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WITTGENSTEIN'S ANALYSIS OF THE PARADOXES
IN HIS
LECTURES ON THE FOUNDATIONS
OF MATHEMATICS

Charles S. Chihara

LUDWIG Wittgenstein is still regarded in many circles as the greatest philosopher of the Twentieth Century. Hence, the publication of his 1939 lectures on the foundations of mathematics will undoubtedly be hailed by many as an important event, which makes a major contribution to the philosophical literature. From the point of view of Wittgensteinian scholarship, there can be little doubt that this publication is valuable, for the text was prepared by the editor from the copious notes of the lectures taken by four students and provides us with a detailed account of what Wittgenstein said and how he responded to questions and comments.

These lectures have been described by Malcolm and also by D. A. T. Gasking and A. C. Jackson; and it is enlightening to find out just what Wittgenstein said that so impressed these

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1This, at any rate, is an impression I got from attending the recent Wittgenstein conference in London, Ontario.

2Wittgenstein's Lectures on the Foundations of Mathematics: Cambridge, 1939, edited by Cora Diamond from the notes of R. G. Bosanquet, Norman Malcolm, Rush Rhees, and Yorick Smythies, Cornell University Press (Ithaca, 1976). In this paper, all page numbers inside parentheses, unless otherwise indicated, are references to the above work. As a rule, single quotation marks are used to indicate that what is being referred to is the expression appearing within the quotes. However, at one point in this paper, I follow the lead of Cora Diamond in using quotation marks to “mention” concepts. Double quotation marks are used as “scare quotes” and for quotation. The reader should note that my use of quotation marks does not coincide with Cora Diamond's (see footnote 13). Conventions governing the use of variables and logical constants are those of Benson Mates, Elementary Logic, 2n ed. (New York, 1972). See especially p. 26.


Thus, Gasking and Jackson wrote in their article that nearly "every single thing said [by Wittgenstein in his lectures] was easy to follow and was usually not the sort of thing anyone would wish to dispute" (p. 51). Having read these notes, I find it hard to understand how anyone who attended these lectures, as Gasking did, could have made that statement. (Wittgenstein is still regarded by many as a "common-sense philosopher," but even a superficial study of these lectures should dispel that conceit).

This publication is useful not so much because of the views expressed in them—students of *PC* and *RFM* will not be surprised by much of what Wittgenstein says—but because Wittgenstein's ideas on the foundations of mathematics are presented within the framework of a series of lectures, where there are interchanges with those in attendance and where Wittgenstein could make use of the reactions of his class to determine when to explain more fully obscure or difficult points.

I found especially interesting the interaction between Wittgenstein and the eponymous logician, Alan Turing, who seems to have been the only person attending the lectures willing or able to challenge him on any significant point. At times, Wittgenstein must have been hard pressed by the logician's challenges. In Lecture VI, for example, we find Wittgenstein trying to convince his class that a mathematical proof of the impossibility of constructing a heptagon by ruler and compass merely explains the use of the word 'analogous'. Turing is skeptical and suggests that when one asserts the impossibility of such a construction, one is not trying to convey information

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5 One gets the impression from Malcolm's memoir (*op. cit.*, pp. 23-4) that it was not so much what Wittgenstein said that so impressed him as how he said it (cf. *infra.*, fn. 7). Also, there seems to be some disagreement about whether Wittgenstein prepared his lectures. Malcolm thinks not (*op. cit.*, p. 24), whereas Gasking and Jackson write: "Each lecture was obviously carefully prepared—its general strategy planned and numerous examples thought up" (*op. cit.*, p. 52).

6 Here, as elsewhere in this paper, I abbreviate references to Wittgenstein's *Philosophical Grammar* (Oxford, 1974) to 'PC' and his *Remarks on the Foundations of Mathematics* (New York, 1956) to 'RFM'.

7 Cf. "I think that I understood almost nothing of the lectures, until I re-studied my notes approximately ten years later." Malcolm, *op. cit.*, p. 23.
about one’s use of the word ‘analogous.’ Wittgenstein replies:

You might say, “No, I’m not trying to convey this information. Mathematicians don’t even use the word ‘analogous’.”—But does one not prove that there is no analogue in the case of the heptagon?

A proof goes in fact step by step by means of analogy—by help of a paradigm [p. 62].

Later in the lecture, he attempts to clarify his position by asserting that such impossibility proofs, in effect, give new meanings to the word ‘analogous.’ The doctrine that mathematical proofs change the meanings of words—a view that I have discussed in some detail elsewhere 8—is an essential part of Wittgenstein’s ideas on mathematical proofs as expounded in PG and RFM. What is remarkable about the position taken in this lecture is this: Wittgenstein was not claiming then that as a result of seeing the proof, a person is made to change his use of such words as ‘heptagon’ and ‘can construct:’ it is the use of the word ‘analogous’ that supposedly is changed! Thus, of the case in which the mathematician proves that a heptadaicagon is constructible and the heptagon is not, Wittgenstein says:

We could say “he has been led to change his use of the word ‘analogous’.” And that is quite true.

At first when he says “analogous” he explains, “Look, this is the construction of the pentagon, and this is the construction of the hexagon; I want to do so-and-so.” But afterwards he would say, “This is the analogue in the case of the heptadaicagon to the construction of the pentagon.” He explains to us a new way of using “analogous” [p. 63-4].

Not surprisingly, Turing is skeptical: he responds that the

8 Charles S. Chihara, “Mathematical Discovery and Concept Formation,” *Philosophical Review* 72 (1963) 17-34. Cf. also C. S. Chihara and J. A. Fodor, “Operationalism and Ordinary Language: A Critique of Wittgenstein”, *American Philosophical Quarterly* 2 (1965) 281-295. The latter article puts forward an interpretation of Wittgenstein’s philosophy of language (according to which he accepted a sort of operationalistic theory of meaning) that receives new support from these lectures. For example, Wittgenstein says on p. 256: “It may seem queer that Euclidean geometry talks of ‘length’ and ‘equality of length’ and yet not of any method of comparing lengths. Especially since “this length is equal to that” changes its meaning when the method of comparison is changed. . . . But we could say that Euclidean geometry gives rules for the application of the words “length” and “equal length”, etc. Not all the rules, because some of these depend on how the lengths are measured and compared.”
mathematician who proves these theorems surely doesn’t invent what ‘analogous’ is to mean since we already know what the word means. Wittgenstein then says:

Yes, certainly it’s not a question merely of inventing what it is to mean. For if that were the problem, we could settle it much easier by making “analogous” mean “cushion”.

The point is indeed to give a new meaning to the word ‘analogous’. But it is not merely that; for one is responsible to certain things. The new meanings must be such that we who have had a certain training will find it useful in certain ways [p. 66].

Shortly thereafter, Wittgenstein asks Turing if he understands what is being claimed. Turing’s answer is sensible, direct, and, I would assume, honest. “I understand” he says, “but I don’t agree that it is simply a question of giving new meanings to words.” Wittgenstein has been quoted as saying about his lectures, “I shan’t say anything that you won’t all immediately agree with, and if you do dispute something I’ll drop it and go on to something else.”

Here, Turing was disputing a view that Wittgenstein had been attempting, with considerable force, to put across—a view that is central to Wittgenstein’s ideas on the nature of mathematics. How would Wittgenstein respond to this challenge? Would he drop this position and go on to something else? Wittgenstein’s recorded response is:

Turing doesn’t object to anything I say. He agrees with every word. He objects to the idea he thinks underlies it. He thinks we’re undermining mathematics, introducing Bolshevism into mathematics [p. 67].

At the time of these lectures, Turing was a respected logician, who had already published his important papers on computable

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9 Gasking and Jackson, op. cit., p. 51.

10 It might be suggested that Wittgenstein was quite right to say that Turing agrees with every word. After all, had not Wittgenstein also said that it is not merely a question of giving new meanings to words? Had he not maintained that the new meanings must be such that we find it useful in certain ways? But had Wittgenstein responded to Turing’s remark by pointing this out, would not Turing have responded with the words ‘I don’t agree that it is simply a question of giving new meanings to words in such a way that we find it useful in certain ways?’ It is possible that Wittgenstein’s comment about Bolshevism stems from a statement of P. F. Ramsey’s in “The Foundations of Mathematics,” in his The Foundations of Mathematics and Other Logical Essays, edited by R. B. Braithwaite (London, 1931), in which he speaks of preserving mathematics “from the Bolshevik menace of Brouwer and Weyl” (p. 56).
functions. That Wittgenstein would reply to him in this cavalier manner suggests to me that he had essentially run out of ideas on the point at issue.

I should now like to focus my discussion of these lectures on the semantical and set-theoretical paradoxes that Russell called “vicious circle paradoxes,” since Wittgenstein’s views on this topic have been sadly neglected. The topic is important, because much of the fruitful research on the foundations of mathematics in the first half of this century was stimulated or motivated by the goal of understanding or, at least, avoiding such paradoxes; and it is hard to imagine what form logic and set theory would have today had the paradoxes not been discovered and taken seriously.

Those familiar with *PG* or *RFM* will not be surprised at the drift of Wittgenstein’s lectures on these paradoxes. Wittgenstein pooh-poohs the concerns that the early logicists and formalists expressed about hidden contradictions in logical systems. He asks his class to suppose that Russell’s logic is used to draw conclusions. Would such a use be vitiated by the fact that a contradiction can be produced somewhere in the system? “And how would it be vitiated?” he asks (p. 209). Later, he gives the example of mathematicians who use what we would consider an inconsistent system of mathematics but who are unaware of the “hidden contradiction” in their system. Then after asking, “But what should we call the hidden contradiction?” Wittgenstein comments, “Is it hidden as long as it hasn’t been noticed? Then as long as it is hidden, I say that it’s as good as gold. And when it comes out in the open it can do no harm” (p. 219).

Now on the face of it, the views Wittgenstein puts forward here are absurd. So one would expect some rather strong reasons for accepting them. But if one is looking for rigorous argumentation to support this position, one will be disappointed. Wittgenstein’s reasons for thinking that a system of logic or mathematics is “as good as gold” so long as the contradiction is hidden are brought out by examples. Thus, on p. 221, we find him developing the following examples. Imagine that a prison is built with the aim of keeping all the prisoners apart. The rooms
corridors are so designed that each prisoner can walk along certain corridors and into certain rooms, but no two prisoners can ever meet. It turns out, however, that whenever two corridors meet at right angles, it never occurs to the prisoners to turn right and they always go straight ahead. But were a prisoner to turn right at some of these intersections, he would be able to get into the room of another prisoner. Since this possibility does not strike any of the prisoners, the prison “functions as good as gold” (p. 221).

Another example clarifies the previous point and also illustrates Wittgenstein’s reason for believing that the inconsistency of the system “can do no harm” when it is uncovered. He asks us to imagine a country with an inconsistent set of statutes. One statute requires the vice-president to sit next to the president on feast days, and other requires him to always sit between two ladies. This inconsistency is unnoticed for some time, because the vice-president is ill on feast days. But on one feast, he is not ill. What are we to do?

I may say, “We must get rid of this contradiction.” All right, but does that vitiate what we did before? Not at all.

Or suppose that we always acted according to the first rule: he is always put next to the president, and we never notice the other rule. That is all right; the contradiction does not do any harm.

When a contradiction appears, then there is time to eliminate it. We may even put a ring round the second rule and say, “This is obsolete” [p. 210].

It can be seen that Wittgenstein’s main idea is this: It is silly for the foundationalists to be worried about hidden inconsistencies in our mathematical and logical system, since so long as no contradiction has been found, the system is perfectly all right; and if an inconsistency is uncovered, “it can do no harm” since it is a simple matter to alter the rules so as to prevent us from drawing any unwanted conclusions. Thus, in reply to the statement “From a contradiction everything would follow” Wittgenstein says:

Well then, don’t draw any conclusions from a contradiction; make that a rule. You might put it: There is always time to deal with a contradiction when we get to it. When we get to it, shouldn’t we simply say, “This is no use—and we won't draw any conclusions from it?” [p. 209].
Lest it be thought that Wittgenstein put forward these amazing views without a period of careful deliberation and assessment, it should be mentioned that he had written much the same thing many years earlier in PG, as the following quotation shows:

Something tells me that a contradiction in the axioms of a system can’t really do any harm until it is revealed. We think of a hidden contradiction as like a hidden illness which does harm even though (and perhaps precisely because) it doesn’t show itself in an obvious way. But two rules in a game which in a particular instance contradict each other are perfectly in order until the case turns up, and it’s only then that it becomes necessary to make a decision between them by a further rule [p. 303].

Applying these ideas specifically to the contradiction Russell discovered in Frege’s system, Wittgenstein suggests that there is no real problem for the Fregean since he can simply add to the rules one that says, “Don’t draw any conclusions from any self-contradiction, i.e. from a sentence of the form ‘P & -P’.” As he puts it:

You might get P & -P by means of Frege’s system. If you can draw any conclusion you like from it [i.e. from P & -P], then that, as far as I can see, is all the trouble you can get into. And I would say, “Well then, just don’t draw any conclusions from a contradiction” [p. 220].

In the usual formal axiomatic systems, any previously proved theorem may be inserted at any point in a derivation of the system; so if φ & -φ were proved, one could then derive any sentence one wanted from the axioms. And this, Wittgenstein suggests in the above quotation, is all the trouble one can get into from the contradiction. He then suggests that we can preclude getting this unwanted result by simply not inferring anything from a contradiction. As he said in the quotation I gave earlier: “make that a rule.”

Turing replies: “But that would not be enough. For if one made that rule, one could get round it and get any conclusion which one liked without actually going through the contradiction” (p. 220). The point Turing is making is an elementary one, for what Russell showed was that there is a sentence φ of Frege’s logical theory such that:

(1) φ is derivable from Σ (the axioms of the system); and

(2) -φ is derivable from Σ;
and hence

\[ (3) \, \phi \& \neg \phi \text{ is derivable from } \Sigma. \]

But one cannot prevent the derivation of unwanted conclusions from the axioms by simply barring inferences from self-contradictory sentences, since by the rules of the propositional calculus we can get:

\[ (4) \, \neg \phi \rightarrow \psi \text{ is derivable from } \Sigma \text{ (from (1))}; \]

So

\[ (5) \, \psi \text{ is derivable from } \Sigma \text{ (from (2) and (4))}. \]

Thus, for any sentence \( \psi \), we would be able to derive \( \psi \) from \( \Sigma \), and without inferring anything from the conjunction \( \phi \& \neg \phi \).

Furthermore, even expanding the set of sentences from which no inferences are to be drawn to include both \( \phi \) and \( \neg \phi \), in addition to \( \phi \& \neg \phi \), will not preclude obtaining unwanted conclusions. Once one understands Russell's paradox, it is easy to construct indefinitely many paradox-producing sentences (and it is not necessary to derive an explicitly self-contradictory sentence with them in order to generate bizarre results). One can also construct sentences that, so far as one can tell at the time, are likely to be paradox-producing—others that are quite possibly paradox-producing.

Turing's point was a simple one, intelligible, one would think, to anyone with an understanding of elementary logic; yet there are reasons for thinking that Wittgenstein failed to grasp it. In the first place, Wittgenstein made no attempt to answer it. Turing's criticism comes at the very end of a lecture, and Wittgenstein begins the next lecture with the words: "We were in a mess at the end of last time and we shall probably get into the same mess again today" (p. 220). But he doesn't explain what the mess was or even try to show that Turing was mistaken in any way. He then goes on to express views in the lecture that can be criticized in the way Turing objected to Wittgenstein earlier. He says, for example:

\[ ^{12} \text{It should be noted that even in } RFM, \text{ we find Wittgenstein thinking along similar lines. For example, of } \text{the Russelian antinomy in Frege's system, he says: } \text{"And suppose the contradiction had been discovered but we were not excited about it, and had settled } \text{e.g. that no conclusions were to be drawn from it. (As no one does draw conclusions from the 'Liar'.) Would this have been an obvious mistake?" (p. 170)} \]
We have seen that if we didn’t recognize a contradiction, or if we allowed a contradiction but, for example, did not draw any further conclusions from it, we could not then say we must come into conflict with any facts [p. 230, italics mine].

In another place, he returns to a discussion of Frege’s logical system, saying:

Why should you say even when the contradiction is discovered, “Now everything is wrong”? Not even the contradiction itself is wrong, or a false mathematical proposition. The only point would be: how to avoid going through the contradiction unawares. [p. 227].

Turing tries again to get Wittgenstein to see his previous point, by returning to the prison example. He asks Wittgenstein to suppose that there is a circus off one of the main corridors of the prison. When the prisoners find that they can turn right, they get into this circus. But, according to Turing, getting into the circus is only a “symptom” of the real difficulty: it is the turning right that is the real problem. “And one cannot get rid of the trouble simply by barring the circus” he says.

At this point, Wittgenstein decides to counterattack not the main point of whether one could eliminate the trouble by “barring the circus,” but Turing’s development of the prison example, suggesting that the case is not appropriately analogous to that of Frege’s logic: he wished to counteract the suggestion that Russell’s paradox showed that one could prove anything in Frege’s system, saying “It isn’t actually that people went through doors into places from which they could go any damn where. It isn’t true that this happened with Frege’s logic” (p. 228). And shortly after, he goes on to say:

He [Frege] was led by the normal rules of logic: the rules of such words as “and”, “not”, “implies”, and so on. He was led also by our normal use of words. As we never ask whether “Fox” is a fox or “predicate” is a predicate, the question didn’t arise and he never got into trouble. . . .

So it is not quite right to say “Frege might have proved anything else.” And this is shown by the fact that Russell, almost immediately on finding the contradiction, found a remedy in the theory of types: “We would never say ‘Fox’ is a fox; so eliminate that” [p. 229].

Notice that Wittgenstein is here suggesting that Russell was able to generate his contradictions in Frege’s system by asking such questions as ‘Is ‘fox’ a fox?’ (he is more explicit on p. 222 where he
describes Russell’s paradox as arising from asking of the predicate “predicate that does not apply to itself” whether it applies to itself; and that Russell was able to fix up the system “almost immediately” by eliminating such questions. There is also a third idea, only hinted at here, that Russell generated his paradox by asking a perverse, unnatural, question. This idea is more clearly set out in another place where he says:

Take Russell’s contradiction: There are concepts which we call predicates—“Man”, “chair”, and “wolf” are predicates, but “Jack” and “John” are not. Some predicates apply to themselves and others don’t. For instance “chair” is not a chair, “wolf” is not a wolf, but “predicate” is a predicate.

You might say this is bosh. And in a sense it is. No one says “‘Wolf’ isn’t a wolf.” We don’t know what it means. Is “Wolf” a name?—in that case Wolf may be a wolf. If someone asked, “Is ‘wolf’ a wolf?”, we simply would not know what to answer [p. 222].

It is important to notice that quotation marks are here being used in two distinct ways: (1) they are used to indicate that the word inside the quotation marks is being referred to; (2) they are used to indicate that the concept expressed by (or denoted by) the word inside the quotation marks is being referred to. But this second use of quotation marks may not have been due to Wittgenstein at all.13 Probably, the above quotation is the result of a purely oral presentation, where an utterance of ‘Wolf is a wolf’ would be indistinguishable from one of ‘Wolf is a wolf,’ unless the utterer signaled the quotation marks in some way or other. And it is quite possible that Wittgenstein was, himself, confused by the above fact. Thus, if the word ‘Wolf’ is a name, say, of John’s pet wolf, then Wolf is a wolf. But it does not follow from this that the concept ‘wolf’ is a wolf or that the word ‘wolf’

13 Indeed the conventions for the use of quotation marks Cora Diamond seems to be following in this work are by no means obvious. She claims in the preface that she is just following the conventions of RFM. But what are they? Given the nature of these lectures, an explicit statement of these conventions is surely called for. I recently received a letter from Professor Diamond informing me that she had planned to include in her preface a description of the conventions she was following, but that she changed her mind when she realized that such a discussion would require a considerable amount of space. Evidently, her main deviation from standard conventions results from an attempt to use single quotation marks as “idea quotes.” Unfortunately, I am not in a position to explain what “idea quotes” are.
is a wolf. To suggest otherwise (as evidently Wittgenstein did) is to invite even more confusion than was already present in the class at that point in the lecture (see p. 223).

Now is it true that Russell “almost immediately” found a way of fixing up Frege’s system via his theory of types? Well, if Principia Mathematica is supposed to be a fixed up version of Frege’s system, then we can see that it took Russell many years of very intensive work, in collaboration with Whitehead, to come up with his “remedy.”14 Furthermore, it is grossly inaccurate to assert that one gets Russell’s extremely complicated ramified theory of types from Frege’s system by merely banning such questions as ‘Is ‘fox’ a fox?’ (Indeed, had such a trivial “remedy” as Wittgenstein suggests been available, is it likely that Frege, himself, would have failed to find it?)15 For one thing, it is not true that the contradiction Russell produced in Frege’s system was generated by asking such questions as ‘Is ‘wolf’ a wolf?’ In his famous letter to Frege,16 Russell did sketch derivations of two contradictions, the first of which was formulated in terms of a predicate $\beta$ that is to apply to all predicates that do not apply to themselves; and he did think a contradiction would result from asking if $\beta$ applies to itself. As essentially the same paradox is described by Russell in his Principles,17 Wittgenstein may have been led to think that the contradiction Russell constructed in Frege’s system was due to asking of a predicate whether it applies to itself. But as Frege pointed out in his reply to Russell, a predicate for him is, as a rule, a first-level function of a certain sort (a first-level concept) which cannot meaningfully take itself as an argument.18 In Frege’s system, it makes no sense to say of any first-level concept that it falls under a first-level concept. So one cannot generate a contradiction in the first way

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14 For a detailed discussion of Russell’s solution to the paradoxes and his Theory of Types, see my Ontology and the Vicious-Circle Principle (Ithaca, 1973), Chapter 1.

15 To see how Frege explored possible remedies to Russell’s paradox, the reader should study the appendix to Frege’s The Basic Laws of Arithmetic, translated and edited by Montgomery Furth (Berkeley, 1964).


18 Jean van Heijenoort, op. cit., p. 128.
WITTGENSTEIN ON THE PARADOXES

suggested by Russell. As Frege’s rules already preclude asking such questions as ‘Does β apply to β?’ or ‘Does β fall under β?’ one could not fix up Frege’s system by banning such questions. Wittgenstein was quite confused in thinking otherwise.

It is well known that Frege’s system is a sort of type theory: concepts, for example, are stratified into types in such a way that a concept is always of a higher level than that of its possible arguments. Russell was able to generate his paradox in Frege’s system because all extensions of concepts are regarded as “objects” and as such belong to one and the same type. If one wishes to regard Russell’s theory of types as a sort of “remedy” for the paradoxes, then one essential part of the remedy is this: in Russell’s system, classes or extensions of concepts are regarded as “logical fictions”: strictly speaking, there are no classes in the ontology of Principia Mathematica, and all talk of classes is translated into talk about propositional functions. And although this basic idea is simple enough, carrying out the details of the idea is no simple task. So Wittgenstein’s conception of Russell’s solution to the paradoxes is both superficial and erroneous.

Let us now reconsider Wittgenstein’s view that an inconsistent logical or mathematical system is “as good as gold” so long as the inconsistency is not noticed. Imagine that the scientists of some nation make use of a formal first-order language—for concreteness, let us say that the language is Mates’ $\mathcal{L}$—and imagine that they formulate some of their scientific theories and even their experimental data in $\mathcal{L}$ whenever possible. Suppose now that their logicians develop the following logical system of inference rules:

P: Any sentence may be entered on a line, with the line number taken as the only premise-number.

T: Any sentence may be entered on a line if it is a tautological consequence of a set of sentences that appear on earlier lines; as premise-numbers of the new line take all premise numbers of those earlier lines.

C: The sentence $(\phi \rightarrow \psi)$ may be entered on a line if $\psi$ appears
on an earlier line; as premise-numbers of the new line take all those of the earlier line, with the exception (if desired) of any that is the premise number of a line on which $\phi$ appears.

US: The sentence $\phi\alpha/\beta$ may be entered on a line if (a)$\phi$ appears on an earlier line; as premise-numbers of the new line take those of the earlier line.

UG: The sentence $(\alpha)\phi$ may be entered on a line if $\phi\alpha/\beta$ appears on an earlier line and $\beta$ occurs neither in $\phi$ nor in any premise of that earlier line; as premise-numbers of the new line take those of the earlier line.

E: The sentence $(\exists \alpha)\phi$ may be entered on a line if $-(\alpha)-\phi$ appears on an earlier line, or vice versa; as premise-numbers of the new line take those of the earlier line.

This system is inconsistent, that is, '$P & -P$ is derivable from the empty set of premises and hence is a theorem of this logic. But these rules are so close to Mates' sound rules that one could easily carry out indefinitely many derivations, getting correct results (that is, deriving sentences that are indeed consequences of their premises) without ever deriving a contradiction from the empty set of premises. So we can easily imagine that the scientists of our example fail to notice the inconsistency of their rules.

Now is this inconsistent system of logic "as good as gold"? Is it as good as any of the sound systems we in fact use? Surely not. A sound system will carry one from truths to truths; the above will not. And the fact that the inconsistency is not noticed will not make it right, any more than not noticing that a person has cancer will make him well. Indeed, it is not hard to see how, by relying on such a system in reasoning about, say, the number of steel beams of such and such tested strength needed in a bridge to support a load of $N$ tons, a disaster could result. Here, we could imagine the engineers working with such a large number of premises and carrying out such an intricate chain of inferences

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22 *Ibid.* pp. 112–3. Mates' system is, of course, a natural deduction system (as is the inconsistent system presented in the above example), unlike the logical systems Wittgenstein explicitly discusses in the lectures in connection with "hidden contradictions." But clearly the main point of the above example does not depend on the fact that the inconsistent predicate logic is a natural deduction system.
that a computer is used to check their work. If we imagine that they carry out their inferences rather mechanically, following set routines in accordance with general rules of strategy (as many students in logic courses do), it is not hard to see how they could start with true premises and end with false conclusions without noticing anything wrong with their logical system (especially when a computer is checking all their work).

The above example is relevant to another strange view advanced by Wittgenstein. In response to Turing's assertion that one could not be confident about applying a system of logic or mathematics until one assured oneself that there was no hidden contradiction in it, because a bridge might collapse or some such thing, Wittgenstein says:

Now it does not sound quite right to say that a bridge might fall down because of a contradiction. We have an idea of the sort of mistake which would lead to a bridge falling.

(a) We've got hold of a wrong natural law—a wrong coefficient.
(b) There has been a mistake in calculation—someone has multiplied wrongly.

The first case obviously has nothing to do with having a contradiction; and the second is not quite clear [p. 211].

Later, Wittgenstein returns to the same point, saying: “I'd say things can go wrong in only two ways: either the bridge breaks down or you have made a mistake in your calculation—for example, you multiplied wrongly. But you seem to think that there may be a third thing wrong: the calculus is wrong” (p. 218). And in another place, we find him considering the question “Why are people afraid of contradictions?”

Turing says, “Because something may go wrong with the applications.” But nothing need go wrong. And if something does go wrong—if the bridge breaks down—then your mistake was of the kind of using a wrong natural law (p. 217).

Suppose that the bridge the engineers design in our example subsequently collapses. Surely, we could distinguish at least three different possible explanations for the disaster: (1) the empirical theories and data the engineers relied upon were inaccurate or incorrect; (2) they made mistakes in calculation or didn’t follow their rules of derivation correctly; (3) the logical system they used was unsound and led them to make invalid inferences (that is,
they followed the rules of derivation correctly, but their calculus was wrong). In the above passages, Wittgenstein seems to be denying this third possibility. But as our example brings out, the collapse of the bridge need not be due to a faulty empirical theory or bad data. In fact, as I have described the situation, if the engineers were to recheck their data and retest their empirical theories, they would find everything in order. Hopefully, there would be some non-Wittgensteinian logicians around to discover the unsoundness of their logical system.

Not surprisingly, Wittgenstein’s attitude towards such semantical paradoxes as The Liar is closely related to his view of contradictions in logical and mathematical systems. In one place, he expresses the belief that it is “queer” that The Liar should have puzzled anyone; and he goes on to say:

Now suppose a man says “I am lying” and I say “Therefore you are not, therefore you are, therefore you are not...” What is wrong? Nothing. Except that it is of no use; it is just a useless language-game, and why should anybody be excited? [p. 207].

To Turing, who suggests that people are puzzled by The Liar because one usually takes the derivation of a self-contradictory sentence from apparently true propositions as a sign that one has done something wrong, Wittgenstein replies: “Yes—and more: nothing has been done wrong” (p. 207).

But is it true that nothing has been done wrong? Suppose we examine a slightly more rigorous version of The Liar. On page 65 of the June 1969 issue of Scientific American, one will find exactly one sentence $\phi$ printed in red, and $\phi$ says: “The sentence printed in red on page 65 of the June 1969 issue of Scientific American is false.” By one form of the law of excluded middle, either $\phi$ is true or $\phi$ is false. Assume the former. Then, since $\phi$ is the very sentence said to be false, we can conclude that $\phi$ is false. But if $\phi$ is false, it follows that the sentence in red on page 65 of the June 1969 issue of Scientific American is not false; hence $\phi$ is both true and false. But how can that be? Surely, we have done something wrong: we have either accepted a false premise or used an unsound rule of inference. What is so puzzling about The Liar is the fact that the argument is so simple and tight. Putting one’s finger on the wrong move is extremely difficult, as practically everyone who has
thought deeply about the paradox will agree. After all, The Liar has remained a paradox without a generally accepted solution for approximately two thousand years, despite strenuous attempts at solving it by some of the very best minds in logic and philosophy. Wittgenstein's hasty dismissal of this ancient and venerable problem is, in my opinion, neither well-reasoned nor insightful.²³

The rationality of Wittgenstein's attitude toward the paradox can be judged more easily, I believe, by returning to the example of the scientists who use an inconsistent logical system. Suppose that one day someone produces a derivation 'P & -P' from a tautology. As in the case of the paradoxes, traditional logicians would see the derivation of a contradiction from a tautology as a symptom that something is wrong with the system—as showing, in fact, that the rules cannot be sound as they had assumed. They would then investigate the rules in an attempt to find out what specifically is wrong. On the other hand, as we have seen from his analysis of The Liar, Wittgenstein thinks this traditional attitude is misguided. He says in an earlier lecture:

One might say, “Finding a contradiction in a system, like finding a germ in an otherwise healthy body, shows that the whole system or body is diseased.”—Not at all. The contradiction does not even falsify anything. Let it lie. Do not go there [p. 138].

Later, he reiterates his position that “it is vitally important to see that a contradiction is not a germ which shows a general illness” (p. 211). Which of these positions, the logician's or Wittgenstein's, is the more rational, I leave to the reader's judgment.

Cora Diamond should be thanked for both doing such a good job of editing (this version is incomparably better than one I read many years ago) and also making these lectures available. It is to be hoped that other lectures of Wittgenstein's that have been

²³ It is ironic that Russell's work on the paradoxes, of which Wittgenstein is so critical, has turned out to be very fruitful, generating much significant research in logic and mathematics. Indeed, I believe that Russell's idea (borrowed from Poincaré) that the paradoxes are due to viciously circular definitions is essentially on the right track, at least, in the case of the semantical paradoxes. See in this regard, my "A Diagnosis of the Liar and Other Semantical Vicious-Circle Paradoxes", forthcoming in G. Roberts (ed.), Bertrand Russell: The Memorial Volumes, Volume I.
circulated, such as his lectures on philosophical psychology, will also be published. As more and more of Wittgenstein's thoughts are made readily available to all philosophers and not just a privileged few, we can expect more realistic appraisals of his contributions to philosophy to emerge than have been given thus far.\(^{24}\)

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\(^{24}\) Thanks are due to Barry Stroud, David Shwayder and Hans Sluga for their comments on an earlier version of this paper. The latter two suggested ways in which the view I attribute to Wittgenstein regarding "hidden contradictions" might be defended or at least made more plausible. I certainly agree with them that much more than I have indicated in this paper can be said to make Wittgenstein's position appear more reasonable, especially by connecting it with his constructivistic views of mathematics. However, lack of space prevents me from going into this more deeply. Besides, as I am convinced that Wittgenstein's position is wrong-headed, it would be best to leave Wittgenstein's defense to others more sympathetic to the view.