_{0i} = M$, we obtain the market clearing price at $t = 0$:

$$
P_0 = \overline{v} - a\sigma_v^2 - a\sigma_{\theta_A}^2 - \frac{2\lambda a\sigma_{\Delta\theta}^2}{a\sigma_v^2 + 2\lambda + a\sigma_{\Delta\theta}^2}
\tag{A.13}
$$

Noting that

$$
\sigma_{\theta_A}^2 = \sigma_\delta^2 + \frac{\sigma_\varepsilon^2}{M}
\tag{A.14}
$$

and

$$
\sigma_{\Delta\theta}^2 = \sigma_\varepsilon^2 \frac{(M-1)}{M}
\tag{A.15}
$$

yields the expression for P_0 in (6).

Q.E.D.

<u>Proof of Lemma 3</u>

The expected size of each individual's trade is given by:

$$
E_0\left[|\Delta X_{1i}|\right] = 2\int_0^\infty \frac{\alpha}{\sigma_X\sqrt{2\pi}} \exp\left(-\frac{\alpha^2}{2\sigma_X^2}\right) d\alpha
\tag{A.16}
$$

where:

$$\sigma_X^2 = \frac{\sigma_\varepsilon^2 \left(\frac{M-1}{M} \right)}{(a\sigma_v^2 + 2\lambda)^2} \tag{A.17}$$

which yields:

$$E_0 \left[| \Delta X_{1i} | \right] = \sigma_X \sqrt{\frac{2}{\pi}} = \frac{\sigma_\varepsilon}{a\sigma_v^2 + 2\lambda} \sqrt{\frac{2}{\pi} \left(\frac{M-1}{M} \right)} \tag{A.18}$$

and the expected total volume of trade in the market, Q_1, is:

$$Q_1 = \frac{M}{2} E_0 \left[| \Delta X_{1i} | \right] = \frac{\sigma_\varepsilon}{a\sigma_v^2 + 2\lambda} \sqrt{\frac{M(M-1)}{2\pi}} \tag{A.19}$$

Q.E.D.

Proof of Lemma 5

When the secondary market at time 1 is perfectly liquid, by following the steps of the proof of Lemma 2 for the general case one can show that the time 0 first order condition of (A.11) with respect to X_{0i} becomes:

$$\overline{v} - a\sigma_v^2 - P_0 - a\gamma_A^2 \sigma_\delta^2 X_{0i} = 0 \tag{A.20}$$

Noting that $\sum_{i=1}^{M} X_{0i} = M$, we can aggregate across the investor base to obtain the following equilibrium results:

$$\begin{aligned} P_0^* &= \overline{v} - a\sigma_v^2 - a\gamma_A^2 \sigma_\delta^2 \\ X_{0i}^* &= 1 \end{aligned} \tag{A.21}$$

Q.E.D.

Proof of Lemma 7

Equating the values for P_0 in (14) and (15) yields:

$$\frac{\mu}{\phi}(\sigma_v^2+\sigma_\delta^2+\sigma_\varepsilon^2)=\frac{(1-\mu)}{(1-\phi)}(\sigma_v^2+\sigma_\varepsilon^2) \tag{A.22}$$

which can be rearranged in the following form:

$$\mu=\phi\left[\frac{1}{\phi+(1-\phi)\left(\dfrac{\sigma_v^2+\sigma_\delta^2+\sigma_\varepsilon^2}{\sigma_v^2+\sigma_\varepsilon^2}\right)}\right]\leq\phi \tag{A.23}$$

Substituting for μ in (14) yields the desired result for P_0.

Q.E.D.

References

Amihud, Y., and H. Mendelson. "Asset Pricing and the Bid-ask Spread." *Journal of Financial Economics*, 17 (1986), 223-249.

Bhattacharya, U. and M. Spiegel. "Insiders, Outsiders and Market Breakdowns." *Review of Financial Studies,* 4 (1991), 255-282.

Brennan, M. and A. Subrahmanyam. "Market Microstructure and Asset Pricing: on the Compensation for Illiquidity in Stock Returns." *Journal of Financial Economics,* 41 (1996), 441-464.

Caballe, J.and M. Krishnan. "Imperfect Competition in a Multi-security Market with Risk Neutrality." *Econometrica* 62 (3) (1994), 695-704.

Chordia, T., R. Roll, and A. Subrahmanyam. "Commonality in Liquidity." *Journal of Financial Economics,* 56 (2000), 3-28.

Diamond, D. and P. Dybvig. "Bank Runs, Deposit Insurance and Liquidity." *Journal of Political Economy,* 91 (1983), 401-419.

Eleswarapu, V.R. and M.C. Reinganum. "The Seasonal Behavior of the Liquidity Premium in Asset Pricing." *Journal of Financial Economics,* 34 (1993), 373-386.

Gibson, R. and N. Mougeot. "The Pricing of Systematic Liquidity Risk: Empirical Evidence from the U.S. Stock Market." Working paper, University of Zurich (2000).

Glosten, L. "Insider Trading, Liquidity and the Role of the Monopolist Specialist." *Journal of Business,* 62 (1989), 211-235.

Grossman, S.J. and J.E. Stiglitz. "On the Impossibility of Informationally Efficient Markets." *American Economic Review*, 70 (1980), 393-408.

Hasbrouck, J. and D.J. Seppi. "Common Factors in Prices, Order Flows and Liquidity." *Journal of Financial Economics,* 59 (2001), 383-411.

Huberman, G. and D. Halka. "Systematic Liquidity." *Journal of Financial Research*, 24 (2001), 161-178.

Karpoff, J.M. "A Theory of Trading Volume." *Journal of Finance,* 41 (1986), 1069-1087.

Lo, A.W. and J. Wang. "Trading Volume: Definitions, Data Analysis, and Implications of Portfolio Theory." *Review of Financial Studies,* 13 (2000), 257-300.

Madhavan, A. “Trading Mechanisms in Securities Markets.” *Journal of Finance,* 47 (1992), 607-641.

Michaely, R. and J. Vila. “Trading Volume with Private Valuation: Evidence from the Ex-dividend Day.” *Review of Financial Studies,* 9 (1995), 471-510.

Michaely, R., J. Vila, and J. Wang. “A Model of Trading Volume with Tax-induced Heterogeneous Valuation and Transaction Costs.” *Journal of Financial Intermediation,* 5 (1996), 340-371.

Milgrom, P. and N. Stokey. “Information, Trade and Common Knowledge.” *Journal of Economic Theory,* 26 (1982), 17-27.

Securities and Exchange Commission. “Discounts Involved in Purchases of Common Stock.” Institutional Investor Study Report (1971).

Silber, W.L. “Discounts on Restricted Stock: the Impact of Illiquidity on Stock Prices.” *Financial Analysts Journal,* 1991, 60-64.

Spiegel, M. and A. Subrahmanyam. “Informed Speculation and Hedging in a Noncompetitive Securities Market.” *Review of Financial Studies,* 5 (1992), 307-330.

Tkac, P.A. “A Trading Volume Benchmark: Theory and Evidence.” *Journal of Financial and Quantitative Analysis,* 34 (1999), 89-114.

Tobin, J. “The Theory of Portfolio Selection.” in Hahn, F. and F. Brechling, (Eds.), The Theory *of Interest Rates*, MacMillan (1965).

Wruck, K.H. “Equity Ownership Concentration and Firm Value: Evidence from Private Equity Financings.” *Journal of Financial Economics,* 23 (1989), 3-28.