

CHAPTER 2

LAGRANGIAN AND EULERIAN FINITE ELEMENTS IN ONE DIMENSION

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2.1 Introduction

In this chapter, the equations for one-dimensional models of nonlinear continua are described and the corresponding finite element equations are developed. The development is restricted to one dimension to simplify the mathematics so that the salient features of Lagrangian and Eulerian formulations can be demonstrated easily. These developments are applicable to nonlinear rods and one-dimensional phenomena in continua, including fluid flow. Both Lagrangian and Eulerian meshes will be considered. Two commonly used types of Lagrangian formulations will be described: updated Lagrangian and total Lagrangian. In the former, the variables are expressed in the current (or updated) configuration, whereas in the latter the variables are expressed in terms of the initial configuration. It will be seen that a variety of descriptions can be developed for large deformation problems. The appropriate description depends on the characteristics of the problem to be solved.

In addition to describing the several types of finite element formulations for nonlinear problems, this Chapter reviews some of the concepts of finite element discretization and finite element procedures. These include the weak and strong forms, the operations of assembly, gather and scatter, and the imposition of essential boundary conditions and initial conditions. Mappings between different coordinate systems are discussed along with the need for finite element mappings to be one-to-one and onto. Continuity requirements of solutions and finite element approximations are also considered. While much of this material is familiar to most who have studied linear finite elements, they are advised to at least skim this Chapter to refresh their understanding.

In solid mechanics, Lagrangian meshes are most popular. Their attractiveness stems from the ease with which they handle complicated boundaries and their ability to follow material points, so that history dependent materials can be treated accurately. In the development of Lagrangian finite elements, two approaches are commonly taken:

1. formulations in terms of the Lagrangian measures of stress and strain in which derivatives and integrals are taken with respect to the Lagrangian (material) coordinates X , called *total Lagrangian formulations*
2. formulations expressed in terms of Eulerian measures of stress and strain in which derivatives and integrals are taken with respect to the Eulerian (spatial) coordinates x , often called *updated Lagrangian formulations*.

Both formulations employ a Lagrangian mesh, which is reflected in the term *Lagrangian* in the names.

Although the total and updated Lagrangian formulations are superficially quite different, it will be shown that the underlying mechanics of the two formulations is identical; furthermore, expressions in the total Lagrangian formulation can be transformed to updated Lagrangian expressions and vice versa. The major difference between the two

formulations is in the point of view: the total Lagrangian formulation refers quantities to the original configuration, the updated Lagrangian formulation to the current configuration, often called the deformed configuration. There are also differences in the stress and deformation measures which are typically used in these two formulations. For example, the total Lagrangian formulation customarily uses a total measure of strain, whereas the updated Lagrangian formulation often uses a rate measure of strain. However these are not inherent characteristics of the formulations, for it is possible to use total measures of strain in updated Lagrangian formulations, and rate measures in total Lagrangian formulation. These attributes of the two Lagrangian formulations are discussed further in Chapter 4.

Until recently, Eulerian meshes have not been used much in solid mechanics. Eulerian meshes are most appealing in problems with very large deformations. Their advantage in these problems is a consequence of the fact that Eulerian elements do not deform with the material. Therefore, regardless of the magnitudes of the deformation in a process, Eulerian elements retain their original shape. Eulerian elements are particularly useful in modeling many manufacturing processes, where very large deformations are often encountered.

For each of the formulations, a weak form of the momentum equation, which is known as the principle of virtual work (or virtual power) will be developed. The weak form is developed by taking the product of a test function with the governing partial differential equation, the momentum equation. The integration is performed over the material coordinates for the total Lagrangian formulation, over the spatial coordinates for the Eulerian and updated Lagrangian formulation. It will also be shown how the traction boundary conditions are treated so that the approximate (trial) solutions need not satisfy these boundary conditions exactly. This procedure is identical to that in linear finite element analysis. The major difference in geometrically nonlinear formulations is the need to define the coordinates over which the integrals are evaluated and to specify the choice of stress and strain measures.

The discrete equations for a finite element approximation will then be derived. For problems in which the accelerations are important (often called dynamic problems) or those involving rate-dependent materials, the resulting discrete finite element equations are ordinary differential equations (ODEs). The process of discretizing in space is called a semidiscretization since the finite element procedure only converts the spatial differential operators to discrete form; the derivatives in time are not discretized. For static problems with rate-independent materials, the discrete equations are independent of time, so the finite element discretization results in a set of nonlinear algebraic equations.

Examples of the total and updated Lagrangian formulations are given for the 2-node, linear displacement and 3-node, quadratic displacement elements. Finally, to enable the student to solve some nonlinear problems, a central difference explicit time-integration procedures is described.

2.2 Governing Equations For Total Lagrangian Formulation

Nomenclature. Consider the rod shown in Fig. 1. The *initial configuration*, also called the *undeformed configuration* of the rod, is shown in the top of the figure. This configuration plays an important role in the large deformation analysis of solids. It is also called the *reference configuration*, since all equations in the total Lagrangian formulation are referred to this configuration. The *current or deformed configuration* is shown at the bottom of the figure. The spatial (Eulerian) coordinate is denoted by x and the coordinates in the reference configuration, or material (Lagrangian) coordinates, by X . The initial cross-sectional area of the rod is denoted by $A_0(X)$ and its initial density by $\rho_0(X)$;

variables pertaining to the reference (initial, undeformed) configuration will always be identified by a subscript or superscript nought. In this convention, we could indicate the material coordinates by x_0 since they correspond to the initial coordinates of the material points, but this is not consistent with most of the continuum mechanics literature, so we will always use X for the material coordinates.

The cross-sectional area in the deformed state is denoted by $A(X, t)$; as indicated, it is a function of space and time. The spatial dependence of this variable and all others is expressed in terms of the material coordinates. The density is denoted by $\rho(X, t)$ and the displacement by $u(X, t)$. The boundary points in the reference configuration are X_a and X_b .

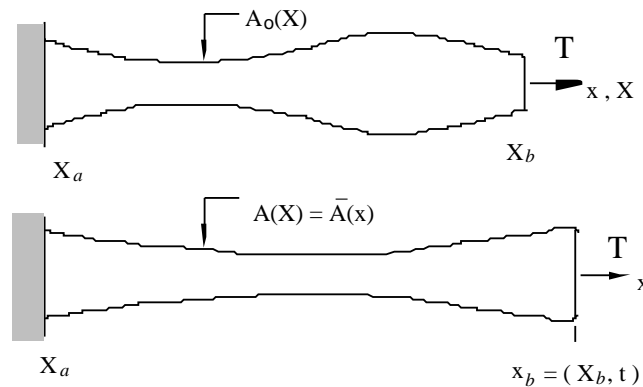


Fig. 1.1. The undeformed (reference) configuration and deformed (current) configurations for a one-dimensional rod loaded at the left end; this is the model problem for Sections 2.2 to 2.8.

Deformation and Strain Measure. The variables which specify the deformation and the stress in the body will first be described. The motion of the body is described by a function of the Lagrangian coordinates and time which specifies the position of each material point as a function of time:

$$x = (X, t) \quad X \in [X_a, X_b] \quad (2.2.1)$$

where (X, t) is called a deformation function. This function is often called a map between the initial and current domains. The material coordinates are given by the deformation function at time $t = 0$, so

$$X = (X, 0) \quad (2.2.2)$$

As can be seen from the above, the deformation function at $t = 0$ is the identity map.

The displacement $u(X, t)$ is given by the difference between the current position and the original position of a material point, so

$$u(X, t) = (X, t) - X \quad \text{or} \quad u = x - X \quad (2.2.3)$$

The deformation gradient is defined by

$$F = \frac{x}{X} = \frac{\partial x}{\partial X} \quad (2.2.4)$$

The second definitions in Eq. (2.2.3) and (2.2.4) can at times be ambiguous. For example, Eq. (2.2.4) appears to involve the partial derivative of an independent variable x with respect to another independent variable X , which is meaningless. Therefore, it should be understood that whenever x appears in a context that implies it is a function, the definition $x = x(X, t)$ is implied.

Let J be the Jacobian between the current and reference configurations. The Jacobian is usually defined by $J(x(X)) = dx/dX$ for one-dimensional mappings; however, to maintain consistency with multi-dimensional formulations of continuum mechanics, we will define the Jacobian as the ratio of an infinitesimal volume in the deformed body, A , to the corresponding volume of the segment in the undeformed body A_0 , so it is given by

$$J = \frac{dx}{dX} \frac{A}{A_0} = \frac{F A}{A_0} \quad (2.2.5)$$

The deformation gradient F is an unusual measure of strain since its value is one when the body is undeformed. We will therefore define the measure of strain by

$$\epsilon(X, t) = F(X, t) - 1 = \frac{x}{X} - 1 = \frac{u}{X} \quad (2.2.6)$$

so that it vanishes in the undeformed configuration. There are many other measures of strain, but this is the most convenient for this presentation. This measure of strain corresponds to what is known as the stretch tensor in multi-dimensional problems. In one dimension, it is equivalent to the engineering strain.

Stress Measure. The stress measure which is used in total Lagrangian formulations does not correspond to the well known physical stress. To explain the measure of stress to be used, we will first define the physical stress, which is also known as the Cauchy stress. Let the total force across a given section be denoted by T and assume that the stress is constant across the cross-section. The Cauchy stress is given by

$$\sigma = \frac{T}{A} \quad (2.2.7)$$

This measure of stress refers to the *current* area A . In the total Lagrangian formulation, we will use the nominal stress. The nominal stress will be denoted by P and is given by

$$P = \frac{T}{A_0} \quad (2.2.8)$$

It can be seen that it differs from the physical stress in that the net resultant force is divided by the *initial, or undeformed, area* A_0 . This is equivalent to the definition of engineering strain; however, in multi-dimensions, the nominal stress is not equivalent to the engineering stress, this is discussed further in Chapter 3.

Comparing Eqs. (2.2.7) and (2.2.8), it can be seen that the physical and nominal stresses are related by

$$= \frac{A_0}{A} P \qquad P = \frac{A}{A_0} \qquad (2.2.9)$$

Therefore, if one of the stresses is known, the other can always be computed if the current and initial cross-sectional areas are known.

Governing Equations. The nonlinear rod is governed by the following equations:

1. conservation of mass;
2. conservation of momentum;
3. conservation of energy;
4. a measure of deformation, often called a strain-displacement equation;
5. a constitutive equation, which describes material behavior and relates stress to a measure of deformation;

In addition, we require the deformation to be continuous, which is often called a compatibility requirement. The governing equations and initial and boundary conditions are summarized in Box 1.

Conservation of mass. The equation for conservation of mass for a Lagrangian formulation can be written as (see Appendix A for an engineering derivation):

$$J = {}_0J_0 \quad \text{or} \quad (X, t)J(X, t) = {}_0(X)J_0(X) \qquad (2.2.10)$$

where the second expression is given to emphasize that the variables are treated as functions of the Lagrangian coordinates. Conservation of matter is an algebraic equation only when expressed in terms of material coordinates. Otherwise, it is a partial differential equation. For the rod, we can use Eq. (2.2.4) to write Eq. (2.2.5) as

$$FA = {}_0A_0 \qquad (2.2.11)$$

where we have used the fact that $J_0 = 1$.

Conservation of momentum. Conservation of momentum is written in terms of the nominal stress and the Lagrangian coordinates as (a derivation is given in Appendix A):

$$(A_0P)_{,X} + {}_0A_0b = {}_0A_0\ddot{u} \qquad (2.2.12)$$

where the superposed dots denote the material time derivative. The material time derivative of the velocity, the acceleration, is written as D^2u/Dt^2 . The subscript following a comma denotes partial differentiation with respect to that variable, i.e.

$$P(X, t)_{,X} = \frac{P(X, t)}{X} \qquad (2.2.13)$$

Equation (2.2.12) is called the *momentum equation*, since it represents conservation of momentum. If the initial cross-sectional area is constant in space, the momentum equation becomes

$$P_{,X} + {}_0b = {}_0\ddot{u} \quad (2.2.14)$$

Equilibrium Equation. When the inertial term ${}_0\ddot{u}$ vanishes, i.e. when the problem is static, the momentum equation becomes the *equilibrium equation*

$$({}_0P)_{,X} + {}_0A_0b = 0 \quad (2.2.15)$$

Solutions of the equilibrium equations are called equilibrium solutions. Some authors call the momentum equation an equilibrium equation regardless of whether the inertial term is negligible; since equilibrium usually connotes a body at rest or moving with constant velocity, this nomenclature is avoided here.

Energy Conservation. The energy conservation equation for a rod of constant area is given by

$${}_0\dot{w}^{int} = \dot{F}P - q_{x,x} + {}_0s \quad (2.2.16)$$

where q_x is the heat flux, s is the heat source per unit mass and \dot{w}^{int} is the rate of change of internal energy per unit mass. In the absence of heat conduction or heat sources, the energy equation gives

$${}_0\dot{w}^{int} = \dot{F}P \quad (2.2.17)$$

which shows that the internal work is given by the product of the rate of the deformation F and the nominal stress P . The energy conservation equation is not needed for the treatment of isothermal, adiabatic processes.

Constitutive Equations. The constitutive equations reflect the stresses which are generated in the material as a response to deformation. The constitutive equations relate the stress to the measures of strain at a material point. The constitutive equation can be written either in total form, which relates the current stress to the current deformation

$$P(X,t) = S^{PF} \left(F(X,\bar{t}), \dot{F}(X,\bar{t}), \text{etc.}, \bar{t} - t \right) \quad (2.2.18)$$

or in rate form

$$\dot{P}(X,t) = S_t^{PF} \left(\dot{F}(X,\bar{t}), F(x,\bar{t}), P(X,\bar{t}), \text{etc.}, \bar{t} - t \right) \quad (2.2.19)$$

Here S^{PF} and S_t^{PF} are functions of the deformation which give the stress and stress rate, respectively. The superscripts are here appended to the constitutive functions to indicate which measures of stress and strain they relate.

As indicated in Eq. (2.2.18), the stress can depend on both F and \dot{F} and on other state variables, such as temperature, porosity; “etc.” refers to these additional variables which can influence the stress. The prior history of deformation can also affect the stress, as in an elastic-plastic material; this is explicitly indicated in Eqs. (2.2.18-2.2.19) by letting the constitutive functions depend on deformations for all time prior to t . The constitutive equation of a solid is expressed in material coordinates because the stress in a solid usually

depends on the history of deformation at that material point. For example, in an elastic solid, the stress depends on strain at the material point. If there are any residual stresses, these stresses are frozen into the material and move with the material point. Therefore, constitutive equations with history dependence should track material points and are written in terms of the material coordinates. When a constitutive equation for a history dependent material is written in terms of Eulerian coordinates, the motion of the point must be accounted for in the evaluation of the stresses, which will be discussed in Chapter 7.

The above functions should be continuous functions of the independent variables. Preferably they should be continuously differentiable, for otherwise the stress is less smooth than the displacements, which can cause difficulties.

Examples of constitutive equations are:

(a) linear elastic material:

$$\text{total form: } P(X, t) = E^{PF} (F(X, t) - 1) \quad (2.2.20)$$

$$\text{rate form: } \dot{P}(X, t) = E^{PF} \dot{F}(X, t) \quad (2.2.21)$$

(b) linear viscoelastic

$$P(X, t) = E^{PF} [(F(X, t) - 1) + \dot{F}(X, t)]$$

$$\text{or } P = E^{PF} (+ \dot{}) \quad (2.2.22)$$

For small deformations the material parameter E^{PF} corresponds to Young's modulus; the constant $\dot{}$ determines the magnitude of damping.

Momentum equation in terms of displacements. A single governing equation for the rod can be obtained by substituting the relevant constitutive equation, i.e. (2.2.18) or (2.2.19), into the momentum equation (2.2.12) and expressing the strain measure in terms of the displacement by (2.2.6). For the total form of the constitutive equation (2.2.18), the resulting equation can be written as

$$\left(A_0 P(u, X, \dot{u}, \ddot{u}, \dots) \right)_{,X} + A_0 b = A_0 \ddot{u} \quad (2.2.23)$$

which is a nonlinear partial differential equation (PDE) in the displacement $u(X, t)$. The character of this partial differential equation is not readily apparent from the above and depends on the details of the constitutive equation. To illustrate one form of this PDE, we consider a linear elastic material. For a linear elastic material, Eq. (2.2.20), the constitutive equation and (2.2.23) yield

$$\left(A_0 E^{PF} u, X \right)_{,X} + A_0 b = A_0 \ddot{u} \quad (2.2.24)$$

It can be seen that in this PDE, the highest derivatives with respect to the material coordinate X is second order, and the highest derivative with respect to time is also second order, so the PDE is second order in X and time t . If the stress in the constitutive equation only depends on the first derivatives of the displacements with respect to X and t as indicated in (2.2.18) and (2.2.19), then the momentum equation will similarly be a second order PDE in space and time.

For a rod of constant cross-section and modulus, if the body force vanishes, i.e. when $b = 0$, the momentum equation for a linear material becomes the well known linear wave equation

$$u_{,XX} = \frac{1}{c^2} \ddot{u} \quad (2.2.25)$$

where c is the wave speed relative to the undeformed configuration and given by

$$c^2 = \frac{E^{PF}}{\rho} \quad (2.2.26)$$

Boundary Conditions. The independent variables of the momentum equation are the coordinate X and the time t . It is an initial-boundary value problem (IBVP). To complete the description of the IBVP, the boundary conditions and initial conditions must be given. The boundary in a one dimensional problem consists of the two points at the ends of the domain, which in the model problem are the points X_a and X_b . From the linear form of the momentum equations, Eq. (2.2.23), it can be seen that the partial differential equation is second order in X . Therefore, at each end, either u or $u_{,X}$ must be prescribed as a boundary condition. In solid mechanics, instead of $u_{,X}$, the traction $t_x^0 = n^0 P$ is prescribed; n^0 is the unit normal to the body which is given by $n^0 = 1$ at X_a , $n^0 = -1$ at X_b . Since the stress is a function of the measure of strain, which in turn depends on the derivative of the displacement by Eq. (2.2.6), prescribing t_x^0 is equivalent to prescribing $u_{,X}$; the superscript "naught" on t indicates that the traction is defined over the undeformed area; the superscript is always explicitly included on the traction t_x^0 to distinguish it from the time t . Therefore either the traction or the displacement must be prescribed at each boundary.

A boundary is called a displacement boundary and denoted by \bar{u} if the displacement is prescribed; it is called a traction boundary and denoted by \bar{t} if the traction is prescribed. The prescribed values are designated by a superposed bar. The boundary conditions are

$$u = \bar{u} \quad \text{on} \quad \bar{u} \quad (2.2.27)$$

$$n^0 P = \bar{t}_x^0 \quad \text{on} \quad \bar{t} \quad (2.2.28)$$

As an example of the boundary conditions in solid mechanics, for the rod in Fig. 2.1, the boundary conditions are

$$u(X_a, t) = 0 \quad n^0(X_b)P(X_b, t) = P(X_b, t) = \frac{T(t)}{A_0(X_b)} \quad (2.2.29)$$

The traction and displacement cannot be prescribed at the same point, but one of these must be prescribed at each boundary point; this is indicated by

$$\bar{u} \quad \bar{t} = 0 \quad \bar{u} \quad \bar{t} = \quad (2.2.30)$$

Thus in a one dimensional solid mechanics problem any boundary is either a traction boundary or a displacement boundary, but no boundary is both a prescribed traction and prescribed displacement boundary.

Initial Conditions. Since the governing equation for the rod is second order in time, two sets of initial conditions are needed. We will express the initial conditions in terms of the displacements and velocities:

$$u(X, 0) = u_0(X) \text{ for } X \in [X_a, X_b] \quad (2.2.31a)$$

$$\dot{u}(X, 0) = v_0(X) \text{ for } X \in [X_a, X_b] \quad (2.2.31b)$$

If the body is initially undeformed and at rest, the initial conditions can be written as

$$u(X, 0) = 0 \quad \dot{u}(X, 0) = 0 \quad (2.2.32)$$

Jump Conditions. In order for the derivative in Eq.(2.2.12) to exist, the quantity $A_0 P$ must be continuous. However, neither A_0 nor P need be continuous in the entire interval. Therefore momentum balance requires that

$$\langle A_0 P \rangle = 0 \quad (2.2.33)$$

where f designates the jump in $f(X)$, i.e.

$$\langle f(X) \rangle = f(X + \epsilon) - f(X - \epsilon) = 0 \quad (2.2.34)$$

In dynamics, it is possible to have jumps in the stress, known as shocks, which can either be stationary or moving. Moving discontinuities are governed by the Rankine-Hugoniot relations, but these are not considered in this Chapter.

2.3 Weak Form for Total Lagrangian Formulation

The momentum equation cannot be discretized directly by the finite element method. In order to discretize this equation, a weak form, often called a variational form, is needed. The principle of virtual work, or weak form, which will be developed next, is equivalent to the momentum equation and the traction boundary conditions (2.2.33). Collectively, these two equations are called the classical *strong form*. The weak form can be used to approximate the strong form by finite elements; solutions obtained by finite elements are approximate solutions to the strong form.

Strong Form to Weak Form. A weak form will now be developed for the momentum equation (2.2.23) and the traction boundary conditions. For this purpose we define trial functions $u(X, t)$ which satisfy any displacement boundary conditions and are smooth enough so that all derivatives in the momentum equation are well defined. The test functions $v(X)$ are assumed to be smooth enough so that all of the following steps are well defined and to vanish on the prescribed displacement boundary. The weak form is obtained by taking the product of the momentum equation expressed in terms of the trial function with the test function. This gives

$$\int_{X_a}^{X_b} u \left[(A_0 P)_{,X} + {}_0A_0 b - {}_0A_0 \ddot{u} \right] dX = 0 \quad (2.3.1)$$

Using the derivative of the product in the first term in (2.3.1) gives

$$\int_{X_a}^{X_b} u (A_0 P)_{,X} dX = \int_{X_a}^{X_b} \left[(u A_0 P)_{,X} - u_{,X} A_0 P \right] dX \quad (2.3.2)$$

Applying the fundamental theorem of calculus to the above gives

$$\begin{aligned} \int_{X_a}^{X_b} u (A_0 P)_{,X} dX &= - \int_{X_a}^{X_b} u_{,X} (A_0 P) dX + \left(u A_0 n^0 P \right) \Big|_t \\ &= - \int_{X_a}^{X_b} u_{,X} (A_0 P) dX + \left(u A_0 \bar{t}_x^0 \right) \Big|_t \end{aligned} \quad (2.3.3)$$

where we obtained the second line using the facts that the test function u vanishes on the prescribed displacement boundary, the complementarity conditions on the boundaries (2.2.30) and the traction boundary conditions. Substituting (2.3.3) into the first term of Eq. (2.3.1) gives (with a change of sign)

$$\int_{X_a}^{X_b} \left[u_{,X} A_0 P - u \left({}_0A_0 b - {}_0A_0 \ddot{u} \right) \right] dX - \left(u A_0 \bar{t}_x^0 \right) \Big|_t = 0 \quad (2.3.4)$$

The above is the weak form of the momentum equation and the traction boundary condition for the total Lagrangian formulation.

Smoothness of Test and Trial functions; Kinematic Admissibility. We shall now investigate the smoothness required to go through the above steps more closely. For the momentum equation (2.2.12) to be well defined in a classical sense, the nominal stress and the initial area must be continuously differentiable, i.e. C^1 ; otherwise the first derivative would have discontinuities. If the stress is a smooth function of the derivative of the displacement as in (2.2.18), then to obtain this continuity in the stresses requires that the trial functions must be C^2 . For Eq. (2.3.2) to hold, the test function $u(X)$ must be C^1 .

However, the weak form is well defined for test and trial functions which are far less smooth, and indeed the test and trial functions to be used in finite element methods will be rougher. The weak form (2.3.4) involves only the first derivative of the test function and the trial function appears directly or as a first derivative of the trial function through the nominal stress. Thus the integral in the weak form is integrable if both functions are C^0 .

We will now define the conditions on the test and trial function more precisely. The weak form is well-defined if the trial functions $u(X,t)$ are continuous functions with piecewise continuous derivatives, which is stated symbolically by $u(X,t) \in C^0(X)$, where the X in the parenthesis following C^0 indicates that it pertains to the continuity in X ; note that this definition permits discontinuities of the derivatives at discrete points. This is the same as the continuity of finite element approximations in linear finite element procedures:

the displacement is continuous and continuously differentiable within elements, but the derivative $u_{,X}$ is discontinuous across element boundaries. In addition, the trial function $u(X,t)$ must satisfy all displacement boundary conditions. These conditions on the trial displacements are indicated symbolically by

$$u(X,t) \in U \quad \text{where} \quad U = \left\{ u(X,t) \mid u(X,t) \in C^0(X), u = \bar{u} \text{ on } \Gamma_u \right\} \quad (2.3.5)$$

Displacement fields which satisfy the above conditions, i.e. displacements which are in U , are called kinematically admissible.

The test functions are denoted by $v(X)$; they are *not* functions of time. The test functions are required to be C^0 in X and to vanish on displacement boundaries, i.e.,

$$v(X) \in U_0 \quad \text{where} \quad U_0 = \left\{ v(X) \mid v(X) \in C^0(X), v = 0 \text{ on } \Gamma_u \right\} \quad (2.3.6)$$

We will use the prefix v for all variables which are test functions and for variables which are related to test functions. This convention originates in variational methods, where the test function emerges naturally as the difference between admissible functions. Although it is not necessary to know variational methods to understand weak forms, it provides an elegant framework for developing various aspects of the weak form. For example, in variational methods any test function is a variation and defined as the difference between two trial functions, i.e. the variation $v(X) = u^a(X) - u^b(X)$, where $u^a(X)$ and $u^b(X)$ are any two functions in U . Since any function in U satisfies the displacement boundary conditions, the requirement in (2.3.6) that $v(X) = 0$ on Γ_u can be seen immediately.

Weak Form to Strong Form. We will now develop the equations implied by the weak form with the less smooth trial and test functions, (2.3.5) and (2.3.6), respectively; the strong form implied with very smooth test and trial functions will also be discussed. The weak form is given by

$$\int_{X_a}^{X_b} v_{,X} A_0 P - v (A_0 b - A_0 \bar{u}) dX - \left(v A_0 \dot{t}_x \right) \Big|_t = 0 \quad v(X) \in U_0 \quad (2.3.7)$$

The displacement fields are assumed to be kinematically admissible, i.e. $u(X,t) \in U$. The above weak form is expressed in terms of the nominal stress P , but it is assumed that this stress can always be expressed in terms of the derivatives of the displacement field through the strain measure and constitutive equation. Since $u(X,t)$ is C^0 and the strain measure involves derivatives of $u(X,t)$ with respect to X , we expect $P(X,t)$ to be C^{-1} in X if the constitutive equation is continuous: $P(X,t)$ will be discontinuous wherever the derivative of $u(X,t)$ is discontinuous.

To extract the strong form, we need to eliminate the derivative of $v(X)$ from the integrand. This is accomplished by integration by parts and the fundamental theorem of calculus. Taking the derivative of the product $v A_0 P$ we have

$$\int_{X_a}^{X_b} (v A_0 P)_{,X} dX = \int_{X_a}^{X_b} v_{,X} A_0 P dX + \int_{X_a}^{X_b} v (A_0 P)_{,X} dX \quad (2.3.8)$$

The second term on the RHS can be converted to point values by using the fundamental theorem of calculus. Let the piecewise continuous function $(A_0P)_{,X}$ be continuous on intervals $[X_1^i, X_2^i]$, $e = 1$ to n . Then by the fundamental theorem of calculus

$$\int_{X_1^e}^{X_2^e} (uA_0P)_{,X} dX = (uA_0P)|_{X_2^i} - (uA_0P)|_{X_1^i} - (uA_0n^0P)|_i \quad (2.3.9)$$

where n^0 is the normal to the segments are $n(X_1^i) = -1$, $n(X_2^i) = +1$, and $|_i$ denotes the two boundary points of the segment i over which the function is continuously differentiable. Let $[X_A, X_B] = \bigcup_i [X_1^i, X_2^i]$; then applying (2.3.2) over the entire domain gives

$$\int_{X_a}^{X_b} (uA_0P)_{,X} dX = (uA_0n^0P)|_t - \sum_i u\langle A_0P \rangle_i \quad (2.3.10)$$

where $|_i$ are the interfaces between the segments in which the integrand is continuously differentiable. The contributions to the boundary points on the right-hand side in the above only appear on the traction boundary $|_t$ since $u = 0$ on $|_u$ and $u = \bar{u} - \bar{t}$ (see Eqs. (2.3.6) and (2.2.30)). Combining Eqs. (2.3.10) and (2.3.2) then gives

$$\int_{X_a}^{X_b} u_{,X} (A_0P) dX = - \int_{X_a}^{X_b} u (A_0P)_{,X} dX + (uA_0n^0P)|_t - \sum_i u\langle A_0P \rangle_i \quad (2.3.11)$$

Substituting the above into Eq. (2.3.7) gives

$$\begin{aligned} & \int_{X_a}^{X_b} u \left[(A_0P)_{,X} + {}_0A_0b - {}_0A_0\ddot{u} \right] dX \\ & + uA_0 \left(n^0P - \bar{t}_x^0 \right) \Big|_t + \sum_i u\langle A_0P \rangle_i = 0 \quad u(X) = U_0 \end{aligned} \quad (2.3.12)$$

The conversion of the weak form to a form amenable to the use of Eq. (2.3.4-5) is now complete. We can therefore deduce from the arbitrariness of the virtual displacement $u(X)$ and Eqs. (2.3.4-5) and (2.3.12) that (a more detailed derivation of this step is given in Chapter 4)

$$(A_0P)_{,X} + {}_0A_0b - {}_0A_0\ddot{u} = 0 \quad \text{for } X \in [X_a, X_b] \quad (2.3.13a)$$

$$n^0P - \bar{t}_x^0 = 0 \quad \text{on } |_t \quad (2.3.13b)$$

$$\langle A_0P \rangle = 0 \quad \text{on } |_i \quad (2.3.13c)$$

These are, respectively, the momentum equation, the traction boundary conditions, and the stress jump conditions. Thus when we admit the less smooth test and trial functions, we have an additional equation in the strong form, the jump condition (2.3.13c).

If the test functions and trial functions satisfy the classical smoothness conditions, the jump conditions do not appear. Thus for smooth test and trial functions, the weak form implies only the momentum equation and the traction boundary conditions.

The less smooth test and trial functions are more pertinent to finite element approximations, where these functions are only C^0 . They are also needed to deal with discontinuities in the cross-sectional area and material properties. At material interfaces, the classical strong form is not applicable, since it assumes that the second derivative is uniquely defined everywhere. This is not true at material interfaces because the strains, and hence the derivatives of the displacement fields, are discontinuous. With the rougher test and trial functions, the conditions which hold at these interfaces. (2.3.13c) emerge naturally.

In the weak form for the total Lagrangian formulation, all integrations are performed over the material coordinates, i.e. the reference configuration, of the body, because total Lagrangian formulations involve derivatives with respect to the material coordinates X , so integration by parts is most conveniently performed over the domain expressed in terms of the material coordinate X . Sometimes this is referred to as integration over the undeformed, or initial, domain. The weak form is expressed in terms of the nominal stress.

Physical Names of Virtual Work Terms. For the purpose of obtaining a methodical procedure for obtaining the finite element equations, the virtual energies will be defined according to the type of work which they represent; the corresponding nodal forces will subsequently carry identical names.

Each of the terms in the weak form represents a virtual work due to the virtual displacement u ; this displacement $u(X)$ is called a often "virtual" displacement to indicate that it is not the actual displacement; according to Webster's dictionary, virtual means "being in essence or effect, not in fact"; this is a rather hazy meaning and we prefer the name test function.

The virtual work of the body forces $b(X,t)$ and the prescribed tractions t_x^0 , which corresponds to the second and fourth terms in (2.3.4), is called the virtual external work since it results from the external loads. It is designated by the superscript "ext" and given by

$$W^{ext} = \int_{X_a}^{X_b} u \cdot b A_0 dX + \left(u A_0 t_x^0 \right) \quad (2.3.16)$$

The first term in (2.3.4) is called the virtual internal work, for it arises from the stresses in the material. It can be written in two equivalent forms:

$$W^{int} = \int_{X_a}^{X_b} u_{,X} P A_0 dX = \int_{X_a}^{X_b} F P A_0 dX \quad (2.3.17)$$

where the last form follows from (2.2.1) as follows:

$$u_{,X}(X) = \left((X) - X \right)_{,X} = \frac{(x)}{X} = F \quad (2.3.18)$$

The variation $\delta X = 0$ because the independent variable X does not change due to an incremental displacement $u(X)$.

This definition of internal work in (2.3.17) is consistent with the internal work expression in the energy conservation equation, Eq. (2.2.16-2.2.17): if we change the rates in (2.2.11) to virtual increments, then $\delta_0 w^{int} = FP$. The virtual internal work W^{int} is defined over the entire domain, so we have

$$W^{int} = \int_{X_a}^{X_b} \delta_0 w^{int} A_0 dX = \int_{X_a}^{X_b} FPA_0 dX \quad (2.3.19)$$

which is the same term that appears in the weak form in (2.2.18).

The term $\delta_0 A_0 \ddot{u}$ can be considered a body force which acts in the direction opposite to the acceleration, i.e. in a d'Alembert sense. We will designate the corresponding virtual work by W^{inert} and call it the virtual inertial work, so

$$W^{inert} = \int_{X_a}^{X_b} \delta u \delta_0 A_0 \ddot{u} dX \quad (2.3.20)$$

This is the work by the inertial forces on the body.

Principle of Virtual Work. The principle of virtual work is now stated using these physically motivated names. By using Eqs. (2.3.16-2.3.20), Eq. (2.3.4) can then be written as

$$W(\delta u, u) = \delta W^{int} - \delta W^{ext} + \delta W^{inert} = 0 \quad \delta u \in U_0 \quad (2.3.21)$$

The above equation, with the definitions in Eqs. (2.3.16-2.3.20), is the weak form corresponding to the strong form which consists of the momentum equation, the traction boundary conditions and the stress jump conditions. The weak form implies the strong form and that the strong form implies the weak form. Thus the weak form and the strong form are equivalent. This equivalence of the strong and weak forms for the momentum equation is called the principle of virtual work.

All of the terms in the principle of virtual work W are energies or virtual work terms, which is why it is called a virtual work principle. That the terms are energies is immediately apparent from W^{ext} : since $\delta_0 b$ is a force per unit volume, its product with a virtual displacement δu gives a virtual work per unit volume, and the integral over the domain gives the total virtual work of the body force. Since the other terms in the weak form must be dimensionally consistent with the external work term, they must also be virtual energies. This view of the weak form as consisting of virtual work or energy terms provides a unifying perspective which is quite useful for constructing weak forms for other coordinate systems and other types of problems: it is only necessary to write an equation for the virtual energies to obtain the weak form, so the procedure we have just gone through can be avoided. The virtual work schema is also useful in memorizing the weak form. However, from a mathematical viewpoint it is not necessary to think of the test functions $u(X)$ as virtual displacements: they are simply test functions which satisfy continuity conditions and vanish on the boundaries as specified by (2.3.6). This second

viewpoint becomes useful when a finite element discretization is applied to equations where the product with a test function does not have a physical meaning. The principle of virtual work is summarized in Box 2.1.

Box 2.1. Principle of Virtual Work for One Dimensional Total Lagrangian Formulation

If the trial functions $u(X, t) \in U$, then

$$(Weak Form) \quad W = 0 \quad u \in U_0 \quad (B2.1.1)$$

is equivalent to

(Strong Form)

$$\text{the momentum equation (2.2.12): } (A_0 P)_{,X} + {}_0A_0 b = {}_0A_0 \ddot{u}, \quad (B2.1.2)$$

$$\text{the traction boundary conditions (2.2.28): } n^0 P = \bar{t}_x^0 \quad \text{on } \Gamma_t, \quad (B2.1.3)$$

$$\text{and the jump conditions (2.2.33): } A_0 P = 0. \quad (B2.1.4)$$

Weak form definitions:

$$W = W^{int} - W^{ext} + W^{inert} \quad (B2.1.5)$$

$$W^{int} = \int_{X_a}^{X_b} u_{,X} P A_0 dX = \int_{X_a}^{X_b} F P A_0 dX, \quad W^{inert} = \int_{X_a}^{X_b} u {}_0A_0 \ddot{u} dX \quad (B2.1.6)$$

$$W^{ext} = \int_{X_a}^{X_b} u {}_0b A_0 dX + \left(u A_0 \bar{t}_x^0 \right)_{\Gamma_t} \quad (B2.1.7)$$

2.4 Finite Element Discretization In Total Lagrangian Formulation

Finite Element Approximations. The discrete equations for a finite element model are obtained from the principle of virtual work by using finite element interpolants for the test and trial functions. For the purpose of a finite element discretization, the interval $[X_a, X_b]$ is subdivided into elements $e=1$ to n_e with n_N nodes. The nodes are denoted by $X_I, I = 1$ to n_N , and the nodes of a generic element by $X_I^e, I = 1$ to m , where m is the number of nodes per element. The domain of each element is $[X_1^e, X_m^e]$, which is denoted by Ω_e . For simplicity, we consider a model problem in which node 1 is a prescribed displacement boundary and node n_N a prescribed traction boundary. However, to derive the governing equations we first treat the model as if there were no prescribed displacement boundaries and impose the displacement boundary conditions in the last step.

The finite element trial function $u(X, t)$ is written as

$$u(X,t) = \sum_{I=1}^{n_N} N_I(X)u_I(t) \quad (2.4.1)$$

In the above, $N_I(X)$ are C^0 interpolants, they are often called shape functions in the finite element literature; $u_I(t)$, $I=1$ to n_N , are the nodal displacements, which are functions of time, and are to be determined in the solution of the equations. The nodal displacements are considered functions of time even in static, equilibrium problems, since in nonlinear problems we must follow the evolution of the load; in many cases, t may simply be a monotonically increasing parameter. The shape functions, like all interpolants, satisfy the condition

$$N_I(X_J) = \delta_{IJ} \quad (2.4.2)$$

where δ_{IJ} is the Kronecker delta or unit matrix: $\delta_{IJ} = 1$ if $I = J$, $\delta_{IJ} = 0$ if $I \neq J$. We note here that if we set $u_I(t) = \bar{u}(0,t)$ then the trial function $u(X,t) = U$, i.e. it is kinematically admissible since it has the requisite continuity and satisfies the essential boundary conditions. Equation (2.4.1) represents a separation of variables: the spatial dependence of the solution is entirely represented by the shape functions, whereas the time dependence is ascribed to the nodal variables. This characteristic of the finite element approximation will have important ramifications in finite element solutions of wave propagation problems.

The test functions (or virtual displacements) depend only on the material coordinates

$$u(X) = \sum_{I=1}^{n_N} N_I(X) u_I \quad (2.4.3)$$

where u_I are the nodal values of the test function; they are not functions of time.

Nodal Forces. To provide a systematic procedure for developing the finite element equations, nodal forces are developed for each of the virtual work terms. These nodal forces are given names which correspond to the names of the virtual work terms. Thus

$$W^{int} = \sum_{I=1}^{n_N} u_I f_I^{int} = \mathbf{u}^T \mathbf{f}^{int} \quad (2.4.4a)$$

$$W^{ext} = \sum_{I=1}^{n_N} u_I f_I^{ext} = \mathbf{u}^T \mathbf{f}^{ext} \quad (2.4.4b)$$

$$W^{inert} = \sum_{I=1}^{n_N} u_I f_I^{inert} = \mathbf{u}^T \mathbf{f}^{inert} \quad (2.4.4c)$$

$$\mathbf{u}^T = [u_1 \quad u_2 \quad \dots \quad u_{n_N}] \quad \mathbf{f}^T = [f_1 \quad f_2 \quad \dots \quad f_{n_N}] \quad (2.4.4d)$$

where \mathbf{f}^{int} are the internal nodal forces, \mathbf{f}^{ext} are the external nodal forces, and \mathbf{f}^{inert} are the inertial, or d'Alembert, nodal forces. These names give a physical meaning to the nodal

forces : the internal nodal forces correspond to the stresses “in” the material, the external nodal forces correspond to the externally applied loads, while the inertial nodal forces correspond to the inertia term due to the accelerations.

Nodal forces are always defined so that they are *conjugate* to the nodal displacements in the sense of work, i.e. so the scalar product of an increment of nodal displacements with the nodal forces gives an increment of work. This rule should be observed in the construction of the discrete equations, for when it is violated many of the important symmetries, such as that of the mass and stiffness matrices, are lost.

Next we develop expressions for the various nodal forces in terms of the continuous variables in the partial differential equation by using (2.3.16-2.3.20). In developing the nodal force expressions, we continue to ignore the displacement boundary conditions and consider u_I arbitrary at all nodes. The expressions for the nodal forces are then obtained by combining Eqs. (2.3.16) to (2.3.20) with the definitions given in Eqs. (2.4.4) and the finite element approximations for the trial and test functions. Thus to define the internal nodal forces in terms of the nominal stress, we use (2.4.4a) and Eq. (2.3.16), and use the finite element approximation of the test function (2.4.3), giving

$$W^{int} = \int_I u_I f_I^{int} = \int_{X_a}^{X_b} u_I u_{,X} P A_0 dX = \int_I u_I \int_{X_a}^{X_b} N_{I,X} P A_0 dX \quad (2.4.5)$$

From the above definition it follows that

$$f_I^{int} = \int_{X_a}^{X_b} N_{I,X} P A_0 dX \quad (2.4.6)$$

which gives the expression for the internal nodal forces. It can be seen that the internal nodal forces are a discrete representation of the stresses in the material. Thus they can be viewed as the nodal forces arising from the resistance of the solid to deformation.

The external and nodal forces are developed similarly. The external nodal forces are obtained by using (2.4.4b) and (2.3.17) in conjunction with the test function:

$$\begin{aligned} W^{ext} &= \int_I u_I f_I^{ext} = \int_{X_a}^{X_b} u_I u_{,0} b A_0 dX + \left(u_I A_0 \dot{t}_x \right) \Big|_t \\ &= \int_I u_I \left\{ \int_{X_a}^{X_b} N_I u_{,0} b A_0 dX + \left(N_I A_0 \dot{t}_x \right) \Big|_t \right\} \end{aligned} \quad (2.4.7)$$

where in the last step (2.4.3) has been used. The above give

$$f_I^{ext} = \int_{X_a}^{X_b} N_I u_{,0} b A_0 dX + \left(N_I A_0 \dot{t}_x \right) \Big|_t \quad (2.4.8)$$

Since $N_I(X_J) = \delta_{IJ}$ the last term contributes only to those nodes which are on the prescribed traction boundary.

The inertial nodal forces are obtained from the inertial virtual work (2.4.4c) and (2.3.20):

$$W^{inert} = \int_I u_I f_I^{inert} = \int_{X_a}^{X_b} u_0 \ddot{u} A_0 dX \quad (2.4.9)$$

Using the finite element approximation for the test functions, Eq. (2.4.3), and the trial functions, Eq. (2.4.1) gives

$$\int_I u_I f_I^{inert} = \int_I u_I \int_{X_a}^{X_b} {}_0N_I N_J A_0 dX \ddot{u}_J \quad (2.4.10)$$

The inertial nodal force is usually expressed as a product of a mass matrix and the nodal accelerations. Therefore we define a mass matrix by

$$M_{IJ} = \int_{X_a}^{X_b} {}_0N_I N_J A_0 dX \quad \text{or} \quad \mathbf{M} = \int_{X_a}^{X_b} {}_0\mathbf{N}^T \mathbf{N} A_0 dX \quad (2.4.11)$$

Letting $\ddot{u}_I = \mathbf{a}_I$ the virtual inertial work is

$$W^{inert} = \int_I u_I f_I^{inert} = \int_I u_I M_{IJ} a_J = \mathbf{u}^T \mathbf{M} \mathbf{a}, \quad \mathbf{a} = \ddot{\mathbf{u}} \quad (2.4.12)$$

The definition of the inertial nodal forces then gives the following expression

$$f_I^{inert} = M_{IJ} a_J \quad \text{or} \quad \mathbf{f}^{inert} = \mathbf{M} \mathbf{a} \quad (2.4.13)$$

Note that the mass matrix as given by Eq. (2.4.11) will not change with time, so it needs to be computed only at the beginning of the calculation. The mass matrix given by (2.4.11) is called the consistent mass matrix.

Semidiscrete Equations. We now develop the semidiscrete equations, i.e. the finite element equations for the model. At this point we will also consider the effect of the displacement boundary conditions. The displacement boundary conditions can be satisfied by the trial and test functions function by letting

$$u_1(t) = \bar{u}_1(t) \quad \text{and} \quad u_1 = 0 \quad (2.4.14)$$

The trial function then meets Eq. (2.3.5). For the test function to meet the conditions of Eq. (2.3.6), it is necessary that $u_1 = 0$, so the nodal values of the test function are not arbitrary at node 1. Our development here, as noted in the beginning, specifies node 1 as the prescribed displacement boundary; this is done only for convenience of notation, and in a finite element model any node can be a prescribed displacement boundary node.

We will now derive the discrete equations. It should be noted that Eqs. (2.4.4a-c) are simply definitions that are made for convenience, and do not constitute the discrete equations. Substituting the definitions (2.4.4a-c) into Eq. (2.3.21) gives

$$\sum_{I=1}^{n_N} u_I (f_I^{int} - f_I^{ext} + f_I^{inert}) = 0 \quad (2.4.15)$$

Since u_I is arbitrary at all nodes except the displacement boundary node, node 1, it follows that

$$f_I^{int} - f_I^{ext} + f_I^{inert} = 0, \quad I = 2 \text{ to } n_N \quad (2.4.16)$$

Substituting (2.4.13) into (2.4.16) gives the discrete equations, which are known as the *equations of motion*:

$$\sum_{J=1}^{n_N} M_{IJ} \frac{d^2 u_J}{dt^2} + f_I^{int} - f_I^{ext} = 0, \quad I = 2 \text{ to } n_N \quad (2.4.17)$$

The acceleration of node 1 is given in this model problem, since node 1 is a prescribed displacement node. The acceleration of the prescribed displacement node can be obtained from the prescribed nodal displacement by differentiating twice in time. Obviously, the prescribed displacement must be sufficiently smooth so that the second derivative can be taken; this requires it to be a C^1 function of time. If the mass matrix is not diagonal, then the acceleration on the prescribed displacement node, node 1, will contribute to the Eq. (2.4.17). The finite element equations can then be written as

$$\sum_{J=2}^{n_N} M_{IJ} \frac{d^2 u_J}{dt^2} + f_I^{int} - f_I^{ext} = M_{I1} \frac{d^2 \bar{u}_1}{dt^2}, \quad I = 2 \text{ to } n_N \quad (2.4.18)$$

In matrix form the equations of motion can be written as

$$\begin{aligned} \mathbf{M}\mathbf{a} &= \mathbf{f}^{ext} - \mathbf{f}^{int} \quad \text{or} \\ \mathbf{f} &= \mathbf{M}\mathbf{a}, \quad \mathbf{f} = \mathbf{f}^{ext} - \mathbf{f}^{int} \end{aligned} \quad (2.4.19)$$

where the matrices have been truncated so that the equations correspond to Eq. (2.4.17), i.e. \mathbf{M} is a $(n_N - 1) \times n_N$ matrix and the nodal forces are column matrices of order $n_N - 1$. The effects of any nonzero nodal prescribed displacements are assumed to have been incorporated in the external nodal forces by letting

$$f_I^{ext} \rightarrow f_I^{ext} + M_{I1} \frac{d^2 \bar{u}_1}{dt^2} \quad (2.4.20)$$

Thus, when the mass matrix is consistent, prescribed velocities make contributions to nodes which are not on the boundary. For a diagonal mass matrix, the accelerations of prescribed displacement nodes have no effect on other nodes and the above modification of the external forces can be omitted.

Equations (2.4.17) and (2.4.19) are two alternate forms of the semidiscrete momentum equation, which is called the *equation of motion*. These equations are called semidiscrete because they are discrete in space but continuous in time. Sometimes they are

called discrete equations, but they are only discrete in space. The equations of motion are systems of $n_N - 1$ second-order *ordinary differential equations*(ODE); the independent variable is the time t . These equations can easily be remembered by the second form in (2.4.19), $\mathbf{f} = \mathbf{M}\mathbf{a}$, which is the well known Newton's second law of motion. The mass matrix in finite element discretizations is often not diagonal, so the equations of motion differ from Newton's second law in that a force at node I can generate accelerations at node J if $M_{IJ} \neq 0$. However, in many cases a diagonal approximation to the mass matrix is used. In that case, the discrete equations of motion are identical to the Newton's equations for a system of particles interconnected by deformable elements. The force $f_I = f_I^{ext} - f_I^{int}$ is the net force on particle I . The negative sign appears on the internal nodal forces because these nodal forces are defined as acting on the elements; by Newton's third law, the forces on the nodes are equal and opposite, so a negative sign is needed. Viewing the semidiscrete equations of motion in terms of Newton's second law provides an intuitive feel for these equations and is useful in remembering these equations.

Initial Conditions. Since the equations of motion are second order in time, initial conditions on the displacements and velocities are needed. The continuous form of the initial conditions are given by Eqs. (2.2.22). In many cases, the initial conditions can be applied by simply setting the nodal values of the variables to the initial values, i.e. by letting

$$u_I(0) = u_0(X_I) \quad I \quad \text{and} \quad \dot{u}_I(0) = v_0(X_I) \quad I \quad (2.4.21)$$

Thus the initial conditions on the nodal variables for a body which is initially at rest and undeformed are

$$u_I(0) = 0 \quad \text{and} \quad \dot{u}_I(0) = 0 \quad I \quad (2.4.22)$$

Least Square Fit to Initial Conditions. For more complex initial conditions, the initial values of the nodal displacements and nodal velocities can be obtained by a least-square fit to the initial data. The least square fit for the initial displacements results from minimizing the square of the difference between the finite element interpolate $N_I(X)u_I(0)$ and the initial data $\bar{u}(X)$. Let

$$M = \frac{1}{2} \int_{X_a}^{X_b} \sum_I u_I(0)N_I(X) - u_0(X) \quad {}_0A_0 dX \quad (2.4.23)$$

The density is not necessary in this expression but as will be seen, it leads to equations in terms of the mass matrix, which is quite convenient. To find the minimum set

$$0 = \frac{\partial M}{\partial u_K(0)} = \int_{X_a}^{X_b} N_K(X) \sum_I u_I(0)N_I(X) - u_0(X) \quad {}_0A_0 dX \quad (2.4.24)$$

Using the definition of the mass matrix, (2.4.11), it can be seen that the above can be written as

$$\mathbf{M}\mathbf{u}(0) = \mathbf{g} \quad (2.4.25a)$$

$$g_k = \int_{x_a}^{x_b} N_k(X) u_0(X) A_0 dX \quad (2.4.25b)$$

The least square fit to the initial velocity data is obtained similarly. This method of fitting finite element approximations to functions is often called an L_2 projection.

Diagonal Mass Matrix. The mass matrix which results from a consistent derivation from the weak form is called a *consistent mass matrix*. In many applications, it is advantageous to use a diagonal mass matrix called a lumped mass matrix. Procedures for diagonalizing the mass matrix are often quite ad hoc, and there is little theory underlying these procedures. One of the most common procedures is the row-sum technique, in which the diagonal elements of the mass matrix are obtained by

$$M_{II}^D = \sum_J M_{IJ}^C \quad (2.4.26)$$

where the sum is over the entire row of the matrix, M_{IJ}^C is the consistent mass matrix and M_{II}^D is the diagonal or lumped, mass matrix.

The diagonal mass matrix can also be evaluated by

$$M_{II}^D = \sum_J M_{IJ}^C = \int_{x_a}^{x_b} N_I \sum_j N_j A_0 dX = \int_{x_a}^{x_b} N_I A_0 dX \quad (2.4.27)$$

where we have used the fact that the sum of the shape functions must equal one; this is a reproducing condition discussed in Chapter 8. This diagonalization procedure conserves the total momentum of a body, i.e. the momentum of the system with the diagonal mass is equivalent to that of the consistent mass, so

$$\sum_{I,J} M_{IJ}^C v_J = \sum_I M_{II}^D v_I \quad (2.4.28)$$

for any nodal velocities.

2.5 Relationships between Element and Global Matrices

In the previous section, we have developed the semidiscrete equations in terms of global shape functions, which are defined over the entire domain, although they are usually nonzero only in the elements adjacent to the node associated with the shape function. The use of global shape functions to derive the finite element equations provides little understanding of how finite element programs are actually structured. In finite element programs, the nodal forces and the mass matrix are usually first computed on an element level. The element nodal forces are combined into the global matrix by an operation called *scatter* or *vector assembly*. The mass matrix and other square matrices are combined from the element level to the global level by an operation called *matrix assembly*. When the nodal displacements are needed for computations, they are extracted from the global

matrix by an operation called *gather*. These operations are described in the following. In addition we will show that there is no need to distinguish element and global shape functions and element and global equations for the nodal forces: the expressions are identical and the element related expressions can always be obtained by limiting the integration to the domain of the element.

The relations between element matrices and the corresponding global matrices will be obtained by the use of the connectivity matrices \mathbf{L}_e . The nodal displacements and nodal forces of element e are denoted by \mathbf{u}_e and \mathbf{f}_e , respectively, and are column matrices of order m , where m is the number of nodes per element. Thus for a 2-node element, the element nodal displacement matrix is $\mathbf{u}_e^T = [u_1, u_2]_e$. The corresponding element nodal force matrix is $\mathbf{f}_e^T = [f_1, f_2]_e$. We will place the element identifier “ e ” as either a subscript or superscript, but will always use the letter “ e ” for the purpose of identifying element-related quantities.

The element and global nodal forces must be defined so that their scalar products with the corresponding nodal displacement increments gives an increment of work. This was used in defining the nodal forces in Section 2.4. In most cases, meeting this requirement entails little beyond being careful to arrange the nodal displacements and nodal forces in the same order in the corresponding matrices. This feature of the nodal force and displacement matrices is crucial to the assembly procedure and symmetry of linear and linearized equations.

The element nodal displacements are related to the global nodal displacements by

$$\mathbf{u}_e = \mathbf{L}_e \mathbf{u} \quad \mathbf{u}_e = \mathbf{L}_e \mathbf{u} \quad (2.5.1)$$

The matrix \mathbf{L}_e is a Boolean matrix, i.e. it consists of the integers 0 and 1. An example of the \mathbf{L}_e matrix for a specific mesh is given later in this Section. The operation of extracting \mathbf{u}_e from \mathbf{u} is called a *gather* because in this operation the small element vectors are *gathered* from the global vector.

The element nodal forces are defined analogously to (2.4.4) as those forces which give the internal work:

$$W_e^{int} = \mathbf{u}_e^T \mathbf{f}_e^{int} = \int_{X_1^e}^{X_m^e} u_{,X} P A_0 dX \quad (2.5.2)$$

To obtain the relations between global and local nodal forces, we use the fact that the total virtual internal energy is the sum of the element internal energies:

$$W^{int} = \sum_e W_e^{int} \quad \text{or} \quad \mathbf{u}^T \mathbf{f}^{int} = \sum_e \mathbf{u}_e^T \mathbf{f}_e^{int} \quad (2.5.3)$$

Substituting (2.5.1) into the (2.5.3) yields

$$\mathbf{u}^T \mathbf{f}^{int} = \sum_e \mathbf{u}^T \mathbf{L}_e^T \mathbf{f}_e^{int} \quad (2.5.4)$$

Since the above must hold for arbitrary \mathbf{u} , it follows that

$$\mathbf{f}^{int} = \sum_e \mathbf{L}_e^T \mathbf{f}_e^{int} \quad (2.5.5)$$

which is the relationship between element nodal forces and global nodal forces. The above operation is called a *scatter*, for the small element vector is *scattered* into the global array according to the node numbers. Similar expressions can be derived for the external nodal forces and the inertial forces

$$\mathbf{f}^{ext} = \sum_e \mathbf{L}_e^T \mathbf{f}_e^{ext}, \quad \mathbf{f}^{inert} = \sum_e \mathbf{L}_e^T \mathbf{f}_e^{inert} \quad (2.5.6)$$

The gather and scatter operations are illustrated in Fig. 2 for a one dimensional mesh of two-node elements. The sequence of gather, compute and scatter is illustrated for two elements in the mesh. As can be seen, the displacements are gathered according to the node numbers of the element. Other nodal variables, such as nodal velocities and temperatures, can be gathered similarly. In the scatter, the nodal forces are then returned to the global force matrix according to the node numbers. The scatter operation is identical for the other nodal forces.

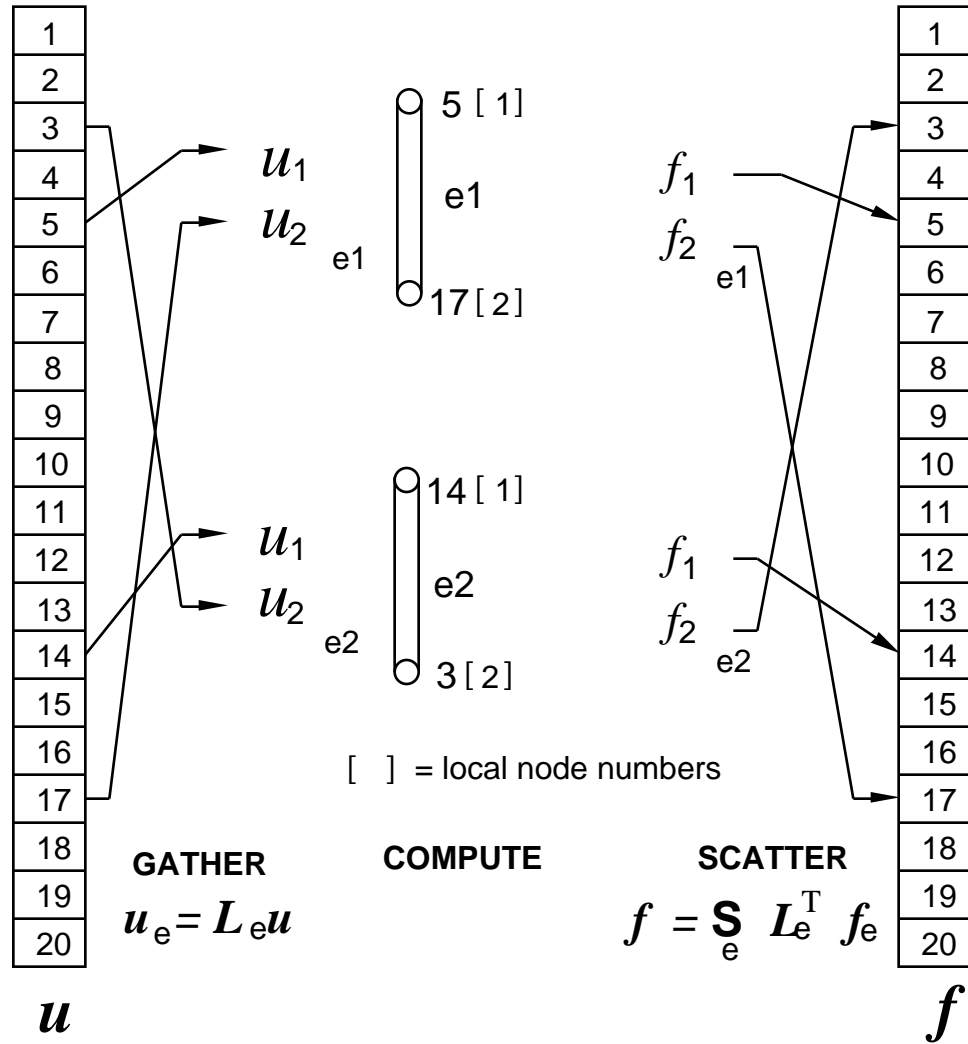


Fig. 2.2. Illustration of gather and scatter for a one-dimensional mesh of two-node elements, showing the gather of two sets of element nodal displacements and the scatter of the computed nodal forces.

In order to describe the assembly of the global mass matrix from the element mass matrices, the element inertial nodal forces are defined as a product of an element mass matrix and the element acceleration, similarly to (2.4.13):

$$\mathbf{f}_e^{inert} = \mathbf{M}_e \mathbf{a}_e \quad (2.5.7)$$

By taking time derivatives of Eq. (2.5.1), we can relate the element and global accelerations by $\mathbf{a}_e = \mathbf{L}_e \mathbf{a}$, (the connectivity matrix does not change with time) and inserting this into the above and using (2.5.6) yields

$$\mathbf{f}^{inert} = \sum_e \mathbf{L}_e^T \mathbf{M}_e \mathbf{L}_e \mathbf{a} \quad (2.5.8)$$

Comparing (2.5.8) to (2.4.13), it can be seen that the global mass matrix is given in terms of the element matrices by

$$\mathbf{M} = \sum_e \mathbf{L}_e^T \mathbf{M}_e \mathbf{L}_e \quad (2.5.9)$$

The above operation is the well known procedure of *matrix assembly*. This is the same operation which is used to assemble the stiffness matrix from element stiffnesses in linear finite element methods.

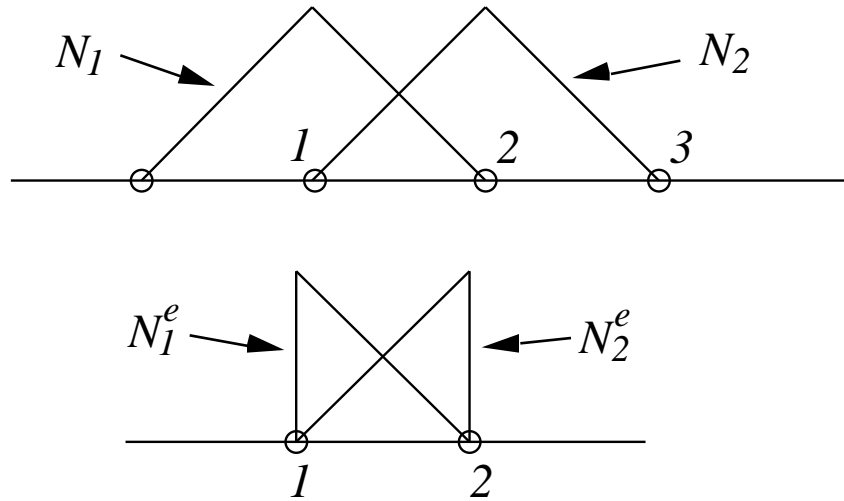


Fig. 2.3. Illustration of element $N^e(X)$ and global shape functions $N(X)$ for a one dimensional mesh of linear displacement, two-node elements.

Relations between element shape functions and global shape functions can also be developed by using the connectivity matrices. However, we shall shortly see that in most cases there is no need to distinguish them. The element shape functions are defined as the interpolants $\mathbf{N}^e(X)$, which when multiplied by the element nodal displacements, give the displacement field in the element, i.e. the displacement field in element e is given by

$$u^e(X) = \mathbf{N}^e(X) \mathbf{u}_e = \sum_{I=1}^m N_I^e(X) u_I^e \quad (2.5.10)$$

The global displacement field is obtained by summing the displacement fields for all elements, which gives

$$u(X) = \sum_{e=1}^{n_e} \mathbf{N}^e(X) \mathbf{L}_e \mathbf{u} = \sum_{e=1}^{n_e} \sum_{I=1}^m \sum_{J=1}^{n_N} N_I^e(X) L_{IJ}^e u_J \quad (2.5.11)$$

where Eq. (2.5.1) has been used in the above. Comparing the above with Eq. (2.4.1), we see that

$$\mathbf{N}(X) = \sum_{e=1}^{n_e} \mathbf{N}^e(X) \mathbf{L}_e \quad \text{or} \quad N_J(X) = \sum_{e=1}^{n_e} \sum_{I=1}^m N_I^e(X) L_{IJ}^e \quad (2.5.12)$$

Thus the global shape functions are obtained from the element shape functions by summing according to the node numbers of the elements. This relationship is illustrated graphically for a two-node linear displacement element in Fig. 2.3.

We will now show that the expressions for the element nodal forces are equivalent to the global nodal forces, except that the integrals are restricted to the elements. Using Eq. (2.5.2) and the element form of the displacement field, we obtain

$$W_e^{int} = \mathbf{u}_e^T \mathbf{f}_e^{int} = \mathbf{u}_e^T \int_{X_1^e}^{X_m^e} \mathbf{N}_{,X}^e P A_0 dX \quad (2.5.13)$$

Invoking the arbitrariness of the virtual nodal displacements, we obtain

$$\mathbf{f}_e^{int} = \int_{X_1^e}^{X_m^e} \mathbf{N}_{,X}^e P A_0 dX \text{ or } f_{I,e}^{int} = \int_{X_1^e}^{X_m^e} N_{I,X}^e P A_0 dX \quad (2.5.14)$$

where the superscript e has been removed from the last expression since in element e , $\mathbf{N}^e(X) = \mathbf{N}(X)$.

Comparing the above with (2.4.6), we can see that (2.5.14) is identical to the global expression (2.4.6) except that integrals here are limited to an element. Identical results can be obtained for the mass matrix and the external force matrix. Therefore, in subsequent derivations we will usually not distinguish element and global forms of the matrices: the element forms are identical to the global forms except that element matrices correspond to integrals over the element domain, whereas global force matrices correspond to integrals over the entire domain.

In finite element programs, global nodal forces are not computed directly but obtained from element nodal forces by assembly, i.e. the scatter operation. Furthermore, the essential boundary conditions need not be considered until the final steps of the procedure. Therefore we will usually concern ourselves only with obtaining the element equations. The assembly of the element equations for the complete model and the imposition of boundary conditions is a standard procedure.

We will often write the internal nodal force expressions for the total Lagrangian formulation in terms of a \mathbf{B}_0 matrix, where \mathbf{B}_0 is in the one-dimensional case a row matrix defined by

$$B_{0I} = N_{I,X} \quad (2.5.15)$$

The nought is specifically included to indicate that the derivatives are with respect to the initial, or material, coordinates. The internal nodal forces (2.5.14) are then given

$$\mathbf{f}_e^{int} = \int_{\hat{0}} \mathbf{B}_0^T P d_0 \text{ or } f_{I,e}^{int} = \int_{\hat{0}} B_{0I} P d_0 \quad (2.5.16)$$

where we have used $d_0 = A_0 dX$ and $\hat{0}$ is the initial domain of the element. In this notation the deformation gradient \mathbf{F} and the one-dimensional strain are given by

$$= \mathbf{B}_0 \mathbf{u}^e \quad (2.5.17)$$

Box 2.2. Discrete Equations in Total Lagrangian Formulation

$$u(X,t) = \mathbf{N}(X) \mathbf{u}_e(t) = \int_I N_I(X) u_I^e(t) \quad (B2.2.1)$$

in each element

$$= \int_I \frac{N_I}{X} u_I^e = \mathbf{B}_0 \mathbf{u}_e \quad (B2.2.2)$$

evaluate the nominal stress P by constitutive equation

$$\mathbf{f}_e^{int} = \int_{\xi_0}^{\xi_1} \frac{\mathbf{N}}{X} P d\xi_0 = \int_{\xi_0}^{\xi_1} \mathbf{B}_0^T P d\xi_0 \quad \text{or} \quad f_{el}^{int} = \int_{\xi_0}^{\xi_1} \frac{N_I}{X} P d\xi_0 \quad (B2.2.3)$$

$$\mathbf{f}_e^{ext} = \int_{\xi_0}^{\xi_1} \mathbf{N}^T b d\xi_0 + (\mathbf{N}^T A_0 \dot{\mathbf{x}}_0) \Big|_{\xi_0}^{\xi_1} \quad (B2.2.4)$$

$$\mathbf{M}_e = \int_{\xi_0}^{\xi_1} \mathbf{N}^T \mathbf{N} d\xi_0 \quad (B2.2.5)$$

$$\mathbf{M} \ddot{\mathbf{u}} + \mathbf{f}^{int} = \mathbf{f}^{ext} \quad (B2.2.6)$$

Example 2.5.1. Two-Node, Linear Displacement Element. Consider a two-node element shown in Fig. 3. The element shown is initially of length ℓ_0 and constant cross-sectional area A_0 . At any subsequent time t , the length is $\ell(t)$ and the cross-sectional area is $A(t)$; the dependence ℓ and A on time t will not be explicitly noted henceforth. The cross-sectional area of the element is taken to be constant, i.e. independent of X .

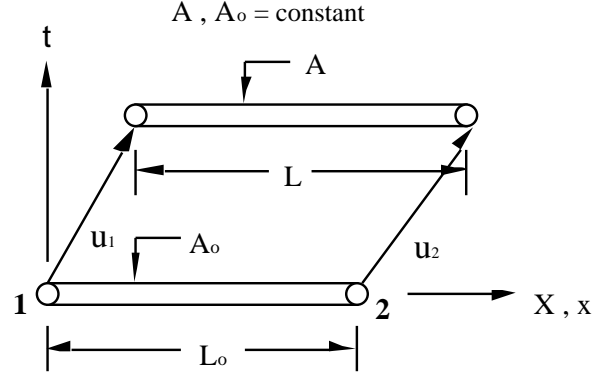


Fig. 2.3. Two node element in one dimension for total Lagrangian formulation showing the initial, undeformed (reference) configuration and the deformed (current) configuration.

Displacement field, strain, and \mathbf{B}_0 matrix. The displacement field is given by the linear Lagrange interpolant expressed in terms of the material coordinate

$$u(X, t) = \frac{1}{\ell_0} [X_2 - X, \quad X - X_1] \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad (2.5.18)$$

where $\ell_0 = X_2 - X_1$. The strain measure is evaluated in terms of the nodal displacements by using Eq. (2.5.18) with (B2.2.2):

$$(X, t) = u_{,X} = \frac{1}{\ell_0} [-1 \quad +1] \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad (2.5.19)$$

The above defines the \mathbf{B}_0 matrix to be

$$\mathbf{B}_0 = \frac{1}{\ell_0} [-1 \quad +1] \quad (2.5.20)$$

Nodal Internal Forces. The internal nodal forces are then given by (2.5.16):

$$\mathbf{f}_e^{int} = \int_{x_1}^{x_2} \mathbf{B}_0^T P d_x = \int_{x_1}^{x_2} \frac{1}{\ell_0} \begin{bmatrix} -1 \\ +1 \end{bmatrix} P A_0 dX \quad (2.5.21a)$$

If we assume that the cross-sectional area and the nominal stress P is constant, the integrand in (2.5.21a) is then constant, so the integral can be evaluated by taking the product of the integrand and the initial length of the element ℓ_0 , which gives

$$\mathbf{f}_e^{int} = \begin{bmatrix} f_1^{int} \\ f_2^{int} \end{bmatrix} = A_0 P \begin{bmatrix} -1 \\ +1 \end{bmatrix} \quad (2.5.21b)$$

From the above, we can see that the nodal internal forces are equal and opposite, so the element internal nodal forces are in equilibrium, even in a dynamic problem. This characteristic of element nodal forces will apply to all elements for which translation results

in no deformation; it does not apply to axisymmetric elements. Since $P = T/A_0$, (see Eq. (2.1.1)) the nodal forces are equal to the load T carried by the element.

Nodal External Forces. The external nodal forces arising from the body force are given by (B2.2.3)

$$\mathbf{f}_e^{ext} = \int_{x_1}^{x_2} \mathbf{N}^T b A_0 dX = \int_{x_1}^{x_2} \frac{X_2 - X}{X_2 - X_1} b A_0 dX \quad (2.5.22a)$$

If we approximate the body forces $b(X,t)$ by a linear Lagrange interpolant

$$b(X,t) = b_1(t) \frac{X_2 - X}{\ell_0} + b_2(t) \frac{X - X_1}{\ell_0} \quad (2.5.22b)$$

and taking A_0 to be constant, the evaluation of the integral in (2.5.22a) gives

$$\mathbf{f}_e^{ext} = \frac{A_0 \ell_0}{6} \begin{bmatrix} 2b_1 + b_2 \\ b_1 + 2b_2 \end{bmatrix} \quad (2.5.22c)$$

The evaluation of the external nodal forces is facilitated by expressing the integral in terms of a parent element coordinate

$$\xi = (X - X_1) / \ell_0, \quad \xi \in [0,1] \quad (2.5.23)$$

Element Mass Matrix. The element mass matrix is given by (B2.2.5):

$$\begin{aligned} \mathbf{M}_e &= \int_{x_1}^{x_2} \mathbf{N}^T \mathbf{N} \rho dV = \int_0^1 \mathbf{N}^T \mathbf{N} A_0 \ell_0 d\xi \\ &= \int_0^1 \begin{bmatrix} 1 - \xi \\ \xi \end{bmatrix} A_0 \ell_0 d\xi = \frac{A_0 \ell_0}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned} \quad (2.5.24a)$$

It can be seen from the above that the mass matrix is independent of time, since it depends only on the initial density, cross-sectional area and length.

The diagonal mass matrix as obtained by the row-sum technique (2.4.26) is

$$\mathbf{M}_e = \frac{A_0 \ell_0}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{A_0 \ell_0}{2} \mathbf{I} \quad (2.5.24b)$$

As can be seen from the above, in the diagonal mass matrix for this element, half of the mass of the element is ascribed to each of the nodes. For this reason, it is often called the lumped mass matrix.

Example 2.5.2. Example of Assembled Equations. Consider a mesh of two elements as shown in Fig. 4. The body force $b(x)$ is constant, b . We will develop the governing equations for this mesh; the equation for the center node is of particular interest since it represents the typical equation for the interior node of any one-dimensional mesh.

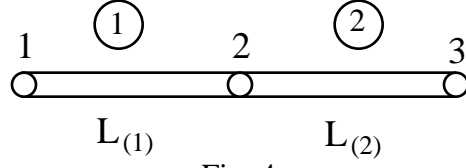


Fig. 4

The connectivity matrices \mathbf{L}_e for this mesh are

$$\mathbf{L}_{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (2.5.25a)$$

$$\mathbf{L}_{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.5.25b)$$

The global internal force matrix by Eq. (2.5.5) is given in terms of the element internal forces by

$$\mathbf{f}^{int} = \mathbf{L}_{(1)}^T \mathbf{f}_{(1)}^{int} + \mathbf{L}_{(2)}^T \mathbf{f}_{(2)}^{int} = \begin{bmatrix} f_1^{int} & 0^{int} \\ f_2^{(1)} & f_1^{(2)} \\ 0 & f_2^{(2)} \end{bmatrix} \quad (2.5.26)$$

which from (2.5.21b) gives

$$\mathbf{f}^{int} = A_0^{(1)} P_{(1)} \begin{bmatrix} -1 & 0 \\ +1 & -1 \\ 0 & +1 \end{bmatrix} + A_0^{(2)} P_{(2)} \begin{bmatrix} 0 & -1 \\ -1 & +1 \\ +1 & 0 \end{bmatrix} \quad (2.5.27)$$

Similarly

$$\mathbf{f}^{ext} = \mathbf{L}_{(1)}^T \mathbf{f}_{(1)}^{ext} + \mathbf{L}_{(2)}^T \mathbf{f}_{(2)}^{ext} = \begin{bmatrix} f_1^{ext} & 0^{ext} \\ f_2^{(1)} & f_1^{(2)} \\ 0 & f_2^{(2)} \end{bmatrix} \quad (2.5.28)$$

and using (2.5.22c) with constant body force gives

$$\mathbf{f}^{ext} = \frac{{}^{(1)}A_0^{(1)} \ell_0^{(1)}}{2} \begin{bmatrix} b & 0 \\ b & b \\ 0 & b \end{bmatrix} + \frac{{}^{(2)}A_0^{(2)} \ell_0^{(2)}}{2} \begin{bmatrix} 0 & b \\ b & b \\ b & 0 \end{bmatrix} \quad (2.5.29)$$

The global, assembled mass matrix is given by (2.5.9)

$$\mathbf{M} = \mathbf{L}_{(1)}^T \mathbf{M}_{(1)} \mathbf{L}_{(1)} + \mathbf{L}_{(2)}^T \mathbf{M}_{(2)} \mathbf{L}_{(2)} \quad (2.5.30)$$

and by (2.5.24a)

$$\mathbf{M} = \mathbf{L}_{(1)}^T \frac{{}_0^{(1)}A_0^{(1)}\ell_0^{(1)}}{6} \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix} \mathbf{L}_{(1)} + \mathbf{L}_{(2)}^T \frac{{}_0^{(2)}A_0^{(2)}\ell_0^{(2)}}{6} \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix} \mathbf{L}_{(2)} \quad (2.5.31)$$

To simplify the form of the assembled equations, we now consider a uniform mesh with constant initial properties, so ${}_0^{(1)} = {}_0^{(2)} = 0$, $A_0^{(1)} = A_0^{(2)} = A_0$, $\ell_0^{(1)} = \ell_0^{(2)} = \ell_0$ and we define $m_1 = \left(\frac{{}_0^{(1)}A_0^{(1)}\ell_0^{(1)}}{6} \right) / 6$, $m_2 = \left(\frac{{}_0^{(2)}A_0^{(2)}\ell_0^{(2)}}{6} \right) / 6$ so the assembled mass matrix is

$$\mathbf{M} = \begin{matrix} 2m_1 & m_1 & 0 \\ m_1 & 2(m_1 + m_2) & m_2 \\ 0 & m_2 & 2m_2 \end{matrix} \quad (2.5.32)$$

Writing out the second equation of motion for this system (which is obtained from the second row of \mathbf{M} , \mathbf{f}^{ext} and \mathbf{f}^{int}) gives

$$\begin{aligned} & \frac{1}{6} {}_0^{(1)}A_0^{(1)}\ell_0^{(1)} \ddot{u}_1 + \frac{1}{3} \left({}_0^{(1)}A_0^{(1)}\ell_0^{(1)} + {}_0^{(2)}A_0^{(2)}\ell_0^{(2)} \right) \ddot{u}_2 + \frac{1}{6} {}_0^{(2)}A_0^{(2)}\ell_0^{(2)} \ddot{u}_3 \\ & - A^{(1)}P^{(1)} + A^{(2)}P^{(2)} = \frac{b}{2} \left({}_0^{(1)}A_0^{(1)}\ell_0^{(1)} + {}_0^{(2)}A_0^{(2)}\ell_0^{(2)} \right) \end{aligned} \quad (2.5.33)$$

Using uniform properties as before and dividing by $A_0\ell_0$, we obtain the following equation of motion at node 2:

$$\left(\frac{1}{6} \ddot{u}_1 + \frac{2}{3} \ddot{u}_2 + \frac{1}{6} \ddot{u}_3 \right) + \frac{P^{(2)} - P^{(1)}}{\ell_0} = {}_0b \quad (2.5.34)$$

If the mass matrix is lumped, the corresponding expression is

$${}_0\ddot{u}_2 + \frac{P^{(2)} - P^{(1)}}{\ell_0} = {}_0b \quad (2.5.35)$$

The above equation is equivalent to a finite difference expression for the momentum equation (2.2.4) with A_0 constant: it is only necessary to use the central difference expression $P_{,x}(X_2) = (P^{(2)} - P^{(1)}) / \ell_0$ to reveal the identity. Thus the finite element procedure appears to be a circuitous way of obtaining what follows simply and directly from a finite difference approximation. The advantage of a finite element approach is that it gives a consistent procedure for obtaining semidiscrete equations when the element lengths, cross-sectional area, and density vary. Furthermore, for linear problems, a finite element solution can be shown to provide the best approximation in the sense that the error is minimized in the energy norm (see Strang and Fix); finite difference approximations for irregular grids and varying areas and densities, on the other hand, are difficult to construct. The finite element method also gives the means of obtaining consistent mass matrices and higher order elements, which are more accurate. But the main advantage of finite element methods, which undoubtedly has been the driving force behind its popularity, is the ease with which it can model complex geometries. This of course is masked in one dimensional problems, but it will become apparent when we study multi-dimensional problems.

Example 2.5.3. Three-node quadratic displacement element. A 3-node element of length L_0 and cross-sectional area A_0 is shown in Fig. 4. Node 2 is placed between nodes 1 and 3; although in this analysis we do not assume it to be midway between the nodes, it is recommended that it be placed midway between the nodes in most models. The mapping between the material coordinates X and the referential coordinate ξ is given by

$$X(\xi) = \mathbf{N}(\xi) \mathbf{X}_e = \frac{1}{2} \begin{pmatrix} 1 - \xi \\ -1 \\ 1 + \xi \end{pmatrix} \begin{matrix} X_1 \\ X_2 \\ X_3 \end{matrix} \quad (2.5.36)$$

where $\mathbf{N}(\xi)$ is the matrix of Lagrange interpolants, or shape functions, and ξ is the element coordinate. The displacement field is given by the same interpolants

$$u(\xi, t) = \mathbf{N}(\xi) \mathbf{u}_e(t) = \begin{bmatrix} \frac{1}{2} (1 - \xi) & -1 & \frac{1}{2} (1 + \xi) \end{bmatrix} \begin{matrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{matrix} \quad (2.5.37)$$

By the chain rule

$$\frac{du}{d\xi} = \frac{du}{dX} \frac{dX}{d\xi} = \frac{du}{dX} \frac{1}{2} [2 - \xi \quad -4 \quad 2 + \xi] \mathbf{u}_e \quad (2.5.38)$$

We have used the fact that in one dimension, $\frac{dX}{d\xi} = X, \xi^{-1}$. We can write the above as

$$\frac{du}{d\xi} = \mathbf{B}_0 \mathbf{u}_e \quad \text{where} \quad \mathbf{B}_0 = \frac{1}{2X} [2 - \xi \quad -4 \quad 2 + \xi] \mathbf{u}_e \quad (2.5.39)$$

The internal nodal forces are given by Eq. (20):

$$\mathbf{f}_e^{int} = \int_0^1 \mathbf{B}_0^T P d \xi = \frac{1}{2X} \begin{bmatrix} 2 - \xi & -4 & 2 + \xi \end{bmatrix} P A_0 X, \quad d = \frac{1}{2} \begin{bmatrix} 2 - \xi & -4 & 2 + \xi \end{bmatrix} P A_0 d \quad (2.5.40)$$

The above integral is generally evaluated by numerical integration. For the purpose of examining this element further, let $P(\xi)$ be linear in ξ :

$$P(\xi) = P_1 \frac{1 - \xi}{2} + P_3 \frac{1 + \xi}{2} \quad (2.5.41)$$

where P_1 and P_3 are the values of P at nodes 1 and 3, respectively. If X, ξ is constant, this is an exact representation for the stress field in a material which is governed by a linear stress-strain relation in these measures, Eq. (2.2.14), since F is linear in ξ by (2.5.40). The internal forces are then given by

$$\mathbf{f}_e^{int} = \begin{matrix} f_1 \\ f_2 \\ f_3 \end{matrix} \begin{matrix} int \\ \\ e \end{matrix} = \frac{A_0}{6} \begin{matrix} -5P_1 - P_2 \\ 4P_1 - 4P_2 \\ P_1 + 5P_2 \end{matrix} \quad (2.5.42)$$

When P is constant, the nodal force at the center node vanishes and the nodal forces at the end nodes are equal and opposite with magnitude A_0P , as in the two node element. In addition, for any values of P_1 and P_2 , the sum of the nodal forces vanishes, which can be seen by adding all the nodal forces. Thus this element is also in equilibrium.

The external nodal forces are

$$\mathbf{f}_e^{ext} = \begin{matrix} +1 & \frac{1}{2} & (-1) \\ & 1-2 & \\ -1 & \frac{1}{2} & (+1) \end{matrix} \begin{matrix} \\ \\ \end{matrix} \begin{matrix} \\ 0bA_0X, d \\ \end{matrix} + \begin{matrix} \frac{1}{2} & (-1) \\ & 1-2 \\ \frac{1}{2} & (+1) \end{matrix} \begin{matrix} \\ \\ \end{matrix} \begin{matrix} \\ A_0\dot{i}_x^0 \\ \end{matrix} \Big|_e \quad (2.5.43)$$

where the shape functions in the last term are either one or zero at a traction boundary. Using $X_e = (X_1 + X_3 - 2X_2) + \frac{1}{2}(X_3 - X_1)$, then

$$\mathbf{f}_e^{ext} = \frac{0bA_0}{6} \begin{matrix} L_0 - 2(X_1 + X_3 - 2X_2) \\ 4L_0 \\ L_0 + 2(X_1 + X_3 - 2X_2) \end{matrix} + \begin{matrix} \frac{1}{2} & (-1) \\ & 1-2 \\ \frac{1}{2} & (+1) \end{matrix} A_0\dot{i}_x^0 \Big|_e \quad (2.5.44)$$

Element Mass Matrix. The element mass matrix is

$$\mathbf{M}_e = \begin{matrix} +1 & \frac{1}{2} & (-1) \\ & 1-2 & \\ -1 & \frac{1}{2} & (+1) \end{matrix} \left[\begin{matrix} \frac{1}{2} & (-1) & 1-2 & \frac{1}{2} & (+1) \end{matrix} \right] \begin{matrix} \\ 0A_0X, d \\ \end{matrix} \quad (2.5.45)$$

$$= \frac{0A_0}{30} \begin{matrix} 4L_0 - 6(X_1 + X_3 - 2X_2) & 2L_0 - 4(X_1 + X_3 - 2X_2) & -L_0 \\ & 16L_0 & 2L_0 + 4(X_1 + X_3 - 2X_2) \\ & & 4L_0 - 6(X_1 + X_3 - 2X_2) \end{matrix}$$

sym

If the node 2 is at the midpoint of the element, i.e., $X_1 + X_3 = 2X_2$, we have

$$\mathbf{M}_e = \frac{0A_0L_0}{30} \begin{matrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{matrix} \quad (2.5.46)$$

If the mass matrix is diagonalized by the row-sum technique, we obtain

$$\mathbf{M}_e = \frac{0A_0L_0}{6} \begin{matrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{matrix} \quad (2.5.47)$$

This results displays one of the shortcomings of diagonal masses for higher order elements: most of the mass is lumped in the center node. This results in rather strange behavior when

high order modes are excited. Therefore, high order elements are usually avoided when a lumped mass matrix is necessary for efficiency.

2.6 Governing Equations for Updated Lagrangian Formulation

In the updated Lagrangian formulation, the discrete equations are formulated in the current configuration. The stress is measured by the Cauchy (physical) stress given by Eq. (2.1.1). In the updated Lagrangian formulation, variables need to be expressed in terms of the spatial coordinates x and the material coordinates X in different equations. The dependent variables are chosen to be the stress $\sigma(X,t)$ and the velocity $v(X,t)$. This choice differs from the total Lagrangian formulation, where we have used the displacement $u(X,t)$ as the independent variable; this is only a formal difference since the displacement and velocities are both computed in a numerical implementation.

In developing the updated Lagrangian formulation, we will need the dependent variables to be expressed in terms of the Eulerian coordinates. Conceptually this is a simple matter, for we can invert (2.2.1) to obtain

$$X = X^{-1}(x,t) \quad X(x,t) \quad (2.6.1)$$

Any variable can then be expressed in terms of the Eulerian coordinates; for example $\sigma(X,t)$ can be expressed as $\sigma(X(x,t),t)$. While the inverse of a function can easily be written in symbolic form, in practice the construction of an inverse function in closed form is difficult, if not impossible. Therefore the standard technique in finite elements is to express variables in terms of element coordinates, which are sometimes called parent coordinates or natural coordinates. By using element coordinates, we can always express a function, at least implicitly, in terms of either the Eulerian and Lagrangian coordinates.

In updated Lagrangian formulations, the strain measure is the rate-of-deformation given by

$$D_x = \frac{v}{x} \quad (2.6.2a)$$

This is also called the velocity-strain or stretching. It is a rate measure of strain, as indicated by two of the names. It is shown in Chapter 3 that

$$\int_0^t D_x(X,\bar{t}) d\bar{t} = \ln F(X,t) \quad (2.6.2b)$$

in one dimension, so the time integral of the rate-of-deformation corresponds to the "natural" or "logarithmic" strain in one dimension; as discussed in Chapter 3, this does not hold for multi-dimensional states of strain.

The governing equations for the nonlinear dimensional continuum are:

1. conservation of mass (continuity equation)

$$J = \rho_0 \quad \text{or} \quad FA = \rho_0 A_0 \quad (2.6.3)$$

2. conservation of momentum

$$-\frac{d}{dx}(A \sigma) + Ab = A\dot{v} \quad \text{or} \quad (A \sigma)_{,x} + Ab = A\dot{v} \quad (2.6.4)$$

3. measure of deformation

$$D_x = \frac{v}{x} \quad \text{or} \quad D_x = v_{,x} \quad (2.6.5)$$

4. constitutive equation
in total form

$$(X, t) = S^D(D_x(X, t), D_x(X, t), \dots), \int_0^t D_x(X, \bar{t}) d\bar{t}, (X, \bar{t}), \bar{t} \quad t, \text{etc.}) \quad (2.6.6a)$$

in rate form

$${}_{,t}(X, t) = S_t^D(D_x(X, \bar{t}), (X, \bar{t}), \text{etc.}, \bar{t} \quad t) \quad (2.6.6b)$$

energy conservation

$$\dot{w}^{int} = D_x - q_{x,x} + s, \quad q_x = \text{heatflux}, \quad s = \text{heatsource} \quad (2.6.7)$$

The mass conservation equation in the updated Lagrangian formulation is the same as in the total Lagrangian formulation. The momentum equation in the updated formulation involves derivatives with respect to the Eulerian coordinates, whereas in the total Lagrangian formulation, derivatives were with respect to Lagrangian coordinates; in addition, the nominal stress is replaced by the Cauchy stress, and that the current values of the cross-sectional area A and density ρ are used. The constitutive equation as written here relates the rate-of-deformation $D_x(X, t)$ or its integral, the logarithmic strain, to the Cauchy stress or its rate. Note that the constitutive equation is written in terms of material coordinates. The subscript "t" on (2.6.6b) indicates that the constitutive equation is a rate equation. We can also use a constitutive equation expressed in terms of the nominal stress and the stretch λ . It would then be necessary to transform the stress to the Cauchy stress before using the momentum equation and use a different measure of strain. Thus in the updated Lagrangian formulation, some of the system equations are in terms of Eulerian coordinates, while others (mass conservation and constitutive equations) are in terms of Lagrangian coordinates.

The subscripts have been appended to the constitutive function to indicate which stress and strain measures are related by the constitutive equation. The constitutive equation depends on the stress and strain measures which are involved. For example, the constitutive equation for a hypoelastic material in terms of the Cauchy stress and rate-of-deformation is

$${}_{,t}(X, t) = E^D D_x(X, t) \quad (2.6.8)$$

where $E^{PF} = E^D$. To see the relationship between the two moduli, we use the relation

$$D_x = \frac{v}{x} = \frac{v}{X} \frac{X}{x} = \frac{v}{X} F^{-1} = \dot{F} F^{-1} \quad (2.6.9)$$

where the first equality is the definition (2.6.5), the second stems from the chain rule, and the third from the definition of F , Eq. (2.2.4). Then inserting Eqs. (2.2.9) and (2.6.9) in (2.6.8) gives

$$A_0 \frac{d}{dt} \frac{P}{A} = E^D \dot{F} F^{-1} \quad (2.6.10)$$

which after some manipulation yields

$$\dot{P} = \frac{A}{A_0 F} E^D \dot{F} + \frac{\dot{A}}{A_0} P \quad (2.6.11)$$

In general, constitutive equations are not easily converted from one stress-strain pair to another. For the above, the cross-sectional area must be known as a function of the elongation to make the conversion.

The boundary conditions are

$$v(X, t) = \bar{v}(t) \text{ on } \Gamma_v \quad (2.6.12)$$

$$n \cdot \bar{t}_x(X, t) = \bar{t}_x(t) \text{ on } \Gamma_t \quad (2.6.13)$$

where $\bar{v}(t)$ and $\bar{t}_x(t)$ are the prescribed velocity and traction, respectively, and n is the normal to the domain. While the boundary condition is specified as applying to the velocity, any velocity boundary condition is also a displacement boundary condition. Note that the traction always carries a subscript to distinguish it from the time t . The relation between the traction and velocity boundaries is the same as in (2.2.30):

$$\Gamma_v \cap \Gamma_t = \emptyset \quad (2.6.14)$$

The boundary over which the velocity is prescribed is denoted by Γ_v ; it is an essential boundary condition and it plays the same role as Γ_u in the total Lagrangian formulation. The tractions in (2.6.13) are physical tractions, force per current area. They are related to the tractions on the undeformed area by

$$\bar{t}_x A = \bar{t}_x^0 A_0 \quad (2.6.15)$$

In addition we have the stress jump conditions, the counterpart of (2.2.33):

$$\langle A \rangle = 0 \quad (2.6.16)$$

The initial conditions are

$$P(X, 0) = P_0(X) \quad (2.6.17)$$

$$v(X, 0) = v_0(X) \quad (2.6.18)$$

Since we have chosen the velocity and the stresses as the dependent variables, the initial conditions are imposed on these variables. In most practical problems, this choice of initial conditions is more practical than conditions on velocities and displacements, as indicated in Chapter 4.

2.7 Weak Form for Updated Lagrangian Formulation

In this Section, the weak form or variational form for the momentum equation is developed. Recall that the dependent variables are the velocity $v(X, t)$ and the stress $\sigma(X, t)$.

The conditions on the trial functions $v(X, t)$ and the test functions $v(X)$ are:

$$v(X, t) \in U \quad U = \{v(X, t) \mid v \in C^0(X), v = \bar{v} \text{ on } \Gamma_v\} \quad (2.7.1)$$

$$v(X) \in U_0 \quad U_0 = \{v(X) \mid v \in C^0(X), v = 0 \text{ on } \Gamma_v\} \quad (2.7.2)$$

These admissibility conditions are identical to those for the test and trial displacements in the total Lagrangian formulation. As in the total Lagrangian formulation, the stress $\sigma(X, t)$ is assumed to be a C^{-1} function in space. The current domain is $[x_a(t), x_b(t)]$, where $x_a = (X_a, t)$, $x_b = (X_b, t)$.

The strong form consists of the momentum equation (2.6.4), the traction boundary conditions and the jump conditions. The weak form is developed by multiplying the momentum equation (2.6.12) by the test function $v(X)$ and integrating over the current domain. The current domain of the body is appropriate, since the momentum equation involves derivatives with respect to the spatial (Eulerian) coordinates. This gives

$$\int_{x_a}^{x_b} v (A \sigma)_{,x} + A b - A \frac{Dv}{Dt} dx = 0 \quad (2.7.3)$$

Integration by parts is performed as in Section 2.3 (see Eqs. (2.3.2) to (2.3.4)), which gives

$$\begin{aligned} \int_{x_a}^{x_b} v (A \sigma)_{,x} dx &= \left[(v A \sigma)_{,x} - v_{,x} A \right] dx \\ &= (v A n)_i \Big|_t - \sum_i v \langle A \rangle_i - \int_{x_a}^{x_b} v_{,x} A dx \end{aligned} \quad (2.7.4)$$

where i are the points of discontinuity of A ; see Eq. (2.6.16). We have used the fundamental theorem of calculus to convert a line (domain) integral to a sum of point (boundary and jump) values, with σ changed to t because $v(X) = 0$ on Γ_v ; see Eq. (2.7.2). Since the strong form holds, the traction boundary condition (2.6.13) gives $n = \hat{t}_x$ and the jump condition $\langle n \rangle = 0$, which when substituted into the above give

$$\int_{x_a}^{x_b} v_{,x} A - v A b - A \frac{Dv}{Dt} dx - (v A \hat{t}_x)_i \Big|_t = 0 \quad (2.7.5)$$

This weak form is often called the principle of virtual power (or principle of virtual velocities, see Malvern (1969), p. 241). If the test function is considered a velocity, then each term in the above corresponds to a variation of power, or rate of work; for example

$\int_{x_a}^{x_b} b dx$ is a force, and when multiplied by $v(X)$ gives a variation in power. Therefore, the terms in the above weak form will be distinguished from the principle of virtual work in Section (2.3) by designating each term by P with the appropriate superscript. However, it should be stressed that this physical interpretation of the weak form is entirely a matter of choice; the test function $v(X)$ need not be attributed any of the properties of a velocity; it can be any function which satisfies Eq. (2.7.2).

We define the virtual internal power by

$$P^{int} = \int_{x_a}^{x_b} v_{,x} A dx = \int_{x_a}^{x_b} D A dx = \int_{x_a}^{x_b} D d \quad (2.7.6)$$

where the second equality is obtained by taking a variation of (2.6.5), i.e., $D_x = v_{,x}$, while the third equality results from the relation $d = A dx$ which parallels (2.5.20). The integral in Eq. (2.7.6) corresponds to the internal energy rate in the energy conservation equation (2.6.7) except that the rate-of-deformation D is replaced by D , so designating this term as a virtual internal power is consistent with the energy equation.

The virtual powers due to external and inertial forces are defined similarly:

$$P^{ext} = \int_{x_a}^{x_b} v b A dx + \left(v A \dot{t}_x \right) \Big|_t = \int_{x_a}^{x_b} v b d + \left(v A \dot{t}_x \right) \Big|_t \quad (2.7.7)$$

$$P^{inert} = \int_{x_a}^{x_b} v \frac{Dv}{Dt} A dx = \int_{x_a}^{x_b} v \frac{Dv}{Dt} d \quad (2.7.8)$$

Using Eqs. (2.7.6-2.7.8). the weak form (2.7.5) can then be written as

$$P = P^{int} - P^{ext} + P^{inert} = 0 \quad (2.7.9)$$

where the terms are defined above. In summary, the principle of virtual power states that

$$\text{if } v(X, t) \text{ is } U \text{ and } P = 0 \text{ then } v(X) \text{ is } U_0 \quad (2.7.10)$$

then the momentum equation (2.6.4), the traction boundary conditions (2.6.13) and the jump conditions are satisfied. The validity of this principle can be established by simply reversing the steps used to obtain Eq. (2.7.5). All of the steps are reversible so we can deduce the strong form from the weak form.

The key difference of this weak form, as compared to the weak form for the total Lagrangian formulation, is that all integrals are over the current domain and are expressed in terms of variables which have a spatial character. However, the two weak forms are just different forms of the same principle; it is left as an Exercise to show that the principle of virtual work can be transformed to the principle of virtual power by using transformations on the integrals and the variables.

Exercise. Replace the virtual displacement in the principle of virtual work by a velocity and use the relations to show that it can be transformed into the principle of virtual power.

2.8. Element Equations for Updated Lagrangian Formulation

We will now develop the updated Lagrangian formulation. As will become clear, the updated Lagrangian formulation is simply a transformation of the total Lagrangian formulation. Numerically, the discrete equations are identical, and in fact, as we shall see, we can use the total Lagrangian formulation for some of the nodal forces and the updated for others in the same program. Students often ask why both methods are presented when they are basically identical. We must confess that the major reason for presenting both formulations today is that both are widely used, so to understand today's software and literature, a familiarity with both formulations is essential. However, in a first course, it is often useful to skip one of these Lagrangian formulations.

The domain is subdivided into elements e , so that $\Omega = \cup_e e$. The coordinates of the nodes in the initial configuration are given by X_1, X_2, \dots, X_{n_e} and the positions of the nodes are given by $x_1(t), x_2(t), \dots, x_m(t)$. The m nodes of element e in the initial configuration be denoted by $X_1^e, X_2^e, \dots, X_m^e$, and the positions of these nodes in the current configuration be given by $x_1^e(t), x_2^e(t), \dots, x_m^e(t)$. The spatial coordinates of the nodes are given by the finite element approximation to the motion

$$x_I(t) = x(X_I, t) \quad (2.8.1)$$

Thus each node of the mesh remains coincident with a material point.

We will develop the equations on an element level and then obtain the global equations by assembly using the scheme described in Section 2.5. As before, the relationships between the terms of the virtual power expression and the corresponding nodal forces along with the physically motivated names will be employed to systematize the procedure.

The dependent variables in this development will be the velocity and the stress. The constitutive equation, combined with the expression for the velocity-strain, and the mass conservation equation are treated in strong form, the momentum equation in weak form. The mass conservation equation can be used to easily compute the density at any point since it is a simple algebraic equation. We develop the equations as if there were no essential boundary conditions and then impose these subsequently.

The velocity field in each element is approximated by

$$\mathbf{v}(X, t) = \sum_{I=1}^m N_I(X) \mathbf{v}_I^e(t) = \mathbf{N}(X) \mathbf{v}_e(t) \quad (2.8.2)$$

Although the shape functions are functions of the material coordinates X , they can be expressed in terms of spatial coordinates. For this purpose, the mapping $x = x(X, t)$ is inverted to give $X = X^{-1}(x, t)$ so the velocity field is

$$v(x, t) = \mathbf{N}(X^{-1}(x, t)) \mathbf{v}^e(t) \quad (2.8.3)$$

Although developing the inverse mapping is often impossible, partial derivatives with respect to the spatial coordinates can be obtained by implicit differentiation, so the inverse mapping need never be calculated.

The acceleration field is given by taking the material time derivative of (2.8.2), which gives

$$\dot{v}(X, t) = \mathbf{N}(X) \dot{\mathbf{v}}(t) + \mathbf{N}(X) \mathbf{a}(t) \quad (2.8.4)$$

It can be seen from this step that it is crucial that the shape functions be expressed as functions of the *material coordinates*. If the shape functions are expressed in terms of the Eulerian coordinates by

$$v(x, t) = \mathbf{N}(x) \mathbf{v}^e(t) = \mathbf{N}(X(x, t)) \mathbf{v}^e(t) \quad (2.8.5)$$

then material time derivative of the shape functions does not vanish and the accelerations cannot be expressed as a product of the same shape functions and nodal accelerations. Therefore, the shape functions are considered to be functions of the material coordinates in the updated Lagrangian method. In fact, expressing the shape functions in terms of spatial coordinates is incompatible with a Lagrangian mesh, since we need to approximate the velocity in an element, which is a material subdomain.

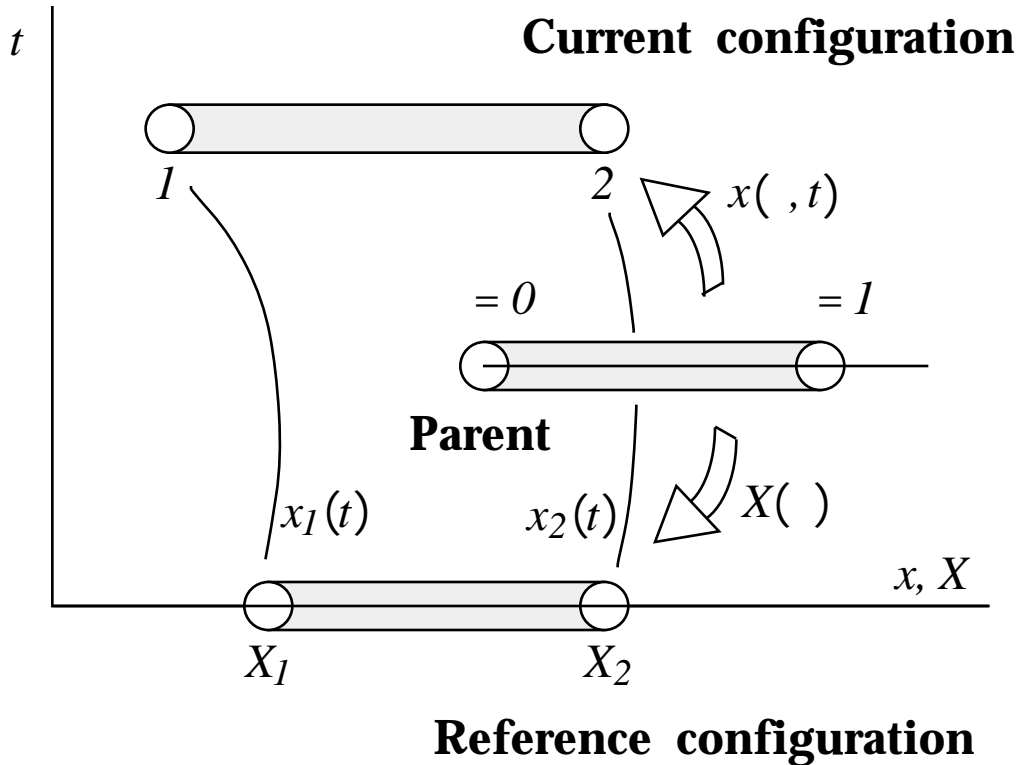


Fig. 2.5. Role of parent configuration, showing mappings to the initial, undeformed configuration and the current, deformed configuration in a Lagrangian mesh.

Element Coordinates. Calculations in the updated Lagrangian formulation are usually performed in the element coordinate system in the parent domain. This is in fact simpler than working in the spatial domain. We have already used element coordinates to simplify the evaluation of element nodal forces in the examples. Element coordinates, such as triangular coordinates and isoparametric coordinates, are particularly convenient for multi-dimensional elements.

Consider Fig. 2.5, which shows a two-node element in the initial and current configurations and the parent domain, which is the interval $0 \leq \xi \leq 1$. The parent domain can be mapped onto the initial and current configurations as shown. For example, in the two-node element, the mapping between the element coordinates and the Eulerian coordinates is given by

$$x(\xi, t) = x_1(t)(1 - \xi) + x_2(t) \tag{2.8.6}$$

or for a general one dimensional element as

$$x(\xi, t) = \mathbf{N}(\xi) \mathbf{x}^e(t) \tag{2.8.7}$$

Specializing the above to the initial time gives the map between the parent domain and the initial configuration

$$\mathbf{X}(\xi) = \sum_{I=1}^m N_I(\xi) \mathbf{X}_I^e = \mathbf{N}(\xi) \mathbf{X}_e \tag{2.8.8}$$

which for the two-node element is

$$X(\xi) = X_1(1 - \xi) + X_2 \quad (2.8.9)$$

The mapping between the Eulerian coordinates and the element coordinates, (2.8.6), changes with time, while the map between the initial configuration and the element domain is time invariant in a Lagrangian mesh. Therefore shape functions expressed in terms of the element coordinates by (2.8.8) will be independent of time. If the initial map is such that every point in the parent element maps onto a unique point of the initial configuration, and for every point X there exists a point ξ , then the parent element coordinates can serve as material labels. Such a map is called one-to-one and onto. The map between the parent domain and the current configuration must be one-to-one and onto for all time; this is discussed further in Example 2.8.3 and Chapter 3.

As shown in Fig. 2.5, at any time the shape functions can be used to map between the current and parent element configurations. Thus the element coordinates provide a link between the initial configuration and the current configuration of the element which can be used in the evaluation of derivatives and integrals.

It follows from Eqs. (2.8.7) and (2.8.8) that the displacements can also be interpolated by the same shape functions since

$$u(\xi, t) = x(\xi, t) - X(\xi) = \mathbf{N}(\xi)(\mathbf{x}^e(t) - \mathbf{X}^e) = \mathbf{N}(\xi)\mathbf{u}^e(t) \quad (2.8.10)$$

The velocities and accelerations are also given by material derivatives of the displacement, while the test function is given by the same shape functions, so

$$v(\xi, t) = \mathbf{N}(\xi)\mathbf{v}_e(t) \quad a(\xi, t) = \mathbf{N}(\xi)\ddot{\mathbf{u}}_e(t) \quad v(\xi, t) = \mathbf{N}(\xi) v_e \quad (2.8.11)$$

since the shape functions are independent of time.

Using Eqs. (2.8.2) and (2.6.5) and noting Eq.(2.8.3), the rate-of-deformation can be expressed in terms of the shape functions by

$$D_x(x, t) = v_{,x}(x, t) = \mathbf{N}_{,x}(X(x, t))\mathbf{v}^e(t) \quad (2.8.12)$$

where we have indicated the implicit dependence of the shape functions on the Eulerian coordinates. The rate-of deformation will be expressed in terms of nodal velocities via a \mathbf{B} matrix by

$$D_x = v_{,x} = \mathbf{B}\mathbf{v}^e = \sum_{I=1}^m B_I v_I^e \quad (2.8.13)$$

where

$$\mathbf{B} = \mathbf{N}_{,x} \quad \text{or} \quad B_I = N_{I,x} \quad (2.8.14)$$

This \mathbf{B} matrix differs from the \mathbf{B}_0 matrix used in the total Lagrangian formulation in that the derivatives are taken with respect to the Eulerian coordinates..

To compute the spatial derivative of the shape function, we use the chain rule

$$\mathbf{N}_{,i} = \mathbf{N}_{,x} x_{,i} \quad \text{so} \quad \mathbf{N}_{,x} = \mathbf{N}_{,i} x_{,i}^{-1} \quad (2.8.15)$$

From the above, it follows that

$$D_x \left(\cdot, t \right) = x_{,i}^{-1} \mathbf{N}_{,i} \left(\cdot \right) \mathbf{v}^e(t) = \mathbf{B} \left(\cdot \right) \mathbf{v}^e(t) \quad \mathbf{B} \left(\cdot \right) = \mathbf{N}_{,x} x_{,i}^{-1} \quad (2.8.16)$$

Internal and External Nodal Forces. We now use the procedure given in Sections 2.4 and 2.5 to determine nodal forces corresponding to each term of the weak form on an element level. The assembled equations and essential boundary conditions are developed subsequently. The internal nodal forces will be developed from the virtual internal power. Defining the element internal nodal forces so that the scalar product with the virtual velocities gives the internal virtual power, then from (2.7.6) and (2.8.13) we can write

$$P_e^{int} = \int_{x_1^e(t)}^{x_m^e(t)} \mathbf{v}_e^T \mathbf{f}_e^{int} = \int_{x_1^e(t)}^{x_m^e(t)} v_{,x}^T A dx = \int_{x_1^e(t)}^{x_m^e(t)} \mathbf{v}_e^T \mathbf{N}_{,x}^T A dx \quad (2.8.17)$$

The transpose is taken of the first term in the integrand even though it is a scalar so that the expression remains consistent when v is replaced by a matrix product. From the arbitrariness of \mathbf{v}_e , it follows that

$$\mathbf{f}_e^{int} = \int_{x_1^e(t)}^{x_m^e(t)} \mathbf{N}_{,x}^T A dx = \int_{x_1^e(t)}^{x_m^e(t)} \mathbf{B}^T A dx \quad \text{or} \quad \mathbf{f}_e^{int} = \mathbf{B}^T d \quad (2.8.18)$$

We have explicitly indicated the time dependence of the limits of integration of the integrals to emphasize that the domain of integration varies with time. The internal nodal forces can then be evaluated in terms of element coordinates by transforming (2.8.18) to the parent domain and using the above with $dx = x_{,d} d$, giving

$$\mathbf{f}_e^{int} = \int_{x_1^e(t)}^{x_m^e(t)} \mathbf{N}_{,x}^T A dx = \int_1^m \mathbf{N}_{,x}^T x_{,i}^{-1} A x_{,i} d = \int_1^m \mathbf{N}_{,i}^T A d \quad (2.8.19)$$

The last form in the above is nice, but this simplification can be made only in one dimension.

The external nodal forces are obtained from the expression for virtual external power (2.7.7):

$$P_e^{ext} = \mathbf{v}_e^T \mathbf{f}_e^{ext} = \int_e v^T b d + \left(v^T A \dot{t}_x \right) \Big|_t \quad (2.8.20)$$

Substituting (2.8.11) into the right hand side of the above and using the arbitrariness of \mathbf{v}_e gives

$$\mathbf{f}_e^{ext} = \int_{x_1^e}^{x_m^e} \mathbf{N}^T b A dx + \left(\mathbf{N}^T A \dot{t}_x \right) \Big|_{x_1^e}^{x_m^e} = \mathbf{N}^T b d + \left(\mathbf{N}^T A \dot{t}_x \right) \Big|_{x_1^e}^{x_m^e} \quad (2.8.21)$$

where the second term contributes only when the boundary coincides with a node of the element.

Mass Matrix. The inertial nodal forces and mass matrix are obtained from the virtual inertial power (2.7.8):

$$P^{inert} = \mathbf{v}_e^T \mathbf{f}_e^{inert} = \int_{x_1(t)}^{x_m(t)} v^T \frac{Dv}{Dt} A dx \quad (2.8.22)$$

Substituting (2.8.11) into the above yields

$$\mathbf{f}_e^{inert} = \int_{x_1(t)}^{x_m(t)} \mathbf{N}^T \mathbf{N} A dx \dot{\mathbf{v}}^e = \mathbf{M}^e \dot{\mathbf{v}}^e \quad (2.8.23)$$

where the inertial force has been written as the product of a mass matrix \mathbf{M} and the nodal accelerations. The mass matrix is given by

$$\mathbf{M}^e = \int_{x_1(t)}^{x_m(t)} \mathbf{N}^T \mathbf{N} A dx = \int_{x_1(t)}^{x_m(t)} \mathbf{N}^T \mathbf{N} d \quad (2.8.24)$$

The above form is inconvenient because it suggests that the mass matrix is a function of time, since the limits of integration and the cross-sectional area are functions of time. However, if we use the mass conservation equation (2.2.10) in the form ${}_0 A_0 dX = A dx$, we can obtain a time invariant form. Substituting the (2.2.10) into (2.8.24) gives

$$\mathbf{M}^e = \int_{x_1}^{x_m} {}_0 \mathbf{N}^T \mathbf{N} A_0 dX \quad (2.8.25)$$

This formula for the mass matrix is identical to the expression developed for the total Lagrangian formulation, (2.4.11). The advantage of this expression is that it clearly shows that the mass matrix in the updated Lagrangian formulation does not change with time and therefore need not be recomputed during the simulation, which is not clear from (2.8.24). We will see shortly that any nodal force for a Lagrangian mesh can be computed by either the total or updated Lagrangian formalism. The one which is chosen is purely a matter of convenience. Since it is more convenient and illuminating to evaluate the mass matrix in the total Lagrangian form, this has been done.

Equivalence of Updated and Total Lagrangian Formulations. The internal and external nodal forces in the updated and total Lagrangian formulations can be shown to be

identical. To show the identity for the nodal internal forces, we express the spatial derivative of the shape function in terms of the material derivative by the chain rule:

$$\mathbf{N}_{,x}(X) = N_{,X} \frac{X}{x} = \mathbf{N}_{,X} F^{-1} = \mathbf{B}_0 F^{-1} \quad (2.8.26)$$

From the first equality we have $\mathbf{N}_{,x} dx = N_{,X} dX$, and substituting this into (2.8.18) gives

$$\mathbf{f}_e^{int} = \int_{x_1(t)}^{x_m(t)} \mathbf{N}_{,x}^T A dx = \int_{X_1}^{X_m} \mathbf{N}_{,X}^T A dX \quad (2.8.27)$$

where the limits of integration in the third expression have been changed to the material coordinates of the nodes since the integral has been changed to the initial configuration. If we now use the identity $A = PA_0$, Eq.(2.2.9), we obtain from the above that

$$\mathbf{f}_e^{int} = \int_{X_1}^{X_m} \mathbf{N}_{,X}^T P A_0 dX \quad (2.8.28)$$

This expression is identical to the expression for the internal nodal forces in the total Lagrangian formulation, (2.5.14). Thus the expressions for the internal nodal forces in the updated and total Lagrangian formulations are simply two ways of expressing the same thing.

The equivalence of the external nodal forces is shown by using the conservation of mass equation, (2.2.10). Starting with (2.8.21) and using the (2.2.10) gives

$$\mathbf{f}_e^{ext} = \int_{x_1^e}^{x_m^e} \mathbf{N}^T b A dx + \left(\mathbf{N}^T A \dot{t}_x \right)_i^e = \int_{X_1}^{X_m} \mathbf{N}^T {}_0 b A_0 dX + \left(\mathbf{N}^T A_0 \dot{t}_x^0 \right)_i^e \quad (2.8.29)$$

where we have used the identity $t_x A = t_x^0 A_0$ in the last term. The above is identical to (2.4.8), the expression in the total Lagrangian formulation.

From this and the identity of the expression for the mass matrix, it can be seen that the total and updated Lagrangian formulations simply provide alternative expressions for the same nodal force vectors. The formulation which is used is simply a matter of convenience. Moreover, it is permissible to use either of these formulations for different nodal forces in the same calculation. For example, the internal nodal forces can be evaluated by an updated Lagrangian approach and the external nodal forces by a total Lagrangian approach in the same calculation. Thus the total and updated Lagrangian formalisms simply reflect different ways of describing the stress and strain measures and different ways of evaluating derivatives and integrals. In this Chapter, we have also used different dependent variables in the two formulations, the velocity and stress in the updated formulations, the nominal stress and the displacement in the total formulation. However, this difference is not tied to the type of Lagrangian formulation, and we have done this only to illustrate how different independent variables can be used in formulating the continuum mechanics problem. We could have used the displacements as the dependent variables in the updated Lagrangian formulation just as well.

Assembly, Boundary Conditions and Initial Conditions. The assembly of the element matrices to obtain the global equations is identical to the procedure described for the total Lagrangian formulation in Section 2.5. The operations of gather are used to obtain the nodal velocities of each element, from which the strain measure, in this case the rate-of-deformation, can be computed in each element. The constitutive equation is then used to evaluate the stresses, from which the nodal internal forces can be computed by (2.8.19). The internal and external nodal forces are assembled into the global arrays by the scatter operation. Similarly, the imposition of essential boundary conditions and initial conditions is identical and described in Section 2.4. The resulting global equations are identical to (2.4.17) and (2.4.15). Initial conditions are now needed on the velocities and stresses. For an unstressed body at rest, the initial conditions are given by

$$v_I = 0, I = 1 \text{ to } n_N \quad \sigma_I = 0, I = 1 \text{ to } n_Q \quad (2.8.30)$$

That initial conditions in terms of the stresses and velocities is more appropriate for engineering problems is discussed in Section 4.2. Nonzero initial values can be fit by an L_2 projection described at the end of Section 2.4.

Box 2.3 Updated Lagrangian Formulation

$$u(X, t) = \mathbf{N}(X) \mathbf{u}_e(t) = N_I(X) u_I^e(t) \quad (\text{B2.3.1})$$

$$\dots v(X, t) = \mathbf{N}(X) \mathbf{v}_e(t) = N_I(X) v_I^e(t) \quad (\text{B2.3.2})$$

note $\mathbf{N}_{,x} = \mathbf{N}, x^{-1}$

in each element

$$D_x = \sum_{I=1}^m \frac{N_I}{x} v_I^e = \mathbf{B} \mathbf{v}^e \quad (\text{B2.3.3})$$

evaluate s by constitutive equation

$$\mathbf{f}_e^{int} = \frac{\mathbf{N}}{x} d \quad \text{or} \quad \mathbf{f}_e^{int} = \mathbf{B}^T d \quad (\text{B2.3.4})$$

$$\mathbf{f}_e^{ext} = \mathbf{N}^T b d + (\mathbf{N}^T A \dot{x}) \Big|_t \quad (\text{B2.3.5})$$

$$\mathbf{M}_e = \int_0 \mathbf{N}^T \mathbf{N} d \quad \text{same as total Lagrangian} \quad (\text{B2.3.6})$$

$$\mathbf{M} \ddot{\mathbf{u}} + \mathbf{f}^{int} = \mathbf{f}^{ext} \quad (\text{B2.3.7})$$

Example 2.8.1. Updated Lagrangian Form of Two-Node Linear Displacement Element. This element is the same as in Example 2.5.1, Fig. 3, except the updated Lagrangian treatment is now used. Recall that A_0 and ℓ_0 are assumed to be constant in each element. The velocity field is the same as for the updated Lagrangian element, (2.5.19):

$$v(X, t) = \frac{1}{\ell_0} \underbrace{[X_2 - X, X - X_1]}_{\mathbf{N}(X)} \begin{matrix} v_1(t) \\ v_2(t) \end{matrix} \quad (\text{2.8.31})$$

In terms of element coordinates, the velocity field is

$$\mathbf{v}(\xi, t) = \underbrace{\begin{bmatrix} 1 - \xi \\ \xi \end{bmatrix}}_{\mathbf{N}(\xi)} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \frac{X - X_1}{\ell_0} \quad (2.8.32)$$

The displacement is the time integrals of the velocity, and since ξ is independent of time

$$\mathbf{u}(\xi, t) = \mathbf{N}(\xi) \mathbf{u}_e(t) \quad (2.8.33)$$

Therefore, since $x = X + u$

$$\mathbf{x}(\xi, t) = \mathbf{N}(\xi) \mathbf{x}_e(t) = \begin{bmatrix} 1 - \xi \\ \xi \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad x_e = x_2 - x_1 = \ell \quad (2.8.34)$$

where ℓ is the current length of the element. For this element, we can express ξ in terms of the Eulerian coordinates by

$$\xi = \frac{x - x_1}{x_2 - x_1} = \frac{x - x_1}{\ell}, \quad \ell = x_2 - x_1, \quad \xi_{,x} = \frac{1}{\ell} \quad (2.8.35)$$

So $\xi_{,x}$ can be obtained directly, instead of through the inverse of $x_{,\xi}$. This is not the case for higher order elements.

The \mathbf{B} matrix is obtained by the chain rule

$$\mathbf{B} = \mathbf{N}_{,\xi} \xi_{,x} = \mathbf{N}_{,\xi} \frac{1}{\ell} = \frac{1}{\ell} \begin{bmatrix} -1 & 1 \end{bmatrix} \quad (2.8.36)$$

so the rate-of-deformation is given by

$$D_x = \mathbf{B} \mathbf{v}^e = \frac{1}{\ell} (v_2 - v_1) \quad (2.8.37)$$

Using (2.8.18) then gives

$$\mathbf{f}_e^{int} = \int_{x_1}^{x_2} \mathbf{B}^T A dx = \frac{1}{\ell} \int_{x_1}^{x_2} \begin{bmatrix} -1 \\ +1 \end{bmatrix} A dx \quad (2.8.38)$$

If the integrand in (2.8.38) is constant, as if often is, then (2.8.38) yields

$$\mathbf{f}_e^{int} = A \begin{bmatrix} -1 \\ +1 \end{bmatrix} \quad (2.8.39)$$

Thus the internal nodal forces for the element correspond to the forces resulting from the stress σ . Note that the internal nodal forces are in equilibrium.

The external nodal forces are evaluated using (2.8.21)

$$\mathbf{f}_e^{ext} = \int_{x_1}^{x_2} \left(b A dx + A \dot{t}_x \right) \mathbf{i}_e \quad (2.8.40)$$

where the last term makes a contribution only if a node of the element is on the traction boundary. Since x is a linear function of ξ and t , Eq. (2.8.16), $b(x,t)$ can always be expressed as a function of ξ and t . It is conventional to fit the data for $b(x,t)$ by linear interpolants for linear displacement elements (the information in higher order interpolations will be beyond the resolution of the mesh). So we let

$$b(\xi, t) = b_1(1 - \xi) + b_2 \xi \quad (2.8.41)$$

Substituting into (2.8.31) and integrating gives

$$\mathbf{f}_e^{ext} = \frac{A \ell}{6} \begin{bmatrix} 2b_1 + b_2 \\ b_1 + 2b_2 \end{bmatrix} \quad (2.8.42)$$

Comparison to Total Lagrangian. We will now compare the nodal forces to those obtained by the total Lagrangian formulation. Replacing \mathbf{f}_e^{ext} in (2.8.39) by the nominal stress using Eq. (2.1.3a), we see that (2.8.39) and (2.5.27) are equivalent. It can easily be shown that (2.8.29) and (2.8.21) lead to the same expression as (2.8.31).

To compare the external nodal forces, we note that by the conservation of matter, $A \ell = {}_0A_0 \ell_0$. Using this in Eq. (2.8.42) gives (2.5.26), the total Lagrangian form of the nodal external forces. In the updated Lagrangian formulation, the mass from the total Lagrangian formulation is used, see Eq. (2.8.25), so the equivalence need not be considered.

Example 2.8.2. Updated Lagrangian of Three Node Element, Quadratic Displacement Element The 3-node element is shown in Fig. 2.7. Node 2 can be placed anywhere between the end-nodes, but we shall see there are restrictions on the placement of this node if the one-to-one condition is to be met. We will also examine the effects of *mesh distortion*.

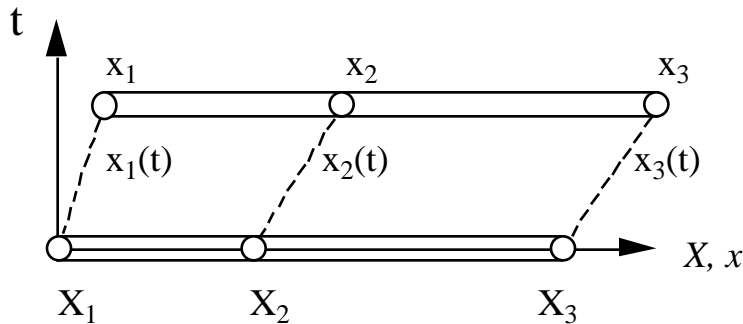


Fig. 2.7. Three node, quadratic displacement element in original and current configurations.

The displacement and velocity fields will be written in terms of the element coordinates

$$\mathbf{u}(\cdot, t) = \mathbf{N}(\cdot) \mathbf{u}_e(t), \quad \mathbf{v}(\cdot, t) = \mathbf{N}(\cdot) \mathbf{v}_e(t), \quad \mathbf{x}(\cdot, t) = \mathbf{N}(\cdot) \mathbf{x}_e(t) \quad (2.8.43)$$

where

$$\mathbf{N}(\cdot) = \begin{bmatrix} \frac{1}{2}(\xi^2 - 1) & 1 - \xi^2 & \frac{1}{2}(\xi^2 + 1) \end{bmatrix} \quad (2.8.44)$$

and

$$\mathbf{u}_e^T = [u_1, u_2, u_3] \quad \mathbf{v}_e^T = [v_1, v_2, v_3] \quad \mathbf{x}_e^T = [x_1, x_2, x_3] \quad (2.8.45)$$

The \mathbf{B} matrix is given by

$$\mathbf{B} = \mathbf{N}_{,x} = \xi^{-1} \mathbf{N}, \quad (2.8.46)$$

$$= \frac{1}{2\xi} \begin{bmatrix} 2\xi & -1 & -4\xi & 2\xi & +1 \end{bmatrix} \quad (2.8.47)$$

where

$$\xi = \mathbf{N}, \quad \mathbf{x}_e = \begin{pmatrix} -\frac{1}{2} \end{pmatrix} x_1 - 2x_2 + \begin{pmatrix} +\frac{1}{2} \end{pmatrix} x_3 \quad (2.8.48)$$

The rate of deformation is given by

$$D_x = \mathbf{N}_{,x} \mathbf{v}_e = \mathbf{B} \mathbf{v}_e = \frac{1}{2\xi} \begin{bmatrix} 2\xi & -1 & -4\xi & 2\xi & +1 \end{bmatrix} \mathbf{v}_e \quad (2.8.49)$$

This rate-of-deformation varies linearly in the element if ξ is constant, which is the case when node 2 is midway between the other two nodes. However, when node 2 moves away from the midpoint due to element distortion, ξ becomes linear and the rate-of-deformation is a rational function. Furthermore, as node 2 moves from the center, it becomes possible for ξ to become negative or vanish. In that case, the mapping between the current spatial coordinates and the element coordinates is no longer one-to-one.

The internal forces are given by (2.8.18):

$$\mathbf{f}_e^{int} = \mathbf{B}^T \int_{x_1}^{x_3} A dx = \frac{1}{\xi} \begin{bmatrix} -\frac{1}{2} & +1 & -\frac{1}{2} \\ -2 & A\xi & d \\ +\frac{1}{2} & -1 & +\frac{1}{2} \end{bmatrix} \begin{bmatrix} +1 \\ A \\ -2 \\ d \end{bmatrix} \quad (2.8.50)$$

where we have used $dx = \xi d\xi$. Using (2.1.3), we can see that this expression is identical to the internal force expression for the total Lagrangian formulation.

Mesh Distortion. We will now examine the effects of mesh distortion on this element. When $x_2 = \frac{1}{4}(x_3 + 3x_1)$, i.e. when node 2 of the element is one quarter of the element length from node 1, then $\xi = \frac{1}{2}(x_3 - x_1)(\xi + 1)$, so $\xi = 0$ at $\xi = -1$. Examining the Jacobian given by Eq. (2.2.3)

$$J = \frac{A}{A_0} x, X = \frac{A}{A_0} x, X^{-1} \quad (2.8.51)$$

we see that it will also vanish. By E. (2.2.4a) this implies that the current density becomes infinite at that point. As node 2 moves closer to node 1, the Jacobian becomes negative in part of the element, which implies a negative density and a violation of the one-to-one condition. This corresponds to a violation of mass conservation and continuity of the displacement field. These situations are often masked by numerical quadrature, because the condition must be more severe to appear at Gauss quadrature points.

The failure to meet the one-to-one condition can also affect the rate-of-deformation, which is given by $D_x = \mathbf{B} \mathbf{v}_e$. From (2.8.37) we can see the potential for difficulties when the denominator x vanishes or becomes negative. When $x_2 = \frac{1}{4}(x_3 + x_1)$, and $x_1 = 0$ at $\xi = -1$, then the rate-of-deformation becomes infinite at node 1. This property of quadratic displacement elements has been exploited in fracture mechanics to develop elements with singular cracktip stresses called quarter-point elements, but in large displacement analysis this phenomenon can be troublesome.

In one-dimensional elements the effects of mesh distortion are not as severe as in multi-dimensional problems. In fact, the effects of mesh distortion can be alleviated somewhat in this element by using F as a measure of deformation, see Eq. (2.5.40). The deformation gradient F never becomes singular in the 3-node element if the initial position of X_2 is at the midpoint. However, any constitutive equation expressed in terms of F will differ markedly from one expressed in terms of the rate-of-deformation D_x when the strains are large.

Example 2.8.3. Axisymmetric 2-Node Element. As an example where the concept of the principle of virtual power or work becomes quite useful, we consider the analysis of an axisymmetric two dimensional disc of constant thickness, a , which is thin compared to its dimensions so $\epsilon_z = 0$. The only nonzero velocity is $v_r(r)$, which as shown, is only a function of the radial coordinate in an axisymmetric problem. The nonzero Cauchy stresses and rate-of-deformations are written in cylindrical coordinates using Voigt notation

$$\{\mathbf{D}\} = \begin{Bmatrix} D_r \\ D \end{Bmatrix} \quad \{\sigma\} = \begin{Bmatrix} \sigma_r \\ \sigma \end{Bmatrix} \quad (2.8.52)$$

The rate-of-deformations are given by

$$D_r = v_{r,r} \quad D = \frac{v_r}{r} \quad (2.8.53)$$

and the momentum equation is

$$-\frac{r}{r} + \frac{r}{r} + b_r = \dot{v}_r \quad (2.8.54)$$

The boundary conditions are

$$r(a) = a \quad r(b) = b \quad (2.8.55)$$

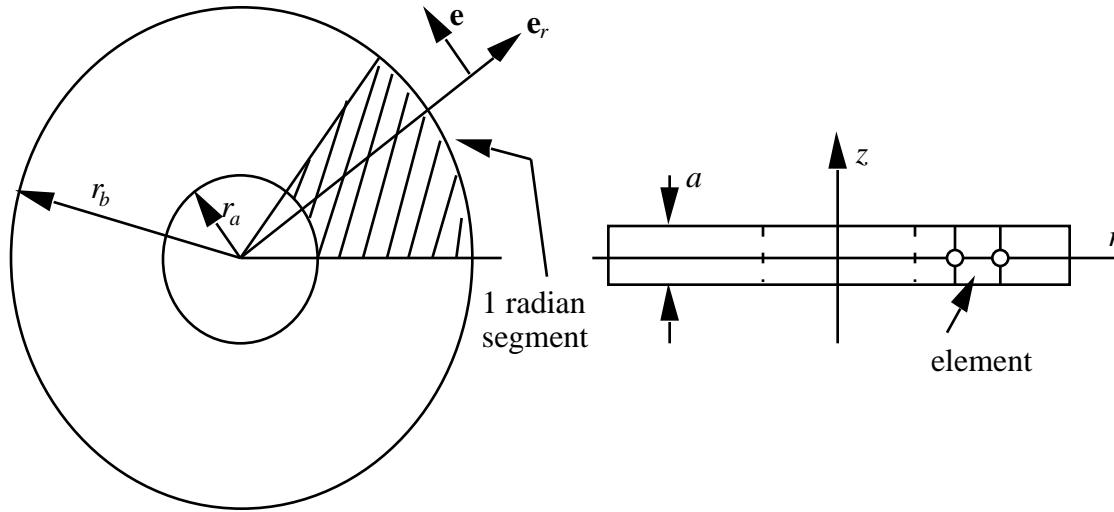


Fig. Schematic of axisymmetric disc
the shaded area is considered in work terms

It is not necessary to integrate the momentum equation to obtain its weak form. By the principle of virtual power the weak form is

$$P = 0 \quad v_r \quad U_0 \quad (2.8.56)$$

The internal virtual power is obtained from the rate-of-deformation and stress

$$P_e^{int} = \int_{r_1^e}^{r_2^e} (D_r \quad r + D) ar dr = \int_e \{ \mathbf{D} \}^T \{ \dot{v} \} d \quad (2.8.57)$$

where $d = ar dr$ because a segment of one radian in the circumferential direction has been chosen to avoid the factor 2 in all terms. The external virtual power is given by

$$P_e^{ext} = \int_e v_r \quad b_r d + (ar \dot{v}_r) \Big|_i \quad (2.8.58)$$

where ar in the last term is the area of a one radian segment. The virtual inertial power is given by

$$P_e^{inert} = \int_e v_r \quad \dot{v}_r d \quad (2.8.59)$$

Consider a two-node finite element with a linear velocity field written in terms of element coordinates

$$v(r, t) = \begin{bmatrix} 1 - \frac{r}{a} \\ \frac{r}{a} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \quad (2.8.60)$$

The rate-of-deformation is evaluated by Eq.() using the above velocity field and immediately put into matrix form

$$\mathbf{D} = \begin{matrix} D_r \\ D \end{matrix} = \begin{matrix} -\frac{1}{r_{21}} & \frac{1}{r_{21}} \\ \frac{1}{r} & -\frac{1}{r} \end{matrix} \begin{matrix} v_1(t) \\ v_2(t) \end{matrix} = \mathbf{B}\mathbf{v}_e \quad (2.8.61)$$

The internal nodal forces are given by an expression identical to () except that the stress is replaced by the column matrix

$$\mathbf{f}_e^{int} = \int_{r_1}^{r_2} \mathbf{B}^T \{ \sigma \} d = \int_{r_1}^{r_2} \begin{matrix} -\frac{1}{r_{21}} & \frac{1}{r_{21}} \\ \frac{1}{r} & -\frac{1}{r} \end{matrix} \sigma r \, ardr \quad (2.8.62)$$

The external nodal forces are given by

$$\mathbf{f}_e^{ext} = \int_{r_1}^{r_2} \begin{matrix} 1 \\ -1 \end{matrix} b_r ardr + (ar\bar{r}_r) \Big|_{r_1}^{r_2} \quad (2.8.63)$$

The element mass matrix is given by

$$\begin{aligned} \mathbf{M}_e &= \int_{r_1}^{r_2} \begin{matrix} 1 \\ -1 \end{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ardr \\ &= \frac{ar_{21}}{12} \begin{bmatrix} 3r_1 + r_2 & r_1 + r_2 \\ r_1 + r_2 & r_1 + 3r_2 \end{bmatrix} \end{aligned} \quad (2.8.64)$$

The lumped mass matrix can be computed by the row sum technique or by lumping half the mass at each node, which gives, respectively

$$\mathbf{M}_e = \frac{ar_{21}}{6} \begin{bmatrix} 2r_1 + r_2 & 0 \\ 0 & r_1 + 2r_2 \end{bmatrix}_{\text{row-sum}} \quad \mathbf{M}_e = \frac{ar_{21}(r_1 + r_2)}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{\text{lump}} \quad (2.8.65)$$

As can be seen the two lumping procedures give slightly different results.

2.9. Governing Equations for Eulerian Formulation

In an Eulerian formulation, the nodes are fixed in space and the independent variables are functions of the Eulerian spatial coordinate x and the time t . The stress measure is the Cauchy (physical) stress $\sigma(x, t)$, the measure of deformation is the rate-of-deformation $D_x(x, t)$. The motion will be described by the velocity $v(x, t)$. In Eulerian formulations, the deformation is not expressed as a function of the reference coordinates since an

undeformed, initial configuration cannot be established, and no counterpart of (2.2.1) is available.

Box. 2.4. Governing Equations for Eulerian Formulation

continuity equation (mass conservation):

$$\frac{\rho}{t} + \frac{(\rho v)}{x} = 0 \quad (\text{B2.4.1})$$

momentum equation

$$A \frac{v}{t} + v \frac{v}{x} = \frac{(A)}{x} + Ab \quad \frac{v}{t} + v \frac{v}{x} = \frac{\sigma}{x} + b \quad (\text{B2.4.2})$$

strain measure (rate-of-deformation): $D_x = v_{,x}$ (B2.4.3)

constitutive equation in rate form:

$$\frac{D}{Dt} = \rho_{,t}(x,t) + \rho_{,x}(x,t)v(x,t) = S_t^D(D_x(X, \bar{t}), (X, \bar{t}), \text{etc.}, \bar{t} - t) \quad (\text{B2.4.4})$$

energy conservation equation
same as before

The governing equations are summarized in Box 2.4. In comparison with the updated Lagrangian formulation we have just discussed, four points are noteworthy:

1. The mass conservation equation is now written as a partial differential equation; the form used with Lagrangian meshes is not applicable because it applies only to material points.
2. The material time derivative for the velocity in the momentum equation has been written out in terms of the spatial time derivative and transport term.
3. The constitutive equation is expressed in rate form; the total form cannot be used since the stress and rate of deformation are functions of material coordinates in a history-dependent material.
4. The boundary conditions are now imposed on spatial points which do not move with time.

The continuity equation has been written as a partial differential equation because it is not possible to obtain an integral form such as Eq. (2.2.4) when the density is a function of spatial coordinates. Therefore, the continuity equation must be treated as a separate partial differential equation, although there are approximations which enable the continuity equation to be omitted when the density changes little, as for a liquid or solid; these are discussed in Chapter 7.

The constitutive equation needs to be expressed in terms of material coordinates for history-dependent materials, so it is treated in rate form in this formulation. It is thus a separate partial differential equation.

In the general case, boundary conditions are required for the density, velocity and stress. As will be seen in Chapter 7, the boundary conditions for the density and stress in an Eulerian mesh depend on whether the material is flowing in or out at the boundary. In this introductory exposition, we consider only boundaries where there is no flow. The boundary points are then Lagrangian, and the density and stress can be determined at these points by the Lagrangian mass conservation equation, Eq. (2.2.10) and the constitutive

equation, respectively. Therefore, there is no need for boundary conditions for these variables.

2.10 WEAK FORMS FOR EULERIAN MESH EQUATIONS

In the Eulerian formulation, we have 3 unknowns or dependent variables: the density $\rho(x, t)$, the velocity $v(x, t)$ and the stress $\sigma(x, t)$. The rate-of-deformation can easily be eliminated from the momentum equations by substituting (B2.4.3) into the constitutive equation (B2.4.4). Therefore, we will need three sets of discrete equations. A weak forms of the momentum equation, the mass conservation equation and the constitutive equation will be developed. We will construct continuous solutions to the governing equations. The equations given in Box 2.4 can in fact have discontinuous solutions, with discontinuities in the density, stress and velocity, as when a shock occurs in the flow. However, we will take the approach of smearing any discontinuities over several elements with a continuous function; this approach is called shock fitting or shock smearing. The trial and test functions will therefore be continuous functions of space.

We consider first the weak form of the continuity equation. The trial functions for the density are denoted by $\rho(x, t)$, the test functions by $w(x)$. The test functions and the trial functions for the continuity equation must be piecewise continuously differentiable, so

$$\rho(x, t) \in R, \quad R = \left\{ \rho(x, t) \mid \rho(x, t) \in C^0(x), \rho(x, t) = \bar{\rho} \text{ on } \Gamma \right\} \quad (2.10.1)$$

$$w(x) \in R_0, \quad R_0 = \left\{ w(x) \mid w(x) \in C^0(x), w(x_a) = 0, w(x_b) = 0 \right\} \quad (2.10.2)$$

In this Section, we do not consider problems with prescribed densities on the boundaries.

The weak form of the continuity equation is obtained by multiplying it by the test function $w(x)$ and integrating over the domain. This gives

$$\int_{x_a}^{x_b} (\rho_{,t} + (v\rho)_{,x}) dx = 0 \quad \forall w(x) \in R_0 \quad (2.10.3)$$

Only first derivatives with respect to the spatial variable of the density and velocity appear in the weak form, so there is no need for integration by parts. The consequence of integrating by parts are interesting and is examined in the Exercises.

The weak form of the constitutive equation is obtained the same way. We express the material derivative in terms of a spatial derivative and a transport term, giving

$$\rho_{,t} + v_{,x}\rho - S(v_{,x}, \text{etc}) = 0 \quad (2.10.4)$$

The test and trial functions, $w(x)$ and $\rho(x, t)$, respectively, are subject to the same continuity and end conditions as for the density in the continuity equation, i.e., we let $w(x_a) = w(x_b) = 0$. The weak form of the constitutive equation is then obtained by multiplying it by the test function and integrating over the domain:

$$\int_{x_a}^{x_b} (\rho_{,t} + v_{,x}\rho - S(v_{,x}, \text{etc})) dx = 0 \quad \forall w(x) \in R_0 \quad (2.10.5)$$

As in the continuity equation, there is no benefit in integrating by parts. Neither this weak form nor the weak continuity equation have a clear physical meaning. They will be referred to as the weak continuity and constitutive equations.

The weak form of the momentum equation is obtained by integrating the test function $v(x)$ over the spatial domain. The procedure is identical to that in the updated Lagrangian formulation in Section 2.7. The test and trial functions are defined by Eqs (2.7.1) and (2.7.2). The resulting weak form is

$$\int_{x_a}^{x_b} v_{,x} A - v Ab - A \frac{Dv}{Dt} dx - \left(v A \bar{t}_x \right) \Big|_i = 0 \quad (2.10.6)$$

or using (??)

$$\int_{x_a}^{x_b} v_{,x} A + v A \frac{v}{t} + v_{,x} v - b dx - \left(v A \bar{t}_x \right) \Big|_i = 0 \quad (2.10.7)$$

Note that the limits of the integration are fixed in space.

The weak form is identical to the principle of virtual power for the updated Lagrangian formulation except that the domain is fixed in space and the material time derivative is expressed in its Eulerian form. Thus the weak form of the momentum equation can be written

$$P = P^{int} - P^{ext} + P^{inert} = 0 \quad v U_0 \quad (2.10.8)$$

where

$$P^{int} = \int_{x_a}^{x_b} v_{,x} A dx = \int_{x_a}^{x_b} D_x A dx = \int_{x_a}^{x_b} D_x \quad (2.10.9)$$

$$P^{ext} = \int_{x_a}^{x_b} v b A dx + \left(v A \bar{t}_x \right) \Big|_i \quad (2.10.10)$$

$$P^{inert} = \int_{x_a}^{x_b} v \frac{v}{t} + v_{,x} v A dx = \int_{x_a}^{x_b} v \frac{v}{t} + v_{,x} v d \quad (2.10.11)$$

All of the terms are identical to the corresponding terms in the principle of virtual power for the updated Lagrangian formulation, except that the limits of integration are fixed in space and the material time derivative in the inertial virtual power has been expressed in terms of the spatial time derivative and the transport term. Similar expressions for the virtual powers also hold on the element level.

2.11. FINITE ELEMENT EQUATIONS

In a general Eulerian finite element formulation, approximations are needed for the pressure, stress and velocity. For each independent variable, test and trial functions are needed. We will develop the equations for the entire mesh. For simplicity, we consider the case where the segment is $0 \leq x \leq L$. As mentioned before, we consider the case where the end points are fixed in space and the velocities on these points vanish. There are then no boundary conditions on the density or stress and the boundary conditions on the velocity are

$$v(0, t) = 0, \quad v(L, t) = 0 \quad (2.11.1)$$

The mapping between spatial and element parent coordinates is given by

$$x = N_I(\xi) x_I \quad (2.11.2)$$

In contrast to the Lagrangian formulations, this mapping is constant in time since the nodal coordinates x_I are not functions of time. The trial and test functions are given by

$$(x, t) = \sum_{I=1}^{n_N} N_I(x) \xi_I(t) \quad (x) = \sum_{I=1}^{n_N} N_I(x) \xi_I \quad (2.11.3)$$

$$(x, t) = \sum_{I=1}^{n_N} N_I(x) \xi_I(t) \quad (x) = \sum_{I=1}^{n_N} N_I(x) \xi_I \quad (2.11.4)$$

$$v(x, t) = \sum_{I=2}^{n_N-1} N_I(x) v_I(t) \quad v(x) = \sum_{I=2}^{n_N-1} N_I(x) v_I \quad (2.11.5)$$

The velocity trial functions have been constructed so the velocity boundary condition is automatically satisfied.

Substituting the test and trial functions for the density into the weak continuity equation gives

$$\sum_{I=1}^{n_N} \sum_{J=1}^{n_N} \int_0^L \left(N_J N_{I,t} + N_J (v)_{,x} \right) dx = 0 \quad (2.11.6)$$

Since this holds for arbitrary ξ_J at interior nodes, we obtain

$$\int_0^L N_I N_J dx \xi_{J,t} + \int_0^L N_I (v)_{,x} dx = 0 \quad I = 1 \text{ to } n_N \quad (2.11.7)$$

We define the following matrices

$$M_{IJ} = \int_0^L N_I N_J dx, \quad \mathbf{M}_e = \int_e (\mathbf{N})^T \mathbf{N} dx \quad (2.11.8)$$

$$g_I = \int_0^L N_I (v)_{,x} dx, \quad \mathbf{g}_e = \int_0^L (\mathbf{N})^T (v)_{,x} dx \quad (2.11.9)$$

The discrete continuity equation can be then be written as

$$M_{IJ} \dot{v}_J + g_I = 0 \text{ for } I = 1 \text{ to } n_N, \text{ or } \mathbf{M} \dot{\mathbf{v}} + \mathbf{g} = 0 \quad (2.11.10)$$

The matrices \mathbf{M} can be assembled from element matrices just like the mass matrix in the momentum equation. The column matrix \mathbf{g} is obtained by a scatter, or vector assembly. The matrix \mathbf{M} is time invariant and closely resembles the mass matrix. However, the column matrix \mathbf{g} varies with time and must be computed in every time step. In most cases, the element matrices are integrated in the parent coordinate system.

The discrete form of the constitutive equation is obtained similarly. The result is

$$M_{IJ} \dot{v}_J + g_I = h_I \text{ for } I = 1 \text{ to } n_N, \text{ or } \mathbf{M} \dot{\mathbf{v}} + \mathbf{g} = \mathbf{h} \quad (2.11.11)$$

where

$$M_{IJ} = \int_0^L N_I N_J dx \quad \text{and} \quad \mathbf{M}_e = \int_e (\mathbf{N})^T \mathbf{N} d \quad (2.11.12)$$

$$g_I = \int_0^L N_I v_{,x} dx \quad \text{and} \quad \mathbf{g}_e = \int_e (\mathbf{N})^T v_{,x} d \quad (2.11.13)$$

where the matrix relations on the right have been extracted from the indicial forms and immediately specialized to elements by the procedure in Section 2.8.

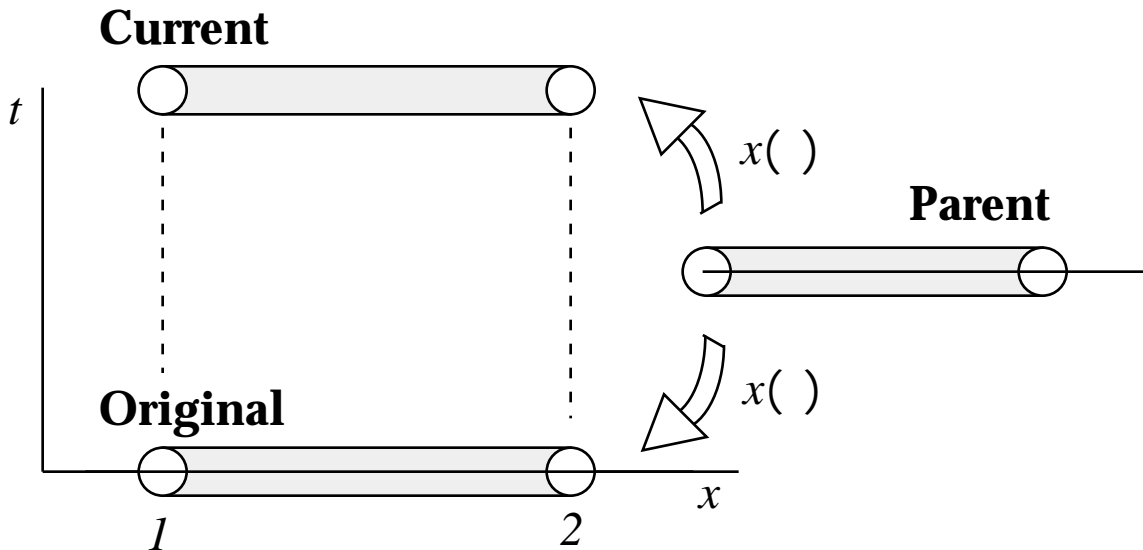


Fig. 2.8 Eulerian element in current and original configurations, which are the same, and the mapping to the parent element.

Momentum Equation. The weak form of the momentum equation is identical to the weak form for the updated Lagrangian formulation except for the inertial term. Therefore the expressions for the internal and external nodal forces are identical. The inertial nodal

forces for the Eulerian formulation are obtained in the following on an element level. We define the inertial nodal forces by Eq. (2.7.8) which gives

$$\mathbf{p}_e^{iner} = \mathbf{v}_e^T \mathbf{f}_e^{inert} = \mathbf{v}_e^T \int_e \mathbf{N}^T (\mathbf{N} \dot{\mathbf{v}} + v_{,x} v) A dx \quad (2.11.14)$$

From the above, it follows that the inertial nodal forces are given by

$$\mathbf{f}_e^{iner} = \mathbf{M}_e \dot{\mathbf{v}}_e + \mathbf{f}_e^{tran} \quad (2.11.15)$$

where

$$\mathbf{M}_e = \int_e \mathbf{N}^T \mathbf{N} A dx, \quad \mathbf{f}_e^{tran} = \int_e v_{,x} v A dx \quad (2.11.16)$$

The transport nodal forces have not been written in matrix form; they are quadratic in the nodal velocities. This term is needed in the Eulerian formulation because the nodes are fixed in space, so the time derivatives of the nodal velocities correspond to spatial derivatives. The mass matrix differs from the mass matrix in the Lagrangian meshes in that it is a function of time: as the density in the element changes, the mass matrix will change correspondingly.

Example. Two-Node Eulerian Finite Element. The finite element equations are developed for a one-dimensional, two node element with linear velocity, density and stress fields. The element, shown in Fig. 2.8, is of length $\ell = x_2 - x_1$ and unit cross-sectional area. As can be seen, the spatial configuration does not change with time since it is an Eulerian element. The map between element and spatial coordinates is given by

$$x(\xi) = \begin{bmatrix} 1 - \xi \\ \xi \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix} = \mathbf{N}(\xi) \mathbf{x}_e \quad (2.11.17)$$

The density, velocity and stress are also interpolated by the same linear shape functions

$$\rho(\xi) = \mathbf{N}(\xi) \rho_e, \quad v(\xi) = \mathbf{N}(\xi) v_e, \quad \sigma(\xi) = \mathbf{N}(\xi) \sigma_e \quad (2.11.18)$$

Superscripts are not appended to the shape functions because all variables are interpolated by the same shape functions.

Density Equation. The element matrices for the discrete continuity equation are given by

$$\mathbf{M}_e = \int_{x_1}^{x_2} \mathbf{N}^T \mathbf{N} dx = \int_0^1 \begin{bmatrix} 1 - \xi \\ \xi \end{bmatrix} \ell d\xi = \frac{\ell}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (2.11.19)$$

$$\mathbf{g}_e = \int_{x_1}^{x_2} \mathbf{N}^T (v)_{,x} dx = \int_0^1 \begin{bmatrix} 1 - \xi \\ \xi \end{bmatrix} (v)_{,x} \ell d\xi \quad (2.11.20)$$

The \mathbf{g}_e vector is usually evaluated by numerical quadrature. For linear interpolants it is given by

$$\mathbf{g}_e = \frac{1}{6} \begin{pmatrix} 2v_1 + v_2 \\ v_1 + 2v_2 \end{pmatrix} + \frac{1}{6} (v_2 - v_1) \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (2.11.21)$$

The above matrix vanishes when the density and velocity are constant in the element.

Stress Equation. The element matrix for the stresses $\mathbf{M}_e = \mathbf{M}_e$. The vector \mathbf{g}_e is given by

$$\begin{aligned} \mathbf{g}_e &= \int_{x_1}^{x_2} \mathbf{N}^T \mathbf{v}_{,x} dx = \int_0^1 \begin{pmatrix} 1-x \\ x \end{pmatrix} (v_1(1-x) + v_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} dx \\ &= \frac{1}{6} \begin{pmatrix} 2v_1 + v_2 \\ v_1 + 2v_2 \end{pmatrix} \end{aligned} \quad (2.11.22)$$

In summary, the finite element equations for the Eulerian formulation consists of three sets of discrete equations: the continuity equation, the constitutive equation, and the momentum equation, or equation of motion. The momentum equation is similar to the updated Lagrangian form, except that the inertial term includes a transport term and varies with time. All nodal forces are defined over fixed intervals in space. The semidiscrete forms of the continuity and constitutive equations are first order ordinary differential equations. We have only developed the discrete equations for the case where the endpoints are fixed.

2.12 Solution Methods

We have seen so far that the momentum equation can be discretized with a Lagrangian mesh in the form

$$\mathbf{M}\ddot{\mathbf{u}} = \mathbf{f}^{\text{ext}} - \mathbf{f}^{\text{int}} = \mathbf{f} \quad (2.12.1)$$

These are ordinary differential equations in time.

In order to enable some nonlinear problems to be solved at this point, we now describe the simplest solution method, explicit time integration of the equations of motion for a Lagrangian mesh. The most widely used explicit method is based on the central difference formulas. Explicit integration can be simplified further by replacing \mathbf{M} by a *diagonal* or *lumped mass* matrix.

We start at time $t=0$ using time steps Δt , so that at time step n , $t = n \Delta t$. The value of a function at $n \Delta t$ is denoted by a superscript n , i.e., $\mathbf{u}^n = \mathbf{u}(n \Delta t)$. In the central difference method, the velocities are approximated by

$$\dot{\mathbf{u}}^n = \mathbf{v}^{n+1/2} = \frac{\mathbf{u}^{n+1/2} - \mathbf{u}^{n-1/2}}{\Delta t} = \frac{\mathbf{u}(t + \Delta t/2) - \mathbf{u}(t - \Delta t/2)}{\Delta t} \quad (2.12.2)$$

where the second equality is included to clarify the notation. Half time step values are used for the velocities. The accelerations are given by

$$\ddot{\mathbf{u}}^n = \mathbf{a}^n = \frac{\mathbf{v}^{n+1/2} - \mathbf{v}^{n-1/2}}{t} \quad (2.12.3)$$

In each case, the value of the derivative at the center of a time interval is obtained from the difference of the function values at the ends of the interval, hence the name *central difference* formulas. The flow chart for an explicit program is then given by the following Box.

Box 2.5 Flowchart for Explicit Time Integration of Lagrangian Mesh

1. Initial conditions and initialization: set $\mathbf{v}^0, \mathbf{u}^0; n = 0, t = 0$; compute \mathbf{M}
2. get \mathbf{f}^n
3. compute accelerations $\mathbf{a}^n = \mathbf{M}^{-1}\mathbf{f}^n$
4. update nodal velocities: $\mathbf{v}^{n+1/2} = \mathbf{v}^{n-1/2} + t\mathbf{a}^n$; $\tau = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ 1 & \text{if } n > 0 \end{cases}$
5. enforce essential boundary conditions: if node I on $\Gamma_v: v_I^n = \bar{v}(x_I, t_n)$
6. update nodal displacements: $\mathbf{u}^{n+1} = \mathbf{u}^n + \tau\mathbf{v}^{n+1/2}$
7. update counter and time: $n = n + 1, t = t + \tau$
8. output, if simulation not complete, go to 2

Module: get \mathbf{f}

1. GATHER element nodal displacements \mathbf{u}_e^n and velocities $\mathbf{v}_e^{n+1/2}$
2. compute measure of deformation
3. compute stress by constitutive equation $\boldsymbol{\sigma}_e^n$
4. compute internal nodal forces by equation in Box.
5. compute external nodal forces on element and $\mathbf{f}_e = \mathbf{f}_e^{ext} - \mathbf{f}_e^{int}$
5. SCATTER element nodal displacements to global matrices

Updating for the displacements by Eq. (6) then does not require *any solution* of algebraic equations. Thus, in a sense, explicit integration is simpler than static linear stress analysis. As can be seen from the flowchart, most of the explicit program is a straightforward interpretation of the governing equations and the time integration formulas. The program begins with the enforcement of the initial conditions; procedures for fitting different initial conditions have already been described. The first time step is somewhat different from the others because only a half-step is taken. This enables the program to correctly account for the initial conditions on the stresses and velocities.

Most of the programming and computation time is in computing the element nodal forces, particularly the internal nodal forces. The nodal forces are computed element-by-element. Prior to starting the element computations, the element nodal velocities and displacements are gathered from the global arrays. As can be seen from the flowchart, the computation of the internal nodal forces involves the application of the equations which are left in strong form, the strain equation and the constitutive equation, followed by the evaluation of the internal nodal forces from the stress, which emanates from the weak form

of the momentum equation. When the computation of the element nodal forces is completed, they are scattered to the global array according to their node numbers.

The essential boundary conditions are enforced quite easily as shown. By setting the nodal velocities equal to the prescribed nodal velocities at all nodes on prescribed velocity boundaries, the correct displacements result, since the velocities are subsequently integrated in time. The placement of this step in the flowchart insures that the correct velocities are available in the nodal force computation. The initial velocities must be compatible with the boundary conditions; this is not checked in this flowchart but would be checked in a production program. The reaction forces can be obtained by outputting the total nodal forces at the prescribed velocity nodes.

It can be seen from the flowchart that the traction boundary conditions enter only through the external nodal forces. Therefore, for a traction-free boundary, nothing need be done: the homogeneous traction boundary condition is enforced naturally in a weak sense by the finite element solution. However, the traction boundary condition is only satisfied approximately.

Stability Criterion. The disadvantage of explicit integration is that the time step must be below a critical value or the solution "blows up" due to a numerical instability. This is described in detail in Chapter 6. Here we limit ourselves to pointing out that the critical time step for the 2-node elements described in this Chapter is given by

$$t_{crit} = \frac{\ell}{c} \quad (2.12.4)$$

where ℓ is the *current* length of the element and c is the wave speed given by

$$c^2 = E / \rho \quad (2.12.5)$$

where E is the modulus in the elastic relation between Cauchy stress and rate-of-deformation. For nonlinear materials, E is replaced by the current tangent modulus

$$E^t_D = \frac{\dot{\sigma}}{\dot{\epsilon}} \quad (2.12.6)$$

A computer program for the explicit integration of the one-dimensional updated and total Lagrangian formulation is given in Appendix B.

Appendix A. Derivation of Conservation Equations in 1D.

In this Appendix, simple derivations of the conservation equations in one dimension will be given. These are "engineering" derivations which develop these equations in simplest terms, and they lack the mathematical rigor and generality associated with the derivations found in texts on continuum mechanics.

We first derive the equation of conservation of mass, also called the continuity equation. Consider a segment of the rod shown in Fig. 2.7, which in the initial, or undeformed, state is of length X , cross-sectional area A_0 and density ρ_0 . In the deformed state this Lagrangian segment has length x , area A and density ρ . The subdomain is a material, or Lagrangian subdomain, in that all material points remain in the subdomain and

the right and left end points are the same material points. No flow of material occurs in or out of the segment. Therefore, by mass conservation, the mass in the undeformed segment must equal the mass of the deformed segment:

$$\rho A(x) dx = \rho_0 A_0(X) dX \quad (A.1)$$

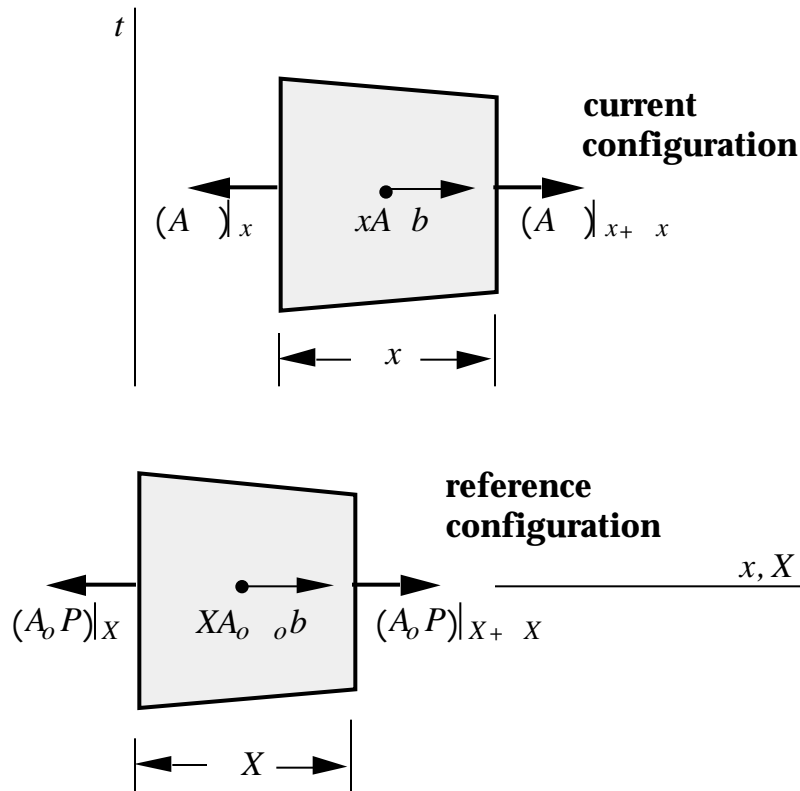


Fig. 2.9. A segment of a rod in the reference (initial, undeformed) and current configurations showing all forces acting on the segment.

Dividing by dX and taking the limit as $dX \rightarrow 0$ gives

$$\rho_0 A_0 = \rho A \frac{dx}{dX} = \rho A F \quad \text{or} \quad \rho_0(X) A_0(X) = \rho(X,t) A(X,t) F(X,t) \quad (A.2)$$

The above is one form of the equation of mass conservation. On the right hand side the independent variables have been indicated to stress that this equation only holds when expressed in terms of the material coordinates; the variables on the RHS are functions of time, whereas the variables on the LHS are independent of time.

To obtain another form of this equation, we note that the volumes of the segment are related by the Jacobian by $dV = J dV_0$. Since $dV = A dx$ and $dV_0 = A_0 dX$, it follows that

$$J = \frac{A}{A_0} F \quad (A.3)$$

Substituting the above into (A.1) gives the another form of the mass conservation equation

$$\rho_0(X) = \rho(X, t) J(X, t) \quad (\text{A.4})$$

The above equation also applies in multi-dimensional problems.

Momentum Equation. The momentum equation is derived by considering the segment of the rod shown in Fig. 2.9. The forces on the deformed segment are shown in Fig. 2.9 and consist of the forces arising from the stress, which act on the right and left hand end of the segment, and the body force; the body force is distributed over the entire segment and its net resultant is placed at the center of the segment. On the left hand end of the segment the force is $(A \sigma)_{|_x}$, i.e. the product of the stress and the current area at the point x . The force due to the stress on the right hand end is given by $(A \sigma)_{|_{x+\Delta x}}$. The resultant force due to the body force is obtained by multiplying $b\left(x + \frac{\Delta x}{2}\right)$ by the mass of the segment $\rho_0 A \Delta x$. So if we write Newton's second law for the segment we have

$$-(A \sigma)_{|_x} + (A \sigma)_{|_{x+\Delta x}} + (\rho_0 A b)_{|_{x+\frac{\Delta x}{2}}} \Delta x = (\rho_0 A \Delta x \ddot{u})_{|_{x+\frac{\Delta x}{2}}} \quad (\text{A.5})$$

where the LHS is the sum of the resultant forces from the stress and the body force and the RHS is the product of the mass of the segment and its acceleration.

The forces due to the stresses are now expanded by a Taylor's series about the midpoint of the segment, with the product $A \sigma$ treated as a single function, which gives

$$(A \sigma)_{|_{x+\Delta x}} = (A \sigma)_{|_{x+\Delta x/2}} + \frac{(A \sigma)_{|_x}}{\Delta x} \frac{\Delta x}{2} + O(\Delta x^2) \quad (\text{A.6a})$$

$$(A \sigma)_{|_x} = (A \sigma)_{|_{x+\Delta x/2}} - \frac{(A \sigma)_{|_x}}{\Delta x} \frac{\Delta x}{2} + O(\Delta x^2) \quad (\text{A.6b})$$

The use of a Taylor series expansion of course presupposes that the function is smooth enough so that the first derivative exists; this is not the case wherever the stress or the area is discontinuous. Substituting (A.6) into (A.5) and dividing by Δx gives

$$(A \sigma)_{|,x} + A b = \rho_0 A \ddot{u} \quad (\text{A.7})$$

The above is the momentum equation for a one-dimensional continuum of varying cross-section.

To derive the momentum equation in the reference configuration, we note that the forces on the sides of the segment are given by multiplying the nominal stress by the initial area, $A_0 P$. The net force due to the body force is $\rho_0 A_0 b \Delta X$ since $\rho_0 b$ is a force per unit initial volume and the initial volume is $A_0 \Delta X$. The mass of the segment is $\rho_0 A_0 \Delta X$. Writing Newton's second law for the segment gives

$$(-A_0 P)_{|_X} + (A_0 P)_{|_{X+\Delta X}} + (\rho_0 A_0 b)_{|_{X+\frac{\Delta X}{2}}} \Delta X = (\rho_0 A_0 \Delta X \ddot{u})_{|_{X+\frac{\Delta X}{2}}} \quad (\text{A.8})$$

where the LHS is the sum of all forces acting on the segment and the RHS is the mass time the acceleration. Expressing the forces due to the nominal stresses by a Taylor series as in (A.6), but in terms of the material coordinate X , substituting into (A.8) and dividing by X gives the momentum equation in Lagrangian form

$$\left(A_0 P\right)_{,X} + {}_0 A_0 b = {}_0 A_0 \ddot{u} \quad (\text{A.9})$$

The above can easily be transformed to the Eulerian form, Eq. (A.7). By the stress transformation (2.1.2), we have $A_0 P = A$, so

$$\left(A_0 P\right)_{,X} = \left(A\right)_{,x} = \left(A\right)_{,x} x_{,X} = \left(A\right)_{,x} F \quad (\text{A.10})$$

where the chain rule has been used in the third step, followed by the definition of F in Eq. (2.2.2). Substituting the (A.10) into (A.9) gives

$$0 = \left(A\right)_{,x} F + {}_0 A_0 b - {}_0 A_0 \ddot{u} = \left(A\right)_{,x} F + F A b - F A \ddot{u}$$

where the continuity equation (A.2) has been used in the last step. Dividing by F then gives the momentum equation in Eulerian form. Note that the body force in the Lagrangian and Eulerian momentum equations is identical. Some authors distinguish the body force in the total form by a subscript naught, i.e., Malvern (1969, p. 224), but this is superfluous if the body force is considered a force per unit mass so that b is a force per unit volume.

SUMMARY

The finite element equations have been developed for one-dimensional continua of varying cross-section. Two mesh descriptions have been used:

1. Lagrangian meshes, where the nodes and elements move with the material;
2. Eulerian meshes, in which nodes and elements are fixed in space.

Two formulations have been developed for Lagrangian meshes:

1. a total Lagrangian formulation, in which the strong form is expressed in spatial coordinates, i.e. the Eulerian coordinates;
2. an updated Lagrangian formulation, where the strong form is expressed in the material, i.e. the Lagrangian coordinates.

In both cases, the element formulation is most conveniently executed in terms of the element coordinates. The mapping of the element coordinates from current and original configuration for a valid finite element discretization is one-to-one and onto. Furthermore, the mapping to the original configuration is time invariant, so the element coordinates can serve as surrogate material coordinates.

It has also been shown that the updated and total Lagrangian formulations are two representations of the same mechanical behavior, and each can be transformed to the other at both the level of partial differential equations and the level of the discrete finite element equations. Thus the internal and external forces obtained by the total Lagrangian formulations are identical to those obtained by the updated formulation, and the choice of formulation is purely a matter of convenience.

The equation of motion corresponds to the momentum equation and is obtained from its weak form. As has been illustrated in the case of explicit time integration, the other equations, measure of deformation and constitutive, are used in the course of computing the internal nodal forces to update the displacements. The weak form and discrete equations have been structured so that their relationship to the corresponding terms in the partial differential equation of momentum conservation is readily apparent: the internal forces correspond to the stress terms, and the internal work (or power); the external forces correspond to the body forces and external work (or power); the terms $\mathbf{M}\mathbf{a}$ correspond to the inertial terms (d'Alembert) forces and the inertial work (or power). This correspondence is summarized in Fig. 8, which shows the steps which are used to convert the partial differential equation of momentum balance to a set of ordinary differential equations which are called the equations of motion. This process is called a *spatial discretization* or *semidiscretization*.

The discretization has been carried out for the general case when inertial forces are not negligible. If the inertial forces can be neglected, the term $\mathbf{M}\mathbf{a}$ is omitted from the discrete equations. The resulting equations are either nonlinear algebraic equations or ordinary differential equations, depending on the character of the constitutive equation.

The governing equations have been developed for a one-dimensional rod of varying cross-section and from these a weak form has been developed by integrating over the domain. When the equations are given in terms of partial derivatives with respect to the material derivatives, it is natural to develop the weak form by integrating over the undeformed domain. This leads to the *total Lagrangian formulation* where all nodal forces are obtained by integrating over the material coordinates. When the partial derivatives are with respect to the spatial coordinates, it is natural to integrate over the current configuration, which leads to the *updated Lagrangian formulation*.

The process of discretization for multidimensional problems is very similar. However, in multi-dimensional problems we will have to deal with the major consequence of geometric nonlinearities, large rotations, which are completely absent in one-dimensional problems.

Exercises

Exercise: Repeat Example 2.8.3 for spherical symmetry, where

$$\mathbf{h} = \begin{matrix} r \\ \end{matrix} \quad \mathbf{s} = \begin{matrix} r \\ \end{matrix}$$

give \mathbf{B} , \mathbf{f}_e^{int} , \mathbf{f}_e^{ext} , \mathbf{M}_e